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ON SPACE-LIKE KÄHLER SUBMANIFOLDS WITH $RS = 0$ IN AN INDEFINITE COMPLEX SPACE FORM

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ABSTRACT. In this paper we give some characterizations of Kähler manifolds and complete space-like complex submanifolds with the Ryan condition $RS = 0$ in an indefinite complex hyperbolic space.

§1. Introduction

As well known, Ryan [11] investigated complex hypersurfaces in a complex space form satisfying the condition

$$R(X,Y)S = 0 \tag{1.1}$$

for any vector fields X and Y tangent to the hypersurface M, where R denote the Riemannian curvature tensor, S is the Ricci tensor on M and $R(X,Y)$ operates on the tensor algebra as a derivation. The condition (1.1) is called the Ryan one. Relative to the Ryan condition, Ryan [11] proved that these hypersurfaces are Einstein manifolds if the holomorphic sectional curvature of the ambient space does not vanish, which was generalized from two distinct directions. One of them is due to Takahashi [12], who verified that such hypersurfaces become cylindrical if the ambient space is complex Euclidean. Another extension is treated by Kon [7] in the case of complex submanifolds in a complex space form of constant negative holomorphic sectional curvature. On the other hand, independently of Kon's work, Aiyama, Kwon and Nakagawa [1] researched about properties on space-like complex submanifolds satisfying the Ryan condition in an indefinite complex space form. In the case of complex submanifolds in a complex space form of constant positive holomorphic sectional curvature, these submanifolds were determined by Nakagawa and Takagi [8].

On the other hand, Ki and Suh [4] observed the Ryan condition from the different point of view and obtained a nice theorem about Kähler manifolds whose totally real bisectional curvature is bounded from below by a positive constant. Thus it seems to us to be interesting to investigate the space-like Kähler submanifolds satisfying the Ryan condition of an indefinite complex space form.

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So, this paper has two purposes, one of which is to give a more generalized property than the theorems by Kon, and Aiyama and et. about the condition $RS = 0$ in terms of totally real bisectional curvatures and another is to prove the theorem related the Nakagawa and Takagi theorem. Namely, the purpose is to prove the following two theorems.

Theorem 1. *Let M be a Kähler manifold whose totally real bisectional curvature is bounded from above(resp. below) by a negative(resp. positive) constant. If it satisfies the condition (1.1), then M is Einstein.*

Theorem 2. *Let M be an $n(\geqq 2)$-dimensional complete space-like complex submanifold in an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c. If M satisfies the condition (1.1) and if the codimension p is less than $n-1$, then M is Einstein.*

§2. Semi-definite complex submanifolds

This section is concerned with semi-definite complex submanifolds of a semi-definite Kähler manifold (see O'Neill [10] for examples). First of all, some basic formulas for the theory of semi-definite complex submanifolds are prepared.

Let M' be an $(n+p)$-dimensional connected semi-definite Kähler manifold of index $2(s+t)$ $(0 \leqq s \leqq n,\ 0 \leqq t \leqq p)$ with semi-definite Kähler structure (g', J'). Let M be an n-dimensional connected complex submanifold of M' and let g be the induced semi-definite Kähler metric tensor of index $2s$ on M from g'. We can choose a local field $\{U_A\} = \{U_i, U_x\} = \{U_1, \cdots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated.

$$A, B, \cdots = 1, \cdots, n, n+1, \cdots, n+p,$$
$$i, j, \cdots = 1, \cdots, n, \qquad x, y, \cdots = n+1, \cdots, n+p.$$

With respect to the unitary frame field $\{U_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kähler metric tensor g' of M' is given by $g' = 2\sum_A \epsilon_A \omega_A \otimes \bar{\omega}_A$, where $\{\epsilon_A\} = \{\epsilon_i, \epsilon_x\}$ satisfy

$$\epsilon_i = 1 \text{ or } -1 \text{ according as } 1 \leqq i \leqq n-s \text{ or } n-s+1 \leqq i \leqq n,$$
$$\epsilon_x = 1 \text{ or } -1 \text{ according as } n+1 \leqq x \leqq n+p-t \text{ or } n+p-t+1 \leqq x \leqq n+p.$$

The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\begin{aligned} &d\omega_A + \sum_B \epsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{AB} = 0, \\ &d\omega_{AB} + \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \quad \Omega'_{AB} = \sum_{C,D} \epsilon_C \epsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{2.1}$$

where $\Omega' = (\Omega'_{AB})$ (*resp.* $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form (*resp.* the components of the semi-definite Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{2.2}$$

and the induced semi-definite Kähler metric tensor g of index $2s$ of M is given by $g = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ its dual frame field, which consists of complex valued 1-forms of type (1.0) on M. It follows from (2.2) and Cartan's lemma that the exterior derivatives of (2.2) give rise to

$$\omega_{xi} = \sum_j \epsilon_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{2.3}$$

The quadratic form $\alpha = \sum_{i,j,x} \epsilon_i \epsilon_j \epsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations of M' it follows that the structure equations for M are similarly given by

$$\begin{aligned} & d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ & d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = \sum_{k,l} \epsilon_k \epsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{2.4}$$

where $\Omega = (\Omega_{ij})$ (*resp.* $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (*resp.* the component of the semi-definite Riemannian curvature tensor R) of M. Furthermore, the first Bianchi identity $\sum_j \epsilon_j \Omega_{ij} \wedge \omega_j = 0$ is given by the exterior differential of the first equation of (2.4). Namely, we get

$$\sum_{j,k,l} \epsilon_j \epsilon_k \epsilon_l R_{\bar{i}jk\bar{l}} \omega_j \wedge \omega_k \wedge \bar{\omega}_l = 0,$$

which implies the further symmetric relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}kj\bar{i}} = R_{\bar{l}jk\bar{i}}. \tag{2.5}$$

Moreover, the following relationships are obtained.

$$d\omega_{xy} + \sum_z \epsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \sum_{k,l} \epsilon_k \epsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.6}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (2.1), (2.3) and (2.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \epsilon_x h^x_{jk} \bar{h}^x_{il},$$

and by means of (2.1), (2.3) and (2.6) we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j \epsilon_j h^{\bar{x}}_{kj} \bar{h}^y_{jl}.$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$(2.7) \qquad S_{i\bar{j}} = \sum_k \epsilon_k R'_{\bar{j}ik\bar{k}} - h^2_{i\bar{j}}, \quad r = 2(\sum_{j,k} \epsilon_j \epsilon_k R'_{\bar{j}jk\bar{k}} - h_2),$$

where $h^2_{i\bar{j}} = h^2_{\bar{j}i} = \sum_{k,x} \epsilon_k \epsilon_x h^{\bar{x}}_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_j \epsilon_j h^2_{j\bar{j}}$.

Now, the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form α on M are given by

$$(2.8) \qquad \sum_k \epsilon_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \sum_k \epsilon_k (h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y \epsilon_y h^y_{ij}\omega_{xy}.$$

Then, substituting dh^x_{ij} into the exterior derivative

$$d\omega_{xi} = \sum_j \epsilon_j (dh^x_{ij} \wedge \omega_j + h^x_{ij} d\omega_{kj})$$

of (2.3) and using (2.1) $\sim$ (2.4) and (2.7), we have

$$(2.9) \qquad h^x_{ijk} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}$$

from the coefficients of $\omega_j \wedge \omega_k$ and $\omega_j \wedge \bar{\omega}_k$. Similarly, the components h^x_{ijkl} and $h^x_{ijk\bar{l}}$ (*resp.* $h^x_{ij\bar{k}l}$ and $h^x_{ij\bar{k}\bar{l}}$) of the covariant derivative of h^x_{ijk} (*resp.* $h^x_{ij\bar{k}}$) can be defined by

$$\sum_l \epsilon_l (h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) = dh^x_{ijk} - \sum_l \epsilon_l (h^x_{ljk}\omega_{li} + h^x_{ilk}\omega_{lj} + h^x_{ijl}\omega_{lk}) + \sum_y \epsilon_y h^y_{ijk}\omega_{xy},$$

$$\sum_l \epsilon_l (h^x_{ij\bar{k}l}\omega_l + h^x_{ij\bar{k}\bar{l}}\bar{\omega}_l) = dh^x_{ij\bar{k}} - \sum_l \epsilon_l (h^x_{lj\bar{k}}\omega_{li} + h^x_{il\bar{k}}\omega_{lj} + h^x_{ij\bar{l}}\bar{\omega}_{lk}) + \sum_y \epsilon_y h^y_{ij\bar{k}}\omega_{xy}.$$

Taking the exterior derivative of (2.8) and using the equations (2.4), (2.6) $\sim$ (2.9), the Ricci formula for the second fundamental form on M are given by

$$h^x_{ijkl} = h^x_{ijlk}, \quad h^x_{ij\bar{k}\bar{l}} = h^x_{ij\bar{l}\bar{k}},$$

$$h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} = \sum_m \epsilon_m (R_{\bar{l}ki\bar{m}} h^x_{mj} + R_{\bar{l}kj\bar{m}} h^x_{im}) - \sum_y \epsilon_y R_{\bar{x}yk\bar{l}} h^y_{ij}.$$

In particular, let the ambient space be an $(n+p)$-dimensional semi-definite complex space form $M^{n+p}_{s+t}(c')$ of constant holomorphic sectional curvature c' and of index $2(s+t)$ $(0 \leqq s \leqq n,\ 0 \leqq t \leqq p)$. Then we get

$$(2.10) \qquad R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \epsilon_x h^x_{jk}\bar{h}^x_{il},$$

$$S_{i\bar{j}} = \frac{(n+1)c'}{2}\epsilon_i\delta_{ij} - h^2_{i\bar{j}}, \quad r = n(n+1)c' - h_2.$$

Next, we introduce here a fundamental property for the generalized maximum principal due to Omori [9] and Yau [13].

Theorem 2.1. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. If a C^2-function f is bounded from above on M, then, for any positive constant ϵ, there exists a point p such that*

$$|\nabla f(p)| < \epsilon, \quad \Delta f(p) < \epsilon, \quad sup\ f - \epsilon < f(p),$$

where sup f denotes the supremum of the function f.

Lastly, we consider the totally real bisectional curvature on a Kähler manifold (M, g) with almost complex structure J. A plane section P in the tangent space T_xM of M at any point x in M is said to be *totally real* if P is orthogonal to JP. For the non-degenerate totally real plane P spanned by orthonormal vectors u and v, the *totally real bisectional curvature* $B(u, v)$ is defined by

$$B(u, v) = g(R(u, Ju)Jv, v).$$

Then, using the first Bianchi identity to the above equation and the fundamental properties of the Riemannian curvature tensor of Kähler manifolds, we get

$$B(u, v) = g(R(u, v)v, u) + g(R(u, Jv)Jv, u) = K(u, v) + K(u, Jv),$$

where $K(u, v)$ means the sectional curvature of the plane spanned by u and v.

Kim, Pyo and Shin [5] have proved the following lemma.

Lemma 2.2. *Let M be an $n(\geqq 3)$-dimensional Kähler manifold. If the totally real bisectional curvature is bounded from above (resp. below) by a constant, and if the scalar curvature on M is bounded from below (resp. above), then the following statements hold true ;*

(1) the Ricci curvature on M is bounded.

(2) the totally real bisectional curvature is bounded.

§3. The Laplacian operator

In this section, we calculate the Laplacian of the squared norm of the Ricci tensor S of an n-dimensional Kähler manifold M. Let f be any smooth function on M. The components f_i and $f_{\bar{i}}$ of the exterior derivative df of f are given by

$$df = \sum_i (f_i \omega_i + f_{\bar{i}} \bar{\omega}_i). \tag{3.1}$$

Moreover, the components f_{ij} and $f_{i\bar{j}}$ (*resp.* $f_{\bar{i}j}$ and $f_{\bar{i}\bar{j}}$) of the covariant derivative of f_i (*resp.* $f_{\bar{i}}$) can be defined by

$$\begin{aligned} \sum_j (f_{ij}\omega_j + f_{i\bar{j}}\bar{\omega}_j) &= df_i - \sum_j f_j \omega_{ji}, \\ \sum_j (f_{\bar{i}j}\omega_j + f_{\bar{i}\bar{j}}\bar{\omega}_j) &= df_{\bar{i}} - \sum_j f_{\bar{j}} \bar{\omega}_{ji}. \end{aligned} \tag{3.2}$$

Taking the exterior derivative of (3.1) and using (3.2), we have

$$f_{ij} + f_{ji} = 0, \quad f_{\bar{i}\bar{j}} + f_{\bar{j}\bar{i}} = 0, \quad f_{i\bar{j}} = f_{\bar{j}i}. \tag{3.3}$$

Hence the Laplacian Δf of the function f is given as

$$\Delta f = 2\sum_j f_{j\bar{j}}. \tag{3.4}$$

We put $f = S_2 = |S|^2 = \sum_{j,k} S_{j\bar{k}} S_{k\bar{j}}$. The components $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$ of the covariant derivative of the Ricci tensor S are obtained by

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{3.5}$$

The components $S_{i\bar{j}kl}$ and $S_{i\bar{j}k\bar{l}}$ (*resp.* $S_{i\bar{j}\bar{k}l}$ and $S_{i\bar{j}\bar{k}\bar{l}}$) of the covariant derivative of $S_{i\bar{j}k}$ (*resp.* $S_{i\bar{j}\bar{k}}$) are expressed as

$$\begin{aligned} \sum_l (S_{i\bar{j}kl}\omega_l + S_{i\bar{j}k\bar{l}}\bar{\omega}_l) &= dS_{i\bar{j}k} - \sum_l (S_{l\bar{j}k}\omega_{li} + S_{i\bar{l}k}\bar{\omega}_{lj} + S_{i\bar{j}l}\omega_{lk}), \\ \sum_l (S_{i\bar{j}\bar{k}l}\omega_l + S_{i\bar{j}\bar{k}\bar{l}}\bar{\omega}_l) &= dS_{i\bar{j}\bar{k}} - \sum_l (S_{l\bar{j}\bar{k}}\omega_{li} + S_{i\bar{l}\bar{k}}\bar{\omega}_{lj} + S_{i\bar{j}\bar{l}}\bar{\omega}_{lk}). \end{aligned} \tag{3.6}$$

Taking the exterior derivative of (3.5), we obtain

$$\begin{aligned} &S_{i\bar{j}kl} = S_{i\bar{j}lk}, \quad S_{i\bar{j}\bar{k}\bar{l}} = S_{i\bar{j}\bar{l}\bar{k}}, \\ &S_{i\bar{j}k\bar{l}} - S_{i\bar{j}\bar{l}k} = \sum_m (R_{\bar{m}ik\bar{l}} S_{m\bar{j}} - R_{\bar{j}mk\bar{l}} S_{i\bar{m}}), \end{aligned} \tag{3.7}$$

where we have used (3.4) $\sim$ (3.6).

Now, we are in a position to calculate the Laplacian ΔS^2 of the squared norm of the Ricci tensor S on M. By (3.4) we have

$$\Delta S_2 = 2\sum_{i,j,k} \{2S_{i\bar{j}k}S_{j\bar{i}\bar{k}} + (S_{j\bar{i}}S_{i\bar{j}k\bar{k}} + S_{i\bar{j}}S_{j\bar{i}k\bar{k}})\},$$

and hence we have

$$\Delta S_2 = 2\{|\nabla S|^2 + \sum_{i,j,k} S_{i\bar{j}}(S_{j\bar{i}k\bar{k}} + S_{j\bar{i}\bar{k}k})\}, \tag{3.8}$$

where $|\nabla S|^2$ is the squared norm of the covariant derivative of the Ricci tensor S, i.e., $|\nabla S|^2 = 2\sum_{i,j,k} S_{i\bar{j}k} S_{j\bar{i}\bar{k}}$. For the scalar curvature r on M the components r_j and $r_{\bar{j}}$ of the exterior differential dr are given by

$$dr = \sum_j (r_j\omega_j + r_{\bar{j}}\bar{\omega}_j),$$

and the components r_{jk} and $r_{j\bar{k}}$ (*resp.* $r_{\bar{j}k}$ and $r_{\bar{j}\bar{k}}$) of the covariant derivative of r_j (*resp.* $r_{\bar{j}}$) are given as

$$\sum_k (r_{jk}\omega_k + r_{j\bar{k}}\bar{\omega}_k) = dr_j - \sum_k r_k\omega_{kj},$$
$$\sum_k (r_{\bar{j}k}\omega_k + r_{\bar{j}\bar{k}}\bar{\omega}_k) = dr_{\bar{j}} - \sum_k r_k\bar{\omega}_{kj}.$$

On the other hand, we have

$$r = 2\sum_{i,j} R_{\bar{i}ij\bar{j}} = 2\sum_j S_{j\bar{j}}, \quad r_i = 2\sum_k S_{k\bar{k}i}, \quad r_{ij} = 2\sum_k S_{k\bar{k}ij}. \tag{3.9}$$

Summing up $j = k$ in (3.7) and using (3.9) and the components of the covariant derivative of the Riemannian curvature tensor R, we get

$$r_{i\bar{l}} - 2\sum_k S_{i\bar{l}\bar{k}k} = 2\sum_m (\sum_k R_{\bar{m}ik\bar{l}}S_{m\bar{k}} - S_{i\bar{m}}S_{m\bar{l}}).$$

Accordingly, we have by (2.5)

$$2\sum_k S_{i\bar{j}\bar{k}k} = r_{i\bar{j}} + 2\sum_m (S_{i\bar{m}}S_{m\bar{j}} - \sum_k R_{\bar{j}ik\bar{m}}S_{m\bar{k}}). \tag{3.10}$$

Next, summing up $i = l$ in (3.7), we obtain

$$\sum_l (S_{l\bar{j}k\bar{l}} - S_{l\bar{j}\bar{l}k}) = \sum_{l,m} (R_{\bar{m}lk\bar{l}}S_{m\bar{j}} - R_{\bar{j}mk\bar{l}}S_{l\bar{m}}),$$

from which it follows that we have similarly

$$2\sum_k S_{i\bar{j}k\bar{k}} = r_{\bar{j}i} + 2\sum_m (S_{i\bar{m}}S_{m\bar{j}} - \sum_k R_{\bar{j}ik\bar{m}}S_{m\bar{k}}), \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.8), we obtain

$$\Delta S_2 = 2|\nabla S|^2 + 2\sum_{i,j} S_{i\bar{j}}r_{j\bar{i}} + 4\sum_{m,i,j} S_{j\bar{i}}(S_{i\bar{m}}S_{m\bar{j}} - \sum_k R_{\bar{j}ik\bar{m}}S_{m\bar{k}}), \tag{3.12}$$

where we have used (3.3). Since $(S_{i\bar{j}})$ is a Hermitian matrix, it can be diagonalized. Thus $S_{i\bar{j}} = \mu_i\delta_{ij}$, where μ_i is a real valued function. From this it follows that we have

$$r = 2\sum_j S_{j\bar{j}} = 2\sum_j \mu_j, \quad S_2 = \sum_{i,j} S_{i\bar{j}}S_{j\bar{i}} = \sum_j \mu_j^2, \tag{3.13}$$

$$S_2 - \frac{r^2}{4n} = \frac{1}{n}\sum_{i,j}(\mu_i - \mu_j)^2. \tag{3.14}$$

And, by (3.12) we get

$$\Delta S_2 \geqq 2\sum_{i,j} S_{i\bar{j}}r_{j\bar{i}} + 2\sum_{i,j}(\mu_i - \mu_j)^2 R_{\bar{i}ij\bar{j}}, \tag{3.15}$$

where the equality holds if and only if the Ricci tensor S is parallel.

The following theorem is originally proved by Ki and Suh [4]. Here, we will give the simple proof of the theorem by using another technique.

Theorem 3.1. *Let M be an $n(\geqq 3)$-dimensional complete Kähler manifold with constant scalar curvature. If the totally real bisectional curvature is bounded from below by a positive constant, then M is Einstein.*

Proof. Suppose that the totally real bisectional curvature is bounded from below by a positive constant a. So, we have

$$R_{\bar{i}ij\bar{j}} \geqq a > 0, \quad i \neq j.$$

Accordingly, (3.15) can be reduced to

$$\Delta S_2 \geqq 2a \sum_{i,j} (\mu_i - \mu_j)^2.$$

Let us consider a non-negative function $f = S_2 - \frac{r^2}{4n}$. Then, from (3.14) and the above inequality it follows that we have

$$\Delta f \geqq 2naf, \tag{3.16}$$

where the equality holds if and only if the Ricci tensor S is parallel on M. Since the totally real bisectional curvature is bounded from below and the scalar curvature is constant, Lemma 2.2 implies that the Ricci curvature is bounded, where the restriction of dimension is used. By the definition of the function f and (3.13), it implies that f is also bounded from above, because the Ricci curvatures on M is bounded from above, and hence we can apply Theorem 2.1 to the function f. For any positive sequence $\{\epsilon_m\}$ in such a way that it converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ in M which satisfies the following properties.

$$|\nabla f(p_m)| < \epsilon_m, \quad \Delta f(p_m) < \epsilon_m, \quad sup\ f - \epsilon_m < f(p_m).$$

By (3.16) and the above property, we have

$$\epsilon_m > \Delta f(p_m) \geqq 2naf(p_m) > 2na(sup\ f - \epsilon_m),$$

which implies $0 \geqq 2na\ sup\ f$. It turns out to be $sup\ f = 0$. Since f is non-negative by (3.14), we see that the function f vanishes identically on M. It means that M is Einstein. It completes the proof. □

§4. The Ryan condition

This section is concerned with Kähler manifolds with the condition $RS = 0$. Namely, it satisfies

$$R(X,Y)S = 0 \tag{4.1}$$

for any vector fields X and Y.

Let M be a complex n-dimensional connected Kähler manifold equipped with Kähler metric tensor g and almost complex structure J, and let $\{U_j\}$ be a local unitary frame field on a neighborhood of M. For the canonical basis $\{U_j, U_{j^*}\}$ the Ricci tensor S and the Riemannian curvature tensor R are given by

$$S(U_j) = \sum_k S_{j\bar{k}} U_k, \quad R(\bar{U}_i, U_j)U_k = -\sum_l R_{\bar{i}jk\bar{l}} U_l,$$

where $U_{j^*} = JU_j$ and $j^* = n + j$. Accordingly, we have

$$\begin{aligned} (R(\bar{U}_i, U_j)S)U_k &= R(\bar{U}_i, U_j)(S(U_k)) - S(R(\bar{U}_i, U_j)U_k) \\ &= -\sum_{m,l} (R_{\bar{i}jm\bar{l}} S_{k\bar{m}} - R_{\bar{i}jk\bar{m}} S_{m\bar{l}}) U_l. \end{aligned} \tag{4.2}$$

On the other hand, we see by (3.7)

$$S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j} = \sum_m (R_{\bar{i}jk\bar{m}} S_{m\bar{l}} - R_{\bar{i}jm\bar{l}} S_{k\bar{m}}),$$

from which together with (4.2), it follows that we have

$$(R(\bar{U}_i, U_j)S)U_k = \sum_l (S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j}) U_l. \tag{4.3}$$

Thus it is easily seen that the following conditions is equivalent to (4.1).

$$S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j} = 0, \tag{4.4}$$

$$\sum_m (R_{\bar{i}jk\bar{m}} S_{m\bar{l}} - R_{\bar{i}jm\bar{l}} S_{k\bar{m}}) = 0. \tag{4.5}$$

By the same argument as that the equation (3.15) is derived, we obtain

$$\sum_{j,k} (S_j - S_k)^2 R_{\bar{j}jk\bar{k}} = 0,$$

where S_j denotes the Ricci curvature on M. It implies that if the totally real bisectional curvature on M is bounded from above (*resp.* below) by a negative (*resp.* positive) constant, then we have

$$\sum_{j,k} (S_j - S_k)^2 = 0,$$

which means that M is Einstein. Thus we can prove

Theorem 4.1. *Let M be a Kähler manifold. If the totally real bisectional curvature on M is bounded from above (resp. below) by a negative (resp. positive) constant, then the following are equivalent.*

(1) M is Einstein.

(2) $RS = 0$.

(3) the Ricci tensor S is parallel.

Proof. The condition (2) is equivalent to the equation (4.4). This means that (3) implies (2). It is trivial that (1) implies (2) and (3). Thus it is sufficient to prove that (2) implies (1). This already showed. □

By this theorem, Theorem 1 in the introduction is verified.

Now, let (M', g') be an $(n+p)$-dimensional connected Kähler manifold and let M be an n-dimensional connected complex submanifold of M' or let (M', g') be an $(n+p)$-dimensional connected indefinite Kähler manifold of index $2p (p > 0)$ and let M be an n-dimensional connected space-like complex submanifold of M'. Then M is the Kähler manifold endowed with the induced metric tensor g. Then, by (2.10) we have

$$R_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x \epsilon_x h^x_{ij} \bar{h}^x_{ij} \quad (i \neq j). \tag{4.6}$$

It implies that if $\epsilon_x = -1$ and if c is positive, then the totally real bisectional curvature is bounded from below by a positive constant. On the other hand, (4.6) implies that if $\epsilon_x = 1$ and if c is negative, then the totally real bisectional curvature is bounded from above by a negative constant. So, as a direct consequence of Theorem 4.1, we can get

Corollary 4.2. *Let M be a space-like complex submanifold of $M^{n+p}_p(c)$. If $c > 0$, then the following statements are equivalent.*

(1) M is Einstein.

(2) $RS = 0$.

(3) the Ricci tensor S is parallel.

Remark 4.1. This result is due to Aiyama, Kwon and Nakagawa [1].

Corollary 4.3. *Let M be a complex submanifold of $M^{n+p}(c)$. If $c < 0$, then the following statements are equivalent.*

(1) M is Einstein.

(2) $RS = 0$.

(3) the Ricci tensor S is parallel.

Remark 4.2. This result is due to Kon [7].

Lastly, we shall prove the following

Theorem 4.4. *Let M be an $n(\geqq 2)$-dimensional space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of index $2p$. If M satisfies the condition (4.1) and if the codimension p is less than $n-1$, then M is Einstein.*

Proof. From the Gauss equation (2.10) and (4.5), we have

$$c(h^2_{i\bar{l}}\delta_{jk} - h^2_{i\bar{j}}\delta_{kl}) + 2\sum_{r,s,x}(h^x_{ik}\bar{h}^x_{lr}h^2_{r\bar{j}} - h^x_{kr}h^2_{r\bar{i}}\bar{h}^x_{jl}) = 0. \tag{4.7}$$

Since $(h^2_{i\bar{j}})$ is a negative semi-definite Hermitian matrix, the eigenvalues $\lambda_1, \cdots, \lambda_n$ are non-positive real valued functions on M. Moreover we have

$$h^2_{i\bar{j}} = \lambda_i\delta_{ij}. \tag{4.8}$$

From (4.8) the equation (4.7) is reformed as

$$c(\lambda_i - \lambda_j)\delta_{il}\delta_{jk} + 2(\lambda_i - \lambda_j)\sum_x h^x_{ik}\bar{h}^x_{jl} = 0,$$

from which it follows that we have

$$\begin{aligned} &(\lambda_i - \lambda_j)(\sum_x h^x_{ij}\bar{h}^x_{ij} + \frac{c}{2}) = 0, \\ &(\lambda_i - \lambda_j)\sum_x h^x_{ik}\bar{h}^x_{jl} = 0 \text{ unless } i = l,\ j = k. \end{aligned} \tag{4.9}$$

We may assume that $\lambda_1, \cdots, \lambda_q$ are all distinct eigenvalues of the matrix $(h^2_{i\bar{j}})$. Let $n_1, \cdots, n_q$ be multiplicities of $\lambda_1, \cdots, \lambda_q$, respectively, where q is the function on M. If $q = 1$ everywhere on M, then M is Einstein. Suppose that there is a point x of M at which $q \geqq 2$. Then, at the point x there exist at least two distinct eigenvalues. For eigenvalues λ_i and λ_j such that $\lambda_i \neq \lambda_j$ it follows from (4.9) that we have

$$\begin{aligned} &\sum_x h^x_{ij}\bar{h}^x_{ij} = -\frac{c}{2} \text{ if } \lambda_i \neq \lambda_j, \\ &\sum_x h^x_{ik}\bar{h}^x_{jl} = 0 \text{ if } \lambda_i \neq \lambda_j \text{ and } (k,l) = (i,j) \text{ or } (j,i). \end{aligned} \tag{4.10}$$

Let h_{ij} be a vector in C^p defined by $h_{ij} = (h^{n+1}_{ij}, \cdots, h^{n+p}_{ij})$. Consider the subspace $\{h_{ij} | \lambda_i \neq \lambda_j\}$ consisting of $\sum_{r<s}^q n_r n_s$ vectors in C^p. The equation (4.10) means that they are linearly independent. Accordingly, because of $\sum_{r=1}^q n_r = n$, we have

$$p \geqq \sum_{r<s}^q n_r n_s \geqq n - 1,$$

where the second equality holds if and only if $q = 2$ and $n_1 = 1$ or $n_2 = 1$. In fact, the first inequality follows from the fact that the vectors h_{ij} are contained in C^p and the second inequality is derived from the following argument.

Let f be a function with variables $n_1, \cdots, n_{q-1}$ defined by $\sum_{r<s}^{q} n_r n_s$ with the condition $\sum_{r=1}^{q} n_r = n$. Namely, f is given by

$$f(n_1, \cdots, n_{q-1}) = \sum_{r<s}^{q-1} n_r n_s + (n_1 + \cdots + n_{q-1})(n - n_1 - \cdots - n_{q-1}).$$

Then it is easily seen that f is monotonically increasing with respect to the first variable n_1 and hence, because of $n_1 \geqq 1$, we have

$$f(n_1, \cdots, n_{q-1}) \geqq f(1, n_2, \cdots, n_{q-1}),$$

where the equality holds if and only if $n_1 = 1$. Similarly, we have

$$f(1, n_2, \cdots, n_{q-1}) \geqq f(1, 1, n_3, \cdots, n_{q-1})$$

where the equality holds if and only if $n_2 = 1$. Inductively, we have

$$f(n_1, \cdots, n_{q-1}) \geqq f(1, \cdots, 1)$$

where the equality holds if and only if $n_1 = \cdots = n_{q-1} = 1$.
On the other hand, we obtain

$$f(\underbrace{1, \cdots, 1}_{r-1}) - f(\underbrace{1, \cdots, 1}_{r-2}) = n - r + 1 > 0,$$

which implies that we have

$$f(1, \cdots, 1) \geqq f(1),$$

where the equality holds if and only if $q = 2$. Thus we obtain

$$f(n_1, \cdots, n_{q-1}) = \sum_{r<s}^{q} n_r n_s \geqq f(1) = n - 1,$$

where the equality holds if and only if $q = 2$ and $n_1 = 1$. It completes the proof. □

Remark 4.3. It is shown that the product manifold of a 1-dimensional complex hyperbolic space $CH^1(c)$ and an $(n-1)$-dimensional complex hyperbolic space $CH^{n-1}(c)$ is an n-dimensional Kähler manifold and it is isometrically imbedded in a $(2n-1)$-dimensional indefinite complex hyperbolic space $CH_{n-1}^{2n-1}(c)$ of index $2(n-1)$(see [2] and [3]). Then it satisfies the condition (1.1), but it is not Einstein if $n \geqq 3$. This implies that the estimate of the codimension is best possible.

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