Nihonkai Math. J.
Vol.11(2000), 87-96

JOINT SPECTRA OF DOUBLY COMMUTING n-TUPLES OF OPERATORS AND THEIR ALUTHGE TRANSFORMS

M. CHŌ*, I. H. JEON**, AND J. I. LEE

Abstract.
We show equalities of various joint spectra of doubly commuting n-tuples of arbitrary operators and their Aluthge transforms, which enable us to extend recent results proved by Jung, Ko, and Pearcy ([12]) for a single operator. Then we give its applications to the class of p-hyponormal operators that the partial isometry part in its polar decomposition is unitary.

1. Introduction. Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable infinite dimentional complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ has an unique polar decoposition $A = U|A|$, where $|A| = (A^*A)^{1/2}$ and U is a partial isometry with the initial space the closure of the range of $|A|$ and the final space the closure of the range of A. In the context of p-hyponormal operators(to be defined later), Aluthge ([1]) has introduced an operator $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$ associated with A. We remark that for a *-homomorphism π from a C^*-algebra containing an operator A to a C^*-algebra, the image $\pi(A) = \pi(U)\pi(|A|)$ is not generally the polar decomposition. Then we need to consider a decomposition with a weaker condition. Let $A = V|A|$ with partial isometry V. Since $|A|V^*V|A| = |A|^2 = |A|U^*U|A|$, V^*V and U^*U are projections, we have $\text{Ker}(V) \subseteq \text{Ker}(U)$. And since $\text{Ker}(U) = \text{Ker}(|A|)$, we have $\text{Ker}(V) \subseteq \text{Ker}(|A|)$. Moreover, we have $\widetilde{A} = |A|^{1/2}U|A|^{1/2} = |A|^{1/2}V|A|^{1/2}$, because $U|A|^{1/2} = V|A|^{1/2}$.

In a vast literature this Aluthge transform have been used as a useful tool to study p-hyponormal operators ([1],[3],[8],[9],[11]). Most of all these studies were accomplished in a special case that the partial isometry U in the polar decomposition $A = U|A|$ is unitary. Recently, however, Jung, Ko, and Pearcy ([12]) derived equalities of various spectra of an arbitrary operator and its associated Aluthge transform, and gave some applications about

1991 *Mathematics Subject Classification.* Primary 47A13, 47B20.
Key words and phrases. Aluthge transform, joint spectrum, p-hyponormal operator.
* Research was partially supported by Grant-in-Aid Scientific Research (No. 09640229)
** Research was partially supported by Brain Ewha Math 21

'the invariant subspace problem' to p-hyponormal operators. Throughout this paper we let $\mathbb{A} = (A_1, \cdots, A_n)$ denote a commuting n-tuple of operators and denote $\mathbb{A}^* = (A_1^*, \cdots, A_n^*)$, $\widetilde{\mathbb{A}} = (\widetilde{A_1}, \cdots, \widetilde{A_n})$ and $(\widetilde{\mathbb{A}})^* = ((\widetilde{A_1})^*, \cdots, (\widetilde{A_n})^*)$. If $A_iA_j = A_jA_i$ and $A_i^*A_j = A_jA_i^*$, for every $i \neq j$, then $\mathbb{A} = (A_1, \cdots, A_n)$ is said to be a *doubly commuting n-tuple.* In this paper we extend all equalities of various spectra of arbitrary operator and its Aluthge transform proved in [12, Theorem 1.3] to doubly commuting n-tuples of arbitrary operators and their Aluthge transforms. Then we give its applications to doubly commuting n-tuples of p-hyponormal operators that the partial isometry part in its polar decomposition is unitary, which enable us to generalize or recapture most of all results proved in [3], [9], and [11]. Let's review definitions ([6], [7]) of joint spectra of a commuting n-tuple $\mathbb{A} = (A_1, \cdots, A_n)$ of operators in $B(\mathcal{H})$. If there exists a non-zero vector x such that

$$(A_i - \lambda_i)x = 0 \quad \text{for every } i = 1, \cdots, n,$$

then $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is called a *joint eigenvalue* of $\mathbb{A}$ and the *joint point spectrum,* denoted by $\sigma_p(\mathbb{A})$, of $\mathbb{A}$ is the set of all joint eigenvalues of $\mathbb{A}$. The *joint approximate point spectrum,* denoted by $\sigma_{ap}(\mathbb{A})$, of $\mathbb{A}$ is the set of all points $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ such that for a sequence $\{x_k\}$ of unit vectors

$$||(A_i - \lambda_i)x_k|| \to 0 \quad \text{as } k \to \infty \quad \text{for every } i = 1, \cdots, n.$$

Sometimes we can deal with questions on the approximate point spectrum by looking at the situation for the point spectrum. To do so, we often use a technique, so called 'Berberian extension', as follows:

PROPOSITION A. *Let $\mathcal{H}$ be a Hilbert space. Then there exist a Hilbert space $\mathcal{H}^\circ \supset \mathcal{H}$ and an isometric *-isomorphism of $B(\mathcal{H})$ into $B(\mathcal{H}^\circ)$:$A_i \mapsto A_i^\circ$ preserving the order such that $\sigma_{ap}(A_i) = \sigma_{ap}(A_i^\circ) = \sigma_p(A_i^\circ)$. Furthermore, if $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{A}^\circ = (A_1^\circ, \cdots, A_n^\circ)$, then $\sigma_{ap}(\mathbb{A}) = \sigma_{ap}(\mathbb{A}^\circ) = \sigma_p(\mathbb{A}^\circ)$*

For a proof, see [2, Proposition 1] or [14].

2. Joint spectra of operators and their Aluthge transforms. In this section we shall give a joint spectral version of Theorem 1.3 in [12]. We first begin with the following lemmas:

LEMMA 1. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be a doubly commuting n-tuple of operators in $B(\mathcal{H})$. Then there exists an n-tuple $(V_1, \cdots, V_n)$ of partial isometries such that $A_i = V_i|A_i|$ $(i = 1, \cdots, n)$, $\widetilde{A} = (\widetilde{A_1}, \cdots, \widetilde{A_n})$ is also a doubly commuting n-tuple and*

(1) *V_i, V_i^* and $|A_i|$ commute with V_j, V_j^* and $|A_j|$ for every $i \neq j$.*

Proof. It immediately follows from Theorem 2 in [10] and the similar arguments of the proof of Lemma 1 in [9]. □

LEMMA 2. *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *be doubly commuting and* $A_i = V_i|A_i|$ *be a polar decomposition of* A_i *with partial isometry satisfying* (1) $(i = 1, \cdots, n)$. *If* $(\prod_{1\le i\le n} |A_i|^{1/2})x \neq 0$, *then* $(\prod_{1\le i\le n} V_i|A_i|^{1/2})x \neq 0$.

Proof. If $V_1(\prod_{1\le i\le n} |A_i|^{1/2})x = 0$, then $|A_1|^{1/2}(\prod_{1\le i\le n} |A_i|^{1/2})x = 0$ because $\text{Ker}(V_1) \subseteq \text{Ker}(|A_1|^{1/2})$. Hence we have

$$|A_1|(\prod_{2\le i\le n} |A_i|^{1/2})x = 0 \text{ and } (\prod_{1\le i\le n} |A_i|^{1/2})x = 0.$$

It's a contradiction. Since

$$V_1(\prod_{1\le i\le n} |A_i|^{1/2})x = |A_2|^{1/2}V_1|A_1|^{1/2}(\prod_{3\le i\le n} |A_i|^{1/2})x \neq 0,$$

similarly we have

$$V_2|A_2|^{1/2}V_1|A_1|^{1/2}(\prod_{3\le i\le n} |A_i|^{1/2})x \neq 0.$$

Repeating this process, we have

$$(\prod_{1\le i\le n} V_i|A_i|^{1/2})x \neq 0. \qquad \square$$

THEOREM 3. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting n-tuple of operators in* $B(\mathcal{H})$, *then*

$$\sigma_p(\mathbb{A}) = \sigma_p(\widetilde{\mathbb{A}}) \text{ and } \sigma_{ap}(\mathbb{A}) = \sigma_{ap}(\widetilde{\mathbb{A}}). \tag{2}$$

Proof. For the first equality of (2), let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \sigma_p(\mathbb{A})$. Let a non-zero vector $x \in \mathcal{H}$ be $A_i x = \lambda_i x$ for every $i = 1, \cdots, n$. We may, without loss of generality, assume that $\lambda_1, \cdots, \lambda_k$ are non-zero and $\lambda_{k+1} = \cdots = \lambda_n = 0$. Let $I_1 = \{1, \cdots, k\}$ and $I_2 = \{k+1, \cdots, n\}$. Put $y = \prod_{s\in I_1} |A_s|^{1/2}x$. Since every $\lambda_i \neq 0$ $(i \in I_1)$, we have $y \neq 0$. Since $\widetilde{A_i}|A_i|^{1/2} = |A_i|^{1/2}A_i$ and $|A_i||A_j| = |A_j||A_i|$ for every $i \neq j$, for each $i \in I_1$,

$$\begin{aligned} \widetilde{A_i}y &= \prod_{s(\neq i)\in I_1} |A_s|^{1/2} \cdot \widetilde{A_i}|A_i|^{1/2}x \\ &= \prod_{s(\neq i)\in I_1} |A_s|^{1/2} \cdot |A_i|^{1/2}A_i x \\ &= \lambda_i \cdot \prod_{s\in I_1} |A_s|^{1/2}x = \lambda_i y. \end{aligned} \tag{3}$$

On the other hand, for each $i \in I_2$,

$$A_i x = 0 \Longrightarrow |A_i|^{1/2} x = 0 \Longrightarrow \widetilde{A_i} x = 0.$$

So, for each $i \in I_2$,

$$\widetilde{A_i} y = \widetilde{A_i} \cdot \prod_{s \in I_1} |A_s|^{1/2} x = \prod_{s \in I_1} |A_s|^{1/2} \cdot \widetilde{A_i} x = 0. \tag{4}$$

Therefore by (3) and (4) we have that $\lambda \in \sigma_p(\widetilde{\mathbb{A}})$.

Conversely, let $\mu = (\mu_1, \cdots, \mu_n) \in \sigma_p(\widetilde{\mathbb{A}})$ and let a non-zero vector $u \in \mathcal{H}$ be $\widetilde{A_i} u = \mu_i u$ for every $i = 1, \cdots, n$. We may, without loss of generality, assume that $\mu_1, \cdots, \mu_k$ are non-zero and $\mu_{k+1} = \cdots = \mu_n = 0$. Let $I_1 = \{1, \cdots, k\}$ and $I_2 = \{k+1, \cdots, n\}$. We may only prove following three cases (a), (b) and (c):

(a) If $|A_t|^{1/2} u = 0$ for every $t \in I_2$, let $v_1 = \prod_{i \in I_1} V_i |A_i|^{1/2} u$. Since every $\mu_i \neq 0$ $(i \in I_1)$, we have $v_1 \neq 0$.

For $j \in I_1$, we have

$$A_j v_1 = \prod_{i \in I_1} V_i |A_i|^{1/2} \cdot (\widetilde{A_j} u) = \mu_j v_1.$$

For $j \in I_2$, it is easy to see that $A_j v_1 = 0$. Hence we have $\mu \in \sigma_p(\mathbb{A})$.

(b) If $\prod_{t \in I_2} |A_t|^{1/2} u \neq 0$, let $v_2 = \prod_{1 \le i \le n} V_i |A_i|^{1/2} u$. Since every $\mu_i \neq 0$ $(i \in I_1)$, we have $v_2 \neq 0$ by Lemma 2.

For every $j = 1, \cdots, n$, we have $A_j v_2 = \prod_{1 \le i \le n} V_i |A_i|^{1/2} \cdot (\widetilde{A_j} u) = \mu_j v_2$. Hence we have $\mu \in \sigma_p(\mathbb{A})$.

(c) If there exists ℓ $(k < \ell < n)$ such that $|A_{k+1}|^{1/2} \cdots |A_\ell|^{1/2} u \neq 0$ and $|A_{k+1}|^{1/2} \cdots |A_\ell|^{1/2} |A_s|^{1/2} u = 0$ for every $s = \ell+1, \cdots, n$, let $v_3 = \prod_{1 \le i \le \ell} V_i |A_i|^{1/2} u$.

Since every $\mu_i \neq 0$ $(i \in I_1)$, we have $v_3 \neq 0$ by Lemma 2.

For $1 \le j \le \ell$, we have $A_j v_2 = \prod_{1 \le i \le \ell} V_i |A_i|^{1/2} \cdot (\widetilde{A_j} u) = \mu_j v_3$.

For $\ell + 1 \le s \le n$,

$$A_s v_3 = V_s |A_s|^{1/2} \cdot \prod_{i \in I_1} V_i |A_i|^{1/2} \cdot \prod_{k+1 \le t \le \ell} V_t \cdot (|A_{k+1}|^{1/2} \cdots |A_\ell|^{1/2} |A_s|^{1/2} u) = 0.$$

Hence we have $A_s v_3 = 0$ and $\mu \in \sigma_p(\mathbb{A})$.

Therefore, by (a), (b) and (c) we have that $\mu \in \sigma_p(\mathbb{A})$. On the other hand, using Proposition A and just proved the first equality of (2), it is easy to prove the second equality of (2). Indeed, we have

$$\sigma_{ap}(\mathbb{A}) = \sigma_p(\mathbb{A}^\circ) = \sigma_p(\widetilde{\mathbb{A}}^\circ) = \sigma_{ap}(\widetilde{\mathbb{A}}) \qquad \square$$

THEOREM 4. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting* n*-tuple of operators in* $B(\mathcal{H})$, *then*

$$\sigma_p(\mathbb{A}^*) \setminus [0] = \sigma_p((\widetilde{\mathbb{A}})^*) \setminus [0] \quad \text{and} \quad \sigma_{ap}(\mathbb{A}^*) \setminus [0] = \sigma_{ap}((\widetilde{\mathbb{A}})^*) \setminus [0], \tag{5}$$

where $[0] = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n \ : \ \lambda_i = 0$ *for at least one* $i \in I = \{1, \cdots, n\}\}$.

Proof. Suppose $\lambda = (\lambda_1, \cdots, \lambda_n) \in \sigma_p(\mathbb{A}^*) \setminus [0]$. Let a non-zero vector $x \in \mathcal{H}$ satisfy $A_i^* x = \lambda_i x$ for every $i \in I$. Put $y = \prod_{j \in I} |A_j|^{1/2} U_j^* x$. Since every $\lambda_i \neq 0$ $(i \in I)$, we have $y \neq 0$. Since $(|A_i|^{1/2} V_i^*) A_i^* = (\widetilde{A_i})^* (|A_i|^{1/2} U_i^*)$, we have

$$(\widetilde{A_i})^* y = \prod_{j(\neq i) \in I} |A_j|^{1/2} V_j^* \cdot (\widetilde{A_i})^* |A_i|^{1/2} V_i^* x$$

$$= \prod_{j(\neq i) \in I} |A_j|^{1/2} U_j^* \cdot |A_i|^{1/2} V_i^* A_i^* x = \prod_{j(\neq i) \in I} |A_j|^{1/2} U_j^* \cdot |A_i|^{1/2} V_i^* \lambda_i x$$

$$= \lambda_i \prod_{j \in I} |A_j|^{1/2} V_j^* x = \lambda_i y.$$

Thus $\lambda \in \sigma_p((\widetilde{\mathbb{A}})^*) \setminus [0]$.

Conversely, suppose that $\mu = (\mu_1, \cdots, \mu_n) \in \sigma_p((\widetilde{\mathbb{A}})^*) \setminus [0]$. Let a non-zero vector $u \in \mathcal{H}$ satisfy $(\widetilde{A_i})^* u = \mu_i u$ for every $i \in I$. Put $v = \prod_{j \in I} |A_j|^{1/2} u$. Since every $\mu_i \neq 0$ $(i \in I)$, we have $v \neq 0$. Similarly above we have

$$A_i^* v = \mu_i v \quad \text{for every } i \in I.$$

Thus $\mu \in \sigma_p(\mathbb{A}^*) \setminus [0]$, and so the first equality of (5) holds. On the other hand, using Propositon A and just the proof of the first equality of (5), it is easy to prove the second equality of (5). Indeed, we have

$$\sigma_{ap}(\mathbb{A}^*) \setminus [0] = \sigma_p(\mathbb{A}^{*\circ}) \setminus [0] = \sigma_p((\widetilde{\mathbb{A}})^{*\circ}) \setminus [0] = \sigma_{ap}((\widetilde{\mathbb{A}})^*) \setminus [0].$$

Hence the proof is complete. □

Recall([6], [7], [9]) that the *left (right) joint spectrum*, denoted by $\sigma^{\ell}(\mathbb{A})$ $(\sigma^r(\mathbb{A}))$, of $\mathbb{A}$ is defined by the set of all points $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ such that $\{A_i - \lambda_i\}_{1 \leq i \leq n}$ generates a proper left (right) ideal in the algebra $B(\mathcal{H})$. It is well known ([6], [7]) that, for a commuting n-tuple $\mathbb{A}$ of operators in $B(\mathcal{H})$,

$$\sigma^{\ell}(\mathbb{A}) = \sigma_{ap}(\mathbb{A}) \quad \text{and} \quad \sigma^r(\mathbb{A}) = \overline{\sigma_{ap}(\mathbb{A}^*)}. \tag{6}$$

Thus the second equalities of (2) and (5), respectively, can be written as follows:

$$\sigma^{\ell}(\mathbb{A}) = \sigma^{\ell}(\widetilde{\mathbb{A}}) \quad \text{and} \quad \sigma^{r}(\mathbb{A}) \setminus [0] = \sigma^{r}(\widetilde{\mathbb{A}}) \setminus [0]. \tag{7}$$

Let $K(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$ and $\mathcal{C}(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$ be the Calkin algebra with the canonical map $\pi : B(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$. Then the *left (right) joint essential spectrum*, denoted by $\sigma_e^{\ell}(\mathbb{A})$ ($\sigma_e^{r}(\mathbb{A})$), of $\mathbb{A}$ is defined by

$$\sigma_e^{\ell}(\mathbb{A}) = \sigma^{\ell}(\pi(\mathbb{A}))\ (\sigma_e^{r}(\mathbb{A}) = \sigma^{r}(\pi(\mathbb{A}))), \text{ where } \pi(\mathbb{A}) = (\pi(A_1), \cdots, \pi(A_n)).$$

COROLLARY 5. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting n-tuple of operators in* $B(\mathcal{H})$, *then*

$$\sigma_e^{\ell}(\mathbb{A}) = \sigma_e^{\ell}(\widetilde{\mathbb{A}}) \quad \textit{and} \quad \sigma_e^{r}(\mathbb{A}) \setminus [0] = \sigma_e^{r}(\widetilde{\mathbb{A}}) \setminus [0]. \tag{8}$$

Proof. For the first equality of (8), let ρ be a faithful representation of the C^*-algebra generated by $\pi(A_1), \cdots, \pi(A_n)$ and $\pi(I)$ on a Hilbert space $\mathcal{H}_{\rho}$. By Theorem 3 and (7), $\sigma^{\ell}(\rho(\pi(\mathbb{A}))) = \sigma^{\ell}(\rho(\pi(\widetilde{\mathbb{A}})))$. Since by the spectral permanence of σ^{ℓ} (see [5]), we have

$$\sigma^{\ell}(\rho(\pi(\mathbb{A}))) = \sigma^{\ell}(\pi(\mathbb{A})) = \sigma_e^{\ell}(\mathbb{A}) \quad \text{and} \quad \sigma^{\ell}(\rho(\pi(\widetilde{\mathbb{A}}))) = \sigma^{\ell}(\pi(\widetilde{\mathbb{A}})) = \sigma_e^{\ell}(\widetilde{\mathbb{A}}),$$

and so $\sigma_e^{\ell}(\mathbb{A}) = \sigma_e^{\ell}(\widetilde{\mathbb{A}})$. For the second equality of (8), using Theorem 3 and (7), a similar proof can be given. □

3. Applications. In this section we shall restrict our view to special classes of operators. First, we let $\mathcal{U}$ denote the class of operators $A \in B(\mathcal{H})$ that the partial isometry U in the polar decomposition $A = U|A|$ is unitary. For operators in $\mathcal{U}$ we have an improvement of Theorem 4 and the second equality in Corollary 5 as follows:

THEOREM 6. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting n-tuple of operators in* $\mathcal{U}$, *then*

$$\sigma_p(\mathbb{A}^*) = \sigma_p((\widetilde{\mathbb{A}})^*), \ \sigma_{ap}(\mathbb{A}^*) = \sigma_{ap}((\widetilde{\mathbb{A}})^*), \ \textit{and}\ \sigma_e^{r}(\mathbb{A}) = \sigma_e^{r}(\widetilde{\mathbb{A}}). \tag{9}$$

Proof. To prove the first equality of (9), in the view of Theorem 4, it suffices to show that

$$\lambda = (\lambda_1, \cdots, \lambda_n) \in [0] \cap \sigma_p(\mathbb{A}^*) \iff \lambda = (\lambda_1, \cdots, \lambda_n) \in [0] \cap \sigma_p((\widetilde{\mathbb{A}})^*).$$

Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in [0] \cap \sigma_p(\mathbb{A}^*)$. Let a non-zero vector $x \in \mathcal{H}$ satisfy $A_i^* x = \lambda_i x$ for every $i = 1, \cdots, n$. Then from the definition of the set $[0]$, without loss of generality, we may let $\lambda_1, \cdots, \lambda_k$ be non-zero and $\lambda_{k+1} = \cdots = \lambda_n = 0$. Let $I_1 = \{1, \cdots, k\}$ and $I_2 = \{k+1, \cdots, n\}$. Let $y = \prod_{s \in I_1} |A_s|^{1/2} U_s^* \cdot \prod_{t \in I_2} U_t^* x$. Since every $\lambda_i \neq 0$ $(i \in I_1)$ and every U_t is unitary $(t \in I_2)$, we have $y \neq 0$. For $1 \leq i \leq k$,

$$(\widetilde{A_i})^* y = |A_i|^{1/2} U_i^* |A_i|^{1/2} y = \prod_{s \in I_1} |A_s|^{1/2} U_s^* \cdot \prod_{t \in I_2} U_t^* (A_i^* x) = \lambda_i y.$$

For $k+1 \leq i \leq n$,

$$(\widetilde{A_i})^* y = \prod_{s \in I_1} |A_s|^{1/2} U_s^* \cdot \prod_{t (\neq i) \in I_2} U_t^* \cdot |A_i|^{1/2} U_i^* (|A_i|^{1/2} U_i^* x) = 0,$$

because $|A_i|^{1/2} U_i^* x = 0$. Therefore, we have $\lambda \in [0] \cap \sigma_p((\widetilde{\mathbb{A}})^*)$ and $\sigma_p(\mathbb{A}^*) \subseteq \sigma_p((\widetilde{\mathbb{A}})^*)$.

Next we prove the reverse inclusion. Let $\mu = (\mu_1, ..., \mu_n) \in [0] \cap \sigma_p((\widetilde{\mathbb{A}})^*)$. And let a non-zero vector u be $(\widetilde{A_i})^* u = \mu_i u$ $(i = 1, ..., n)$. We may assume that $\mu_1, ..., \mu_k$ are non-zero and $\mu_{k+1} = \cdots = \mu_n = 0$. Let $I_1 = \{1, \cdots, k\}$ and $I_2 = \{k+1, \cdots, n\}$. We may only prove following three cases (a), (b) and (c):

(a) If $|A_j|^{1/2} u = 0$ for every $j \in I_2$, let $v_1 = \prod_{i \in I_1} |A_i|^{1/2} \cdot \prod_{t \in I_2} U_t u$. Since every U_t is unitary $(t \in I_2)$, we have $v_1 \neq 0$.

For $j \in I_1$,

$$A_j^* v_1 = \prod_{i \in I_1} |A_i|^{1/2} \cdot \prod_{t \in I_2} U_t \cdot ((\widetilde{A_j})^* u) = \mu_j v_1.$$

For $j \in I_2$,

$$A_j^* v_1 = \prod_{i \in I_1} |A_j|^{1/2} \cdot \prod_{t (\neq j) \in I_2} U_t \cdot |A_j|^{1/2} (|A_j|^{1/2} u) = 0.$$

Hence we have $\mu \in \sigma_p(\mathbb{A}^*)$.

(b) If $\prod_{i \in I_2} |A_i|^{1/2} u \neq 0$, let $v_2 = \prod_{1 \leq i \leq n} |A_i|^{1/2} u$. If $v_2 = 0$, then we have $\prod_{i \in I_2} |A_i|^{1/2} u = 0$ since every $\mu_j \neq 0$ $(j \in I_1)$. Hence, we have $v_2 \neq 0$.

For every $j = 1, ..., n$, we have

$$A_j^* v_2 = \prod_{1 \leq i \leq n} |A_i|^{1/2} ((\widetilde{A_j})^* u) = \mu_j v_2.$$

Hence we have $\mu \in \sigma_p(\mathbb{A}^*)$.

(c) If there exists ℓ $(k < \ell < n)$ such that $|A_{k+1}|^{1/2} \cdots |A_\ell|^{1/2} u \neq 0$ and

$|A_{k+1}|^{1/2}\cdots|A_\ell|^{1/2}|A_s|^{1/2}u=0$ for every $s=\ell+1,\cdots,n$, let

$$v_3=\prod_{i\in I_1}|A_i|^{1/2}\cdot\prod_{k+1\le j\le\ell}|A_j|^{1/2}\cdot\prod_{\ell+1\le t\le n}U_tu.$$

Since every $\mu_i\neq 0 (i\in I_1)$ and every U_t is unitary $(\ell+1\le t\le n)$, we have $v_3\neq 0$. For $1\le s\le\ell$,

$$A_s^*v_3=\prod_{1\le i\le\ell}|A_i|^{1/2}\cdot\prod_{\ell+1\le j\le n}U_j\cdot((\widetilde{A_s})^*u)=\mu_s v.$$

For $\ell+1\le s\le n$,

$$A_s^*v_3=\prod_{i\in I_1}|A_i|^{1/2}\cdot\prod_{\ell+1\le j\le n(j\neq s)}U_j\cdot|A_s|^{1/2}(|A_{k+1}|^{1/2}\cdots|A_\ell|^{1/2}|A_s|^{1/2}u)=0.$$

Therefore we have

$$(\mu_1,...,\mu_n)\in[0]\cap\sigma_p(\mathbb{A}^*).$$

Therefore, we have that $\sigma_p(\mathbb{A}^*)\supseteq\sigma_p((\widetilde{\mathbb{A}})^*)$. Hence we have the first equality of (9). For the second equality of (9), using Proposition A, we can easily prove it. Also, the third equality of (9) immediately follows from Corollary 5. □

An operator $A\in B(\mathcal{H})$ is called *p-hyponormal* if $(A^*A)^p-(AA^*)^p\ge 0$ for some $p\in(0,\infty)$. If $p=1$, A is said to be hyponormal and if $p=\frac{1}{2}$, A is said to be semi-hyponormal ([14]). By the consequence of Löwner's inequality ([13]) if A is p-hyponormal for some $p\in(0,\infty)$, then A is also q-hyponormal for every $q\in(0,p]$. Thus we assume, without loss of generality, that $p\in(0,1/2)$. Let $\mathcal{HU}(p)$ denote the class of p-hyponormal operators A that the partial isometry U in the polar decomposition $A=U|A|$ is unitary. Since the set $\mathcal{U}$ properly contains the set $\mathcal{HU}(p)$, we have the following corollary:

COROLLARY 7. *If $\mathbb{A}=(A_1,\cdots,A_n)$ is a doubly commuting n-tuple of operators in $\mathcal{HU}(p)$, then*

$$\omega(\mathbb{A})=\omega(\widetilde{\mathbb{A}})\text{ and }\omega(\mathbb{A}^*)=\omega((\widetilde{\mathbb{A}})^*),$$

where $\omega(\cdot)$ denotes $\sigma_p(\cdot),\sigma_{ap}(\cdot),\sigma_e^\ell(\cdot)$, and $\sigma_e^r(\cdot)$, respectively.

REMARK. Now, we consider applicatons of results above to the calss of operators in $\mathcal{HU}(p)$. It is well known([12, Lemma 1.10]) that if A is p-hyponormal for some $p\in(0,1/2)$, then its Aluthge transform $\widetilde{A}$ is $(p+1/2)$-hyponormal and its double Aluthge transform $\widetilde{\widetilde{A}}$ is hyponormal. Hence combining this fact with Corollary 7, we can easily generalize or recapture most of all results proved in [3], [9], and [11], and so we shall abbreviate the details.

Acknowledgments. Authors would like to express their thanks to the referee for his useful advice.

REFERENCES

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th., 13(1990), 307-315.

[2] M.-D. Choi and C. Davis, *The spectral mapping theorem for joint approximate spectrum*, Bull. Amer. Math. Soc., 80(1974), 317-321.

[3] M. Chō, *Spectral properties of p-hyponormal operators*, Glasgow Math. J., 36(1994), 117-122.

[4] M. Chō and T. Huruya, *p-hyponormal operators for* $0 < p < 1/2$, Commentationes Math., 33(1993), 23-29.

[5] R. E. Curto, *Spectral permanence for joint spectra*, Trans. Amer. Math. Soc. 270(1982), 659-665.

[6] A. T. Dash, *Joint essential spectra* , Pacific J. Math., 64(1976), 119-128.

[7] A. T. Dash, *Joint spectra*, Studia Math., 45(1973), 225-237.

[8] B. P. Duggal, *Quasi-similar p-hyponormal operators*, Integr. Equat. Oper. Th., 26(1996), 338-345.

[9] B. P. Duggal, *On the spectrum of n-tuples of p-hyponormal operators*, Glasgow Math. J., 40(1998), 123-131.

[10] T. Furuta, *On the polar decomposition of an operator*, Acta Sci. Math. (Szeged), 46(1983), 261-268.

[11] I. H. Jeon, *On joint essential spectra of doubly commuting n-tuples of p-hyponormal operators*, Glasgow Math. J., 40(1998), 353-358.

[12] I. B. Jung, E. Ko, and C. Pearcy, *Invariant subspace theorems for p-hyponormal operators*, (preprint).

[13] K. Löwner, *Über monotone matrix funtionen*, Math. Z., 38(1983), 507-514.

[14] D. Xia, *Spectral theory of hyponormal operators*, Birkhäuser, Basel, 1983.

Muneo Chō
Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan

E-mail address: m-cho@cc.kanagawa-u.ac.jp

I. H. JEON
DEPARTMENT OF MATHEMATICS, EWHA WOMAN'S UNIVERSITY, SEOUL 120-750, KOREA
E-mail address: jih@math.skku.ac.kr

J. I. LEE
DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, KOREA
E-mail address: jilee@math.skku.ac.kr

Received November 12, 1999 Revised January 4, 2000