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ON THE RATE OF CONVERGENCE OF A POSITIVE APPROXIMATION PROCESS

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ABSTRACT. In this paper we are dealing with a class of summation integral operators on unbounded interval generated by a sequence $(L_n)_{n\geq 1}$ of linear and positive operators. We study the degree of approximation in terms of the moduli of smoothness of first and second order. Also we present the relationship between the local smoothness of functions and the local approximation. By using probabilistic methods, new features of $L_n f$ are pointed out such as the approximation property at discontinuity points and the monotonicity property under some additional assumptions of the function f. Also the rate of convergence of these operators for functions of bounded variation is given.

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1. Introduction

In [7] Lupaş proposed to study the following sequence of linear and positive operators

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0, \quad f : [0,\infty) \to \mathbf{R}, \tag{1}$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$, $k \geq 1$.

We can consider that L_n, $n \geq 1$, are defined on E where $E = \bigcup_{a>0} E_a$ and E_a is the subspace of all real valued continuous functions f on $[0,\infty)$ such as $e(f;a) := \sup_{x\geq 0}(\exp(-ax)|f(x)|)$ is finite. The space E_a is endowed with the norm $\|f\|_a = e(f;a)$ with respect to which it becomes a Banach lattice.

Concerning the raised problem, in [1] some quatitative estimates for the rate of convergence were given. Among the results below we mention the following.

Let b be a positive number.

(i) $|(L_nf)(x) - f(x)| \le (3 + 2b\max(1, b/n))\omega_2(f; 1/\sqrt{n})$, $x \in [0, b]$;

(ii) If f has a continuous derivative on $[0, b]$ then

$$|(L_nf)(x) - f(x)| \le \sqrt{2b}\left(\sqrt{2b} + 1\right) n^{-1/2}\omega_1(f'; 1/\sqrt{n}), \quad x \in [0, b];$$

(iii) $\lim_{n\to\infty} L_nf = f$ uniformly on $[0, b]$.

We recall the usual first and second moduli of smoothness of a function g as defined by

$$\omega_1(g; \delta) = \sup_{0<h\le\delta} \sup_{x\ge 0} |g(x + h) - g(x)| \quad \text{respectively}$$

$$\omega_2(g; \delta) = \sup_{0<h\le\delta} \sup_{x\ge 0} |g(x) - 2g(x + h) + g(x + 2h)|, \quad \delta > 0.$$

In the present paper we modify the operators defined by (1) into integral form in Kantorovich sense, see also G.G. Lorentz [6, Ch.II, p.30]. Actually, we replace $f(k/n)$ by an integral mean of $f(x)$ over a small interval around the point k/n as follows

$$(T_nf)(x) = n\sum_{k=0}^{\infty} l_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt, \tag{2}$$

where

$$l_{n,k}(x) = 2^{-nx}\frac{(nx)_k}{2^k k!}, \quad k \in \mathbf{N}_0, \quad x \in [0, \infty),$$

and f belongs to the class of local integrable functions defined on $[0, \infty)$.

The focus of the paper is to investigate these linear and positive operators.

Section 2 provided results in connection with the rate of convergence for T_nf under different assumptions of the function f. In section 3 we present new properties of L_n operator.

2. Approximation properties

In what follows, for any integer $s \ge 0$ we denote by e_s the test function, $e_s(x) = x^s$, $x \ge 0$, and we also introduce the s-th order central moment of the operator T_n, that is

$$\Omega_{n,s}(x) = (T_n\psi_{x,s})(x) \text{ where } \psi_{x,s}(t) = (t - x)^s, \quad x \ge 0, \quad t \ge 0.$$

Lemma 1. *The operators T_n, $n \in \mathbf{N}$, defined by (2) verify*

$$(T_ne_0)(x) = 1, \quad (T_ne_1)(x) = x + \frac{1}{2n}, \quad (T_ne_2)(x) = x^2 + \frac{3x}{n} + \frac{1}{3n^2}.$$

Proof. Starting from the identity

$$2^{nx} = \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} 2^{-k} \tag{3}$$

the following relations $L_n e_0 = e_0$, $L_n e_1 = e_1$, $L_n e_2 = e_2 + 2e_1/n$ are valid [1].

In this respect we have $T_n e_0 = L_n e_0$, $T_n e_1 = L_n e_1 + (2n)^{-1}$, $T_n e_2 = L_n e_2 + n^{-1} L_n e_1 + (3n^2)^{-1}$. Our assertions follow. □

Lemma 1 implies the following identities

$$\Omega_{n,0}(x) = 1, \quad \Omega_{n,1}(x) = \frac{1}{2n}, \quad \Omega_{n,2}(x) = \frac{6nx+1}{3n^2}. \tag{4}$$

Theorem 1. *Let T_n be defined by (2). Then for $f \in C[0,\infty)$ one has*

$$\lim_{n\to\infty} T_n f = f \text{ uniformly on any compact } K \subset [0,\infty).$$

Proof. By making use of Lemma 1 we have $\lim\limits_{n\to\infty} T_n e_r = e_r$, $r = 0, 1, 2$, uniformly on any compact $K \subset [0,\infty)$. Consequently, our assertion follows directly from the well-known theorem of Bohman-Korovkin. □

Theorem 2. *If T_n is defined by (2) then for each $x \geq 0$ the following inequality*

$$|(T_n f)(x) - f(x)| \leq \frac{4}{3}\omega_1\left(f; \frac{\sqrt{6nx+1}}{n}\right)$$

holds.

Proof. Since $(T_n e_0)(x) = 1$ and $l_{n,k}(x) \geq 0$ we can write

$$|(T_n f)(x) - f(x)| \leq n \sum_{k=0}^{\infty} l_{n,k}(x) \int_{k/n}^{(k+1)/n} |f(t) - f(x)| dt. \tag{5}$$

On the other hand $|f(t) - f(x)| \leq \omega_1(f; |t-x|) \leq (1 + \delta^{-2}(t-x)^2)\omega_1(f;\delta)$. For $|t - x| < \delta$ the last increase is clear. For $|t - x| \geq \delta$ we use the following properties

$$\omega_1(f; \lambda\delta) \leq (1+\lambda)\omega_1(f;\delta) \leq (1+\lambda^2)\omega_1(f;\delta)$$

where we choose $\lambda = \delta^{-1}|t-x|$. This way the relation (5) implies

$$|(T_n f)(x) - f(x)| \leq n \sum_{k=0}^{\infty} l_{n,k}(x) \int_{k/n}^{(k+1)/n} (1 + \delta^{-2}(x-t)^2)\omega_1(f;\delta) dt =$$

$$= (\Omega_{n,0}(x) + \delta^{-2}\Omega_{n,2}(x))\omega_1(f;\delta).$$

Taking into account (4) and choosing $\delta = (3\Omega_{n,2}(x))^{1/2}$ we obtain the desired result.□

Further, we estimate the rate of convergence for smooth functions.

Theorem 3. *Let T_n be defined by (2). Then for $f \in C^1[0,\infty)$ and $a > 0$ one has*

$$|(T_n f)(x) - f(x)| \le \frac{1}{2n}\left(\|f'\|_{C[0,a]} + \alpha_n \omega_1\left(f'; \frac{1}{\sqrt{n}}\right)\right),$$

where $\alpha_n = 2\sqrt{2na+1}\,(1 + \sqrt{2a + n^{-1}})$.

Proof. We can write

$$f(x) - f(t) = (x-t)f'(x) + (x-t)(f'(\xi) - f'(x)),$$

where $\xi = \xi(t,x)$ is a point of the interval determinated by x and t. If we multiply both members of this inequality by $n l_{n,k}(x) \int_{k/n}^{(k+1)/n} dt$ and sum over k, there follows

$$|(T_n f)(x) - f(x)| \le |f'(x)|\Omega_{n,1}(x) + n\sum_{k=0}^{\infty} l_{n,k}(x) \int_{k/n}^{(k+1)/n} |x-t|\cdot|f'(\xi) - f'(t)|dt \le$$

$$\le \frac{1}{2n}\max_{x\in[0,a]}|f'(x)| + n\sum_{k=0}^{\infty} l_{n,k}(x)\int_{k/n}^{(k+1)/n} |x-t|(1+\delta^{-1}|t-x|)\omega_1(f';\delta)dt.$$

According to Cauchy's inequality we have

$$n\sum_{k=0}^{\infty} l_{n,k}(x)\int_{k/n}^{(k+1)/n}|x-t|dt \le \sqrt{n}\sum_{k=0}^{\infty} l_{n,k}(x)\left\{\int_{k/n}^{(k+1)/n}(x-t)^2dt\right\}^{1/2} \le$$

$$\le \sqrt{n}\left\{\left(\sum_{k=0}^{\infty} l_{n,k}(x)\right)\left(\sum_{k=0}^{\infty} l_{n,k}(x)\int_{k/n}^{(k+1)/n}(x-t)^2dt\right)\right\}^{1/2} = \Omega_{n,2}^{1/2}(x). \tag{6}$$

The above inequalities enable us to write

$$|(T_n f)(x) - f(x)| \le \frac{1}{2n}\|f'\|_{C[0,a]} + \Omega_{n,2}^{1/2}(x)(1 + \delta^{-1}\Omega_{n,2}^{1/2}(x))\omega_1(f';\delta).$$

Inserting $\delta = 1/\sqrt{n}$ and using $\Omega_{n,x}^{1/2}(x) < \dfrac{\sqrt{2na+1}}{n}$, $x \in [0,a]$, the proof of our theorem is complete. □

Since the operators defined by (1) verify $(L_n\psi_{x,2})(x) = 2x/n$ we introduce the function φ, $\varphi(x) = \sqrt{2x}$, $x \ge 0$, representing the step weight function of the Lupaş operators and controling their rate of convergence. For T_n we can prove another estimation of the rate of convergence which implies the function φ.

Theorem 4. *Let T_n be defined by (2). Then for $f \in C^2[0,\infty)$ such that f' and $\varphi^2 f''$ are bounded on $[0,\infty)$ one has*

$$|(T_n f)(x) - f(x)| \leq \frac{4}{3n}(\|f'\|_\infty + \|\varphi^2 f''\|_\infty), \quad x \geq 0,$$

where $\|\cdot\|_\infty$ is defined by $\|h\|_\infty = \sup\limits_{x\geq 0} |h(x)|$.

Proof. Case 1. $\varphi^2(x) \geq 1/n$. We start from the identity

$$f(t) = f'(x)(t-x) + \int_x^t (t-v)f''(v)dv. \tag{7}$$

For every v situated between the positive numbers t and x we have

$$\frac{|t-v|}{\varphi^2(v)} \leq \frac{|t-x|}{\varphi^2(x)}. \tag{8}$$

The identities (7) and (4) lead us to the following relation

$$|(T_n f)(x) - f(x)| \leq \frac{1}{n}|f'(x)| + T_n\left(\left|\int_{xe_0}^{e_1}(e_1 - ve_0)f''(v)dv\right|; x\right).$$

Using (8) we have

$$\left|\int_x^t (t-v)f''(v)dv\right| \leq \left|\int_x^t \frac{\varphi^2(v)}{\varphi^2(x)}|t-x||f''(v)|dv\right| \leq \frac{\|\varphi^2 f''\|_\infty}{\varphi^2(x)}(t-x)^2,$$

and consequently

$$|(T_n f)(x) - f(x)| \leq \frac{1}{n}\|f'\|_\infty + \frac{\|\varphi^2 f''\|_\infty}{\varphi^2(x)}\Omega_{n,2}(x) =$$

$$= \frac{1}{n}\|f'\|_\infty + \|\varphi^2 f''\|_\infty\left(\frac{1}{n} + \frac{1}{3n^2\varphi^2(x)}\right) \leq \frac{1}{n}\|f'\|_\infty + \frac{4}{3n}\|\varphi^2 f''\|_\infty.$$

Case 2. $\varphi^2(x) < 1/n$. We can write successively

$$|(T_n f)(x) - f(x)| = \left(T_n\left(\int_{xe_0}^{e_1} f'(v)dv; x\right)\right| \leq \|f'\|_\infty T_n(|e_1 - xe_0|; x) \leq$$

$$\leq \|f'\|_\infty \Omega_{n,2}^{1/2}(x) \leq \frac{2}{\sqrt{3}\,n}\|f'\|_\infty.$$

We have also used the relation (6).

Analysing the two above cases the conclusion of our theorem follows. □

Further, it is easy to notice that T_n can be written as a singular integral of the type

$$(T_n f)(x) = \int_0^\infty H_n(x,t)f(t)dt, \quad x \geq 0,$$

with the non-negative kernel $H_n(x,t) = nl_{n,k}(x)$ for $k/n \le t < (k+1)/n$, $k \in \mathbf{N}_0$.

Lemma 1 guarantees that our kernel satisfies

$$\int_0^\infty H_n(x,t)dt = \sum_{k=0}^{\infty} l_{n,k}(x) = 1. \tag{9}$$

We present a direct result of Hölder's inequality.

Lemma 2. *If T_n is defined by (2) then for every $0 < \alpha \le 1$ and $h \ge 0$ we have*

$$T_n(h^\alpha; x) \le (T_n(h^2; x))^{\alpha/2}.$$

Proof. Considering $r := 2/\alpha$ in the relation $1/r + 1/s = 1$, $r > 0$, $s > 0$, which characterizes Hölder's inequality, from (9) we get the claimed result. □

As a consequence of Lemma 2 we obtain

$$T_n(|e_1 - xe_0|^\alpha; x) \le \Omega_{n,2}^{\alpha/2}(x), \quad x \ge 0. \tag{10}$$

At this point we recall that a continuous function f defined on J is locally *Lip*α on E $(0 < \alpha \le 1,\ E \subset J)$ if it satisfies the condition

$$|f(x) - f(y)| \le M_f|x-y|^\alpha,\ (\forall)\ (x,y) \in J \times E, \tag{11}$$

where M_f is a constant depending only on α and f.

Theorem 5. *Let T_n be given by (2), $0 < \alpha \le 1$ and E be any subset of $[0,\infty)$. If f is locally Lipα on E then we have*

$$|(T_nf)(x) - f(x)| \le M_f(\lambda_n(x,\alpha)n^{-\alpha/2} + 2d^\alpha(x,E)),$$

where $\lambda_n(x,\alpha) = (2x)^{\alpha/2} + (3n)^{-\alpha/2}$ and $d(x,E)$ is the distance between x and E defined as $d(x,E) = \inf\{|x-y| :\ y \in E\}$.

Proof. By using the continuity of f it is obvious that (11) holds for any $x \ge 0$ and $y \in \overline{E}$, the closure in $\mathbf{R}$ of the set E. Let $(x,x_0) \in [0,\infty)\times\overline{E}$ be such that $|x-x_0| = d(x,E)$. On the other hand, we can write $|f - f(x)| \le |f - f(x_0)| + |f(x_0) - f(x)|$ and applying the linear and positive operator T_n we have

$$|(T_nf)(x) - f(x)| \le T_n(|f - f(x_0)|; x) + |f(x) - f(x_0)| \le$$

$$\le T_n(M_f|e_1 - x_0e_0|^\alpha; x) + M_f|x-x_0|^\alpha. \tag{12}$$

In the classical inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ $(a \ge 0,\ b \ge 0,\ 0 < \alpha \le 1)$ we put $a = |t-x|$, $b = |x - x_0|$ and further by using (10) we have

$$T_n(M_f|e_1 - x_0e_0|^\alpha; x) \le M_f(T_n|e_1 - xe_0|^\alpha; x) + |x - x_0|^\alpha) \le$$

$$\le M_f(\Omega_{n,2}^{\alpha/2}(x) + |x - x_0|^\alpha) \le M_f\left(\left(\frac{2x}{n}\right)^{\alpha/2} + \left(\frac{1}{3n^2}\right)^{\alpha/2} + |x - x_0|^\alpha\right).$$

Returning to (12), we obtain the aimed result. □

In particular, when $E = [0, \infty)$ the following proposition can be stated.

Corollary. *Let T_n be given by (2) and $0 < \alpha \le 1$. If f satisfies $\omega_1(f,t) = \mathcal{O}(t^\alpha)$ then it exists a constant M_f independent of n and x such that*

$$|(T_nf)(x) - f(x)| \le M_f((2nx)^{\alpha/2} + 1)n^{-\alpha}, \quad x \ge 0. \tag{13}$$

Remark. The inverse result given by this Corollary is not true, which can be seen from the following example. Let f be defined by $f(x) = (x+1)(\ln(x+1) - 1)$, $x \ge 0$. Then $\omega_1(f;t) \ne \mathcal{O}(t^\alpha)$ for $\alpha = 1/2$. However (13) is satisfied. For $x \ge 1/n$ we have

$$|(T_nf)(x) - f(x)| = \left|f'(x)T_n(e_1 - xe_0; x) + T_n\left(\int_{xe_0}^{e_1}(e_1 - ue_0)f''(u)du; x\right)\right| \le$$

$$\le \frac{\ln(x+1)}{2n} + \|e_1f''\|_\infty T_n\left(\frac{(e_1 - xe_0)^2}{x}; x\right) \le \frac{C_1x^{1/4}}{n} + \frac{2}{n} + \frac{1}{3n^2x} \le C_2((2nx)^{1/4} + 1)n^{-1/2}.$$

For $x < 1/n$ we have

$$|(T_nf)(x) - f(x)| = \left|T_n\left(\int_{xe_0}^{e_1}\ln(u+1)du; x\right)\right| \le T_n\left(\int_{xe_0}^{e_1}udu; x\right) \le$$

$$\le T_n(e_2 + x^2e_0; x) = 2x^2 + \frac{3x}{n} + \frac{1}{3n^2} \le C_3((2nx)^{1/4} + 1)n^{-1/2}.$$

Here C_1, C_2, C_3 are constants independent of n and x.

3. A probabilistic look over L_n

In this section we return to the operator L_n and by using probabilistic tools we present further approximation properties different from those quoted in the first section of this paper.

It is known that many classical linear and positive operators such as Bernstein, Szász, Baskakov, Gamma and Weierstrass are special cases of an operator due to Feller [2]. To define the Feller operator let $(X_n)_{n\geq 1}$ be a sequence of random variables having distribution function $F^*_{n,x}$ with expectation $EX_n = x$ and variance $Var(X_n) = \sigma_n^2(x)$ where $x \in I$ is a real parameter. For a continuous function f on the real line define the operator

$$F_n(f;x) = Ef(X_n) = \int_{\mathbf{R}} f(t)dF^*_{n,x}(t) \quad \text{if } E|f(X_n)| < \infty.$$

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with mean $x \in I$ and variance $\sigma^2(x)$ and set $S_n = \sum_{i=1}^{n} Y_i$. Then Feller operator is equivalent to

$$F_n(f;x) = Ef(S_n/n) = \int_{\mathbf{R}} f(t/n)dF_{n,x}(t), \tag{14}$$

where $F_{n,x}$ is the distribution function of S_n.

We point out that the approximation properties of F_n for continuous function were investigated by D.D. Stancu [8].

If we take

$$P(Y_1 = k) = 2^{-x-k}\frac{(x)_k}{k!}, \quad k = 0, 1, \ldots, \quad x \geq 0, \tag{15}$$

then (14) reduces to the operator defined at (1).

Lemma 3. *Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with the distribution given by (15). For $j \in \mathbf{N}$ the following relations*

$$E(Y_j) = x, \quad E(Y_j^2) = x^2 + 2x, \quad E(Y_j^3) = x^3 + 6x^2 + 6x,$$

hold.

Proof. Taking into account the relation (3) and the recurrence formula $(\alpha)_k = \alpha(\alpha+1)_{k-1}$, $k \geq 1$, after few calculations our identities follow. □

Theorem 6. *Let L_n be defined by (1).*

(i) If $x_0 > 0$ is a discontinuity point of the first kind for f then

$$\lim_{n\to\infty}(L_n f)(x_0) = \frac{1}{2}(f(x_0^+) + f(x_0^-)).$$

(ii) If f is a continuous convex and bounded function then

$$(L_n f)(x) \geq (L_{n+1} f)(x) \geq \ldots \geq f(x).$$

Proof. (i) In view of Lemma 3, our statement is a consequence of a result due to B. Levikson [5, Theorem 1].

(ii) Based on the identity $E(n^{-1}S_n|S_{n+1}) = (n+1)^{-1}S_{n+1}$ a.s., R.A. Khan [4] had given an elementary probabilistic proof of monotonic convergence for the Feller operator. Our assertion follows. □

Using probabilistic approach we give an estimate of the rate of convergence for L_n operators for a function f of bounded variation ($f \in BV[0,\infty)$). We denote by $V_{[a,b]}(h)$ the total variation of h on $[a,b]$.

Theorem 7. *Let L_n be defined by (1) and $f \in BV[0,\infty)$. For every $x > 0$ and all $n = 1, 2, \ldots$ we have*

$$|(L_nf)(x) - \frac{1}{2}(f(x^+) + f(x^-))| \le \frac{1}{n}\left\{(4x+1)\sum_{k=0}^{n} V_{I_k}(g_x) + \sqrt{\frac{2n}{x}}(4x^2+6x+3)\tilde{f}(x)\right\},$$

where $I_0 = (-\infty,\infty)$, $I_k = \left[x - 1/\sqrt{k}, x + 1/\sqrt{k}\right]$, $k = 1, 2, \ldots, n$, $\tilde{f}(x) = |f(x^+) - f(x^-)|$ and

$$g_x(t) = \begin{cases} f(t) - f(x^+), & t > x \\ 0, & t = x \\ f(t) - f(x^-), & t < x. \end{cases}$$

Proof. If $f \in BV[0,\infty)$ then one can extend f over $(-\infty,\infty)$ by $f(t) = f(0)$ for $t < 0$. Therefore the extended f belongs to $BV(-\infty,\infty)$. Throughout we shall use the notation f for both f and its extended version. Clearly, $\tilde{f}(x)$ indicates the size of the saltus of f at x. At this point we need a result established by S.S. Guo and M.K. Khan [3, Theorem 2]. For the Feller operator (14) we have

$$|(F_nf)(x) - \frac{1}{2}(f(x^+) + f(x^-))| \le \frac{2\sigma^2(x)+1}{n}\sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2E|Y_1 - x|^3}{\sqrt{n}\sigma^3(x)}\tilde{f}(x), \qquad (16)$$

where $\sigma^2(x) = Var(Y_1)$.

In our case Lemma 3 implies $\sigma^3(x) = E(Y_1^2) - E^2(Y_1) = 2x$, $x > 0$, and for $E|Y_1 - x|^3$ we can give the following simple bound

$$E|Y_1 - x|^3 \le E(Y_1^3) + 3xE(Y_1^2) + 3x^2E(Y_1) + x^3 = 8x^3 + 12x^2 + 6x.$$

Thus, the relation (16) leads us to the aimed result. □

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