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On an Invariant Subspace Whose Common Zero Set is the Zeros of Some Function

By

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Abstract. Let F be a nonzero function in $H^2(D^n)$ such that if ϕ is a function in $L^\infty(T^n)$ and ϕF is in $H^2(D^n)$, then ϕ belongs to $H^\infty(D^n)$. We study the set of multipliers of an invariant subspace M of $H^2(D^n)$ whose common zero set of M is just a zero set of F.

§1. Introduction

Let D^n be the open unit polydisc in $\mathbf{C}^n$ and T^n be its distinguished boundary. The normalized Lebesgue measure on T^n is denoted by dm. For $0 < p \leq \infty$, $H^p(D^n)$ is the Hardy space and $L^p(T^n)$ is the Lebesgue space on T^n. Let $N(D^n)$ denote the Nevanlinna class. Each f in $N(D^n)$ has radial limits f^* defined on T^n *a.e.*. Moreover, there is a singular measure $d\sigma_f$ on T^n determined by f such that the least harmonic majorant $u(\log|f|)$ of $\log|f|$ is given by $u(\log|f|)(z) = P_z(\log|f^*| + d\sigma_f)$ where P_z denotes Poisson integration and $z = (z_1, z_2, \cdots, z_n) \in D^n$. Put $N_*(D^n) = \{f \in N(D^n)\ ;\ d\sigma_f \leq 0\}$, then $H^p(D^n) \subset N_*(D^n) \subset N(D^n)$ and $H^p(D^n) = N_*(D^n) \cap L^p(T^n) \subsetneq N(D^n) \cap L^p(T^n)$. These facts are shown in [10, Theorem 3.3.5].

A closed subspace M of $H^p(D^n)$ is said to be invariant if $z_jM \subset M$ for $j = 1, 2, \cdots, n$. For an invariant subspace M of $H^2(D^n)$, set

$$\mathcal{M}(M) = \{\phi \in L^\infty(T^n)\ ;\ \phi M \subseteq H^2(D^n)\}.$$

$\mathcal{M}(M)$ is called the set of multipliers of M and $\mathcal{M}(M) \supseteq H^\infty(D^n)$. $\mathcal{M}(M)$ has been studied in [1],[2],[3],[7],[8] and [9]. In the previous paper [7], the author studied $\mathcal{M}(M)$ in general and gave a necessary and sufficient condition for $\mathcal{M}(M) = H^\infty(D^n)$. It is easy to see that $\mathcal{M}(M) = H^\infty(D^n)$ when the condimension of M in $H^2(D^n)$ is finite.

R.G.Douglas and K.Yan [1] generalized this result. They introduced the common zero set $Z(M)$ and the singular measure $Z_\partial(M)$ for an invariant subspace M of $H^2(D^n)$, that is,

$$Z(M) = \{z \in D^n \; ; \; f(z) = 0 \text{ for } f \in M\}$$

and

$$Z_\partial(M) = \inf\{-d\sigma_f \; ; \; f \in M, f \neq 0\}.$$

If F is a nonzero function in $H^2(D^n)$ and M_F is an invariant subspace generated by F, then

$$Z(M_F) = \{z \in D^n \; ; \; F(z) = 0\} = Z(F)$$

and $Z_\partial(M_F) = -d\sigma_F$. If $h_{2n-2}(Z(M)) = 0$ and $Z_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$ where h_{2n-2} is real $2n-2$ dimensional Hausdorff measure [1]. In the previous paper [8], the author studied an invariant subspace whose common zero set is the common zero set of the kernel of a slice map. The real $2n-2$ dimensional Hausdorff measure of such a common zero set may be positive when $n = 2$. K.Izuchi [2] showed that $\mathcal{M}(M_F) = H^\infty(D^n)$ for an outer function F. In this case, $Z(M_F) = \emptyset$ and $Z_\partial(M_F) = 0$. In the previous paper [9], the author studied the function F with $\mathcal{M}(M_F) = H^\infty(D^n)$ when $n = 2$. He gave two necessary and sufficient conditions for $\mathcal{M}(M_F) = H^\infty(D^2)$. Moreover he showed that some function F (it is neither an outer function nor a weakly outer function) satisfies $\mathcal{M}(M_F) = H^\infty(D^2)$.

In Section 2, we give several factorization lemmas which will be used in the latter sections. In Section 3, we generalize (3) of Theorem 4 in [9] to an arbitrary n. Moreover we study when a function f with $d\sigma_f = 0$ satisfies $\mathcal{M}(M_f) = H^\infty(D^n)$ under a condition on $Z(f)$. Fix $\alpha \in \overline{D^n}$. For f in $H^2(D^n)$, put

$$(\Phi_\alpha f)(\lambda) = f(\alpha_1\lambda, \cdots, \alpha_n\lambda) \quad (\lambda \in D).$$

Φ_α is called a slice map. When $n = 2, \Phi_\alpha$ maps $H^2(D^n)$ into $L^2_a(D)$, where $L^2_a(D)$ is the Bergman space (*cf.* [10, p.53],[8]). In Section 4, in case $n \geq 3$, we show that if M is an invariant subspace with $Z(M) = Z(\ker \Phi_\alpha)$ and $Z_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$. In case $n = 2$, we determine α with $\mathcal{M}(M) = H^\infty(D^n)$ when M is finitely generated, $Z(M) = Z(\ker \Phi_\alpha)$ and $Z_\partial(M) = 0$. This improves Theorem 4 in [8]. When $n = 2, Z(\ker \Phi_\alpha) = Z(F)$ for $F(z) = \alpha_2 z_1 - \alpha_1 z_2$. Let F be a homogeneous polynomial of arbitrary degree. We are interested in $\mathcal{M}(M)$ when M is an invariant subspace with $Z(M) = Z(M_F)$ and $Z_\partial(M) = 0$. In Section 5, we study it when F is a Weierstrass polynomial.

In this paper, we use the following notations.

$z = (z_j, z'_j), z'_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$.

$D^n = D_j \times D'_j, D'_j = \prod_{\ell \neq j} D_\ell$ where $D^n = \prod_{\ell=1}^n D_\ell$ and $D_\ell = D$.

$T^n = T_j \times T'_j$, $T'_j = \prod_{\ell \neq j} T_\ell$ where $T^n = \prod_{\ell=1}^n T_\ell$ and $T_\ell = T$.

$m = m_j \times m'_j$, $m'_j = \prod_{\ell \neq j} m_\ell$ where $m = \prod_{\ell=1}^n m_\ell$ and m_ℓ is the normalized Lebesgue measure on T_ℓ.

§2. Factorization lemmas

For f in $N(D^n)$, $f(z) = \sum_{j=0}^{\infty} F_j(z)$ is a homogeneous expansion of f and F_j is a polynomial which is homogeneous of degree j. The smallest $j = j(f)$ such that F_j is not the zero-polynomial is called the order of the zero which f has at $(0, \cdots, 0)$. For $p \in D^n$, the order of the zero of f at $p, O(f,p)$, is simply the order of the zero of $f_p(z) = f(z+p)$ at $z = (0, \cdots, 0)$. In this section, we give a factorization of f under a condition on $O(f,p)$ $(p \in D^n)$. This will be used in the latter sections. Put

$$F(z) = z_1^\ell + a_{\ell-1}(z_1')z_1^{\ell-1} + \cdots + a_1(z_1')z_1 + a_0(z_1')$$

where $\{a_j\}_{j=0}^{\ell-1}$ are analytic on D_1' and $a_j(0, \cdots, 0) = 0$ for $0 \le j \le \ell - 1$, then we call F a Weierstrass polynomial of degree ℓ. In this section we give several factorization lemmas which will be used in this paper. Lemma 1 is well known. In fact, it is valid for simply connected regions which are Cousin II domain (*cf.* [4],[5]).

Lemma 1. *Let F and f be nonzero holomorphic functions on D^n. If $O(F,p) \le O(f,p)$ for every $p \in D^n$ then $f = Fg$ where g is holomorphic on D^n. When $O(F,p) = O(f,p)$ for every $p \in D^n$, $Z(g) = \emptyset$.*

Lemma 2. *Let F and f be nonzero functions in $N(D^n)$.*

(1) $O(F,p) = O(f,p)$ for every $p \in D^n$, then $f = Fg$ where g and g^{-1} are in $N(D^n)$.

(2) If $O(F,p) = O(f,p)$ for every $p \in D^n$ and $d\sigma_F \ge d\sigma_f$, then $f = Fg$ where g is in $N_*(D^n)$.

Proof. By Lemma 1, we have a factorization $f = Fg$. Hence

$$\int_{T^n} |\log |g(rz)||dm \le \int_{T^n} |\log |f(rz)||dm + \int_{T^n} |\log |F(rz)||dm$$

implies that g belongs to $N(D^n)$. This implies (1). Since $d\sigma_f = d\sigma_F + d\sigma_g$, if $d\sigma_F \ge d\sigma_f$ then $d\sigma_g \le 0$ and so g belongs to $N_*(D^n)$. This implies (2).

Lemma 3. *Let F be a function in $N_*(D^n)$ and $d\sigma_F = 0$. If f is a nonzero function in $N(D^n)(N_*(D^n))$ and $O(F,p) \le O(f,p)$ for every $p \in D^n$, then $f = Fg$ where g is in $N(D^n)(N_*(D^n))$.*

Proof. By Lemma 1, we have a factorization $f = Fg$. By the proof of Lemma 2, g belongs to $N(D^n)$. Since $d\sigma_F = 0$, $d\sigma_g = d\sigma_f \le 0$ and so g belongs to $N_*(D^n)$ if f is in $N_*(D^n)$.

Lemma 4. *Let F be a Weierstrass polynomial of degree 1 in the Nevanlinna class and $d\sigma_F = 0$. If f is a nonzero function in $N(D^n)$ $(N_*(D^n))$ such that $f_p(z_1, 0')$ has a zero of order $O(f,p)$ at $z_1 = 0$ for each p in $Z(f)$, $Z(f) \subseteq Z(F)$ and $Z(f) \ne \emptyset$ then $f = F^\ell g$ where g is in $N(D^n)$ $(N_*(D^n))$ and ℓ is a positive integer.*

Proof. Suppose $F(z) = z_1 - \alpha(z_1')$ is a Weierstrass polynomial. By hypothesis, there exists $f \in N(D^n)(N_*(D^N))$ such that $f_p(z_1, 0')$ has a zero of order $O(f,p) \ne 0$

at $z_1 = 0$ and so by the Weierstrass preparation theorem, there exists a polydisc $\triangle$ in $\mathbf{C}^n$, centered at $(0,\cdots,0)$, such that $f_p(z) = w(z)h(z)$ for $z \in \triangle$, where h is analytic in $\triangle, h$ has no zero in $\triangle$ and $w(z)$ is a Weierstrass polynomial of degree ℓ. We can write $w(z) = \prod_{j=1}^{\ell}(z_1 - \alpha_j(z_1'))$ for $z = (z_1, z_1') \in \triangle$. Since $Z(f) \subseteq Z(F)$, if $(\alpha_j(z_1'), z_1') \in \triangle$, then $\alpha_j(z_1') = \alpha(z_1' + p_1') - p_1$. Hence $w(z) = (z_1 - \alpha(z_1' + p_1') + p_1)^{\ell}$ on some polydisc $\tilde{\triangle}$ which is contained in $\triangle$. Hence $f(z) = F(z)^{\ell} h(z-p)$ for $z \in \tilde{\triangle} + p$. This implies Lemma 4.

Lemma 5. *Let F be a nonzero homogeneous polynomial such that $F(z) = F(z_1, z_2)$. If f is a nonzero function in $N(D^n)(N_*(D^n))$ such that $Z(F) = Z(f)$ and $Z(f) \neq \emptyset$, then there exists a homogeneous polynomial $Q(z) = Q(z_1, z_2)$ of degree 1 such that $f = Qg$ and $F = QG$ where g is in $N(D^n)(N_*(D^n))$ and $G(z) = G(z_1, z_2)$ is a homogeneous polynomial. When $n = 2$, the same conclusion is valid under the weaker condition : $Z(F) \supseteq Z(f)$ and $Z(f) \neq \emptyset$.*

Proof. Since $F(z) = \sum_{j=0}^{\ell} a_j z_1^{\ell-j} z_2^j$ because $F(z) = F(z_1, z_2)$,

$$\begin{aligned} F(z) &= z_1^{\ell} \sum_{j=0}^{\ell} a_j \left(\frac{z_2}{z_1}\right)^j \\ &= c \prod_{j=0}^{\ell} (b_j z_2 - c_j z_1) \text{ where } b_j = 1 \text{ or } c_j = 1, \text{ and } |b_j| \leq 1,\ |c_j| \leq 1. \end{aligned}$$

Let $Q(z) = b_0 z_2 - c_0 z_1$, then $Z(Q) \subseteq Z(F) = Z(f)$. Hence $O(Q,p) \leq O(f,p)$ for every $p \in D^n$. Lemma 3 implies this lemma. Suppose $n = 2$ and $Z(F) \supseteq Z(f) \neq \emptyset$. For each j, put

$$h_j(\lambda) = f(b_j \lambda, c_j \lambda) \quad (\lambda \in D),$$

then $h_j \equiv 0$ on D or $Z(h_j)$ is a descrete set in D. If there exist at least a j $(0 \leq j \leq \ell)$ such that $h_j \equiv 0$ on D, then $O(f,p) \geq O(F_j, p)$ for every $p \in D^2$ and $F_j(z) = b_j z_2 - c_j z_1$. Then as $Q = F_j$ the lemma follows. If there does not exist any j such that $h_j \equiv 0$ on D, then $\bigcup_{j=0}^{\ell} Z(h_j)$ is descrete. Since $Z(f) \subseteq Z(F)$, this implies that $Z(f)$ is descrete and hence $Z(f) = \emptyset$. This contradicts $Z(f) \neq \emptyset$.

§3. $\mathcal{M}(M_F) = H^\infty(D^n)$

Let F be a nonzero function in $H^2(D^n)$. Then $\mathcal{M}(M_F) = H^\infty(D^n)$ if and only if F has the following property : If $|F| \geq |f|$ *a.e.* on T^n and f is a function in $H^2(D^n)$, then $|F| \geq |f|$ on D^n.

This was shown in [9, (1) of Theorem 4] only for $n = 2$ but the proof works for arbitrary $n \geq 2$. In this section, we study a function F with $\mathcal{M}(M_F) = H^\infty(D^n)$. Put for each $1 \leq j \leq n$,

$$H_j^p = \{f \in L^p(T^n)\ ;\ \hat{f}(m_j, m_j') = 0 \ \text{ if } \ m_j < 0\} \ \text{ and } \ H_j^p \cap \bar{H}_j^p = \mathcal{L}_{(j)}^p,$$

then $H_j^\infty \cap \bar{H}_j^\infty = \mathcal{L}_{(j)}^\infty$ is a commutative von Neumann algebra. If $\mathcal{E}^{(j)}$ denotes the conditional expectation from $L^\infty(T^n)$ to $\mathcal{L}_{(j)}^\infty$, then $\mathcal{E}^{(j)}$ is multiplicative on H_j^∞ and $H_j^\infty + \bar{H}_j^\infty$ is weak star dense in $L^\infty(T^n)$. This implies that H_j^∞ is an extended weak-$*$Dirichlet algebra with respect to $\mathcal{E}^{(j)}$. Hence we can use the general theory of an extended weak-$*$Dirichlet algebra in [6].

Suppose h is a nonzero function in $H^p(D^n)$. For some measurable set E in T_j', if h satisfies the following equality ;

$$\int_{T_j \times E} \log |h| dm = \int_E (\log |\int_{T_j} h dm_j|) dm_j',$$

h is called j-outer for $E \subset T_j'$. The left side in the above equality is always bigger than or equal to the right one for arbitrary function in $H^p(D^n)$. h is j-outer for $E \subset T_j'$ if and only if

$$\mathcal{E}^{(j)}(\log |h|) = \log |\mathcal{E}^{(j)}(h)| \quad a.e. \text{ on } T_j \times E.$$

We call h simply j-outer when $E = T_j'$. The following Theorem 1 is a generalization of (3) of Theorem 4 in [9] for arbitrary n. The proof is parallel to that in [9].

Theorem 1. *Suppose h is a function in $H^p(D^n)$. If h is ℓ-outer for any ℓ ($\neq j$) and j-outer for $E \subset T_j'$ with $m_j'(E) > 0$, then $\mathcal{M}(M_h) = H^\infty(D^n)$.*

If $h = \prod_{\ell=1}^{t} h_\ell$ and each h_ℓ in $H^\infty(D^n)$ satisfies $\mathcal{M}(M_{h_\ell}) = H^\infty(D^n)$, then it is clear that $\mathcal{M}(M_h) = H^\infty(D^n)$. By [9, p.495] there exists a function h in $H^\infty(D^n)$ which does not satisfy the conditon in Theorem 4 but $\mathcal{M}(M_h) = H^\infty(D^n)$. This was pointed to me privately by Professor K.Takahashi.

Lemma 6. *([1, Corollary 4]). For a function ϕ in $N(D^n) \cap L^\infty(T^n)$ and an invariant subspace M of $H^2(D^n)$, we have $\phi \in \mathcal{M}(M)$ if and only if $d\sigma_\phi \leq Z_\partial(M)$.*

Theorem 2. *Suppose F is a nonzero function in $H^\infty(D^n)$ and $\mathcal{M}(M_F) = H^\infty(D^n)$. If f is a nonzero function in $H^2(D^n)$ and it satisfies one of the following (1) $\sim$ (3), then $\mathcal{M}(M_f) = H^\infty(D^n)$.*

(1) $O(F,p) = O(f,p)$ for every $p \in D^n$ and $d\sigma_f = 0$

(2) $n = 2$ and F is a homogeneous polynomial with $Z(F) \supseteq Z(f)$ and $d\sigma_f = 0$.

(3) F is a Weierstrass polynomial of degree 1, $Z(F) \supseteq Z(f)$, $d\sigma_f = 0$ and $f_p(z_1, 0')$ has a zero of order $O(f,p)$ at $z_1 = 0$ for each p in $Z(f)$.

Proof. (1) If $\phi \in \mathcal{M}(M_f)$, then $\phi f \in H^2(D^n)$ and so by Lemma 2, $\phi F g \in H^2(D^n)$ where g and g^{-1} are in $N(D^n)$. Hence $\psi = \phi F$ is analytic on D^n and so $\psi \in N(D^n) \cap L^\infty(T^n)$. ψ is also in $\mathcal{M}(M_f)$ because $F \in H^\infty(D^n)$. By Lemma 6,

$$d\sigma_\psi \leq Z_\partial(M_f) = d\sigma_f = 0$$

by hypothesis on f and so ψ belongs to $H^\infty(D^n)$. Thus $F\mathcal{M}(M_f) \subseteq H^\infty(D^n)$ and so $\mathcal{M}(M_f) = H^\infty(D^n)$ because $\mathcal{M}(M_F) = H^\infty(D^n)$.

(2) We may assume that $Z(f) \neq \emptyset$ by [1]. Since $n = 2$ and F is a homogeneous polynomial, by the proof of Lemma 5, $F(z) = c\prod_{j=0}^{\ell}(b_j z_2 - c_j z_1), |b_j| = 1$ or $|c_j| = 1$ and $|b_j| \leq 1$, $|c_j| \leq 1$. By Lemma 5, there exists at least j $(0 \leq j \leq \ell)$ such that $f = (b_j z_2 - c_j z_1)g_j$ and $d\sigma_{g_j} = 0$. If $Z(g_j)$ is not empty, then $Z(g_j) \subseteq Z(F)$. By repeating the argument above, we can prove that $f = \prod_{j=0}^{\ell}(b_j z_2 - c_j z_1)^{\ell(j)} g$ where $Z(g) = \emptyset$ and $\ell(j)$ is a nonnegative integer $(0 \leq j \leq \ell)$. Since $\mathcal{M}(M_F) = H^\infty(D^n)$, $|b_j| = |c_j| \neq 0$ for any j $(0 \leq j \leq \ell)$. For if there exists a j such that $|b_j| \neq |c_j|$, then $(b_j z_2 - c_j z_1)^{-1} \notin H^\infty(D^n)$, and $(b_j z_2 - c_j z_1)^{-1} \in \mathcal{M}(M_F)$. This contradicts that $\mathcal{M}(M_F) = H^\infty(D^n)$. By [8, (4) of Proposition 3], $\mathcal{M}(M_Q) = H^\infty(D^n)$ where $Q = \prod_{j=0}^{\ell}(b_j z_2 - c_j z_1)^{\ell(j)}$. By (1), $\mathcal{M}(M_f) = H^\infty(D^n)$ because $f = Qg$ and $Z(g) = \emptyset$.

(3) By Lemma 4, $f = F^j g$ and $g \in N(D^n)$. If $Z(g)$ is not empty, $Z(g) \subseteq Z(F)$ and so by Lemma 4, $g = F^k g'$. We can repeat this process and get $f = F^\ell h$ for some positive integer ℓ where h and h^{-1} are in $N(D^n)$. We can prove (3) as in the proof of (1) and (2).

§4. Slice map

In this section, when $Z(M) = Z(\ker \Phi_\alpha)$ and $Z_\partial(M) = 0$, we give a necessary and sufficient condition for that $\mathcal{M}(M) = H^\infty(D^n)$.

Theorem 3. *Suppose $n \geq 3$. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \bar{D}^n$ and M be an invariant subspace in $H^2(D^n)$.*

(1) If $M \supseteq \ker \Phi_\alpha$, then $\mathcal{M}(M) = H^\infty(D^n)$.

(2) If $Z(M) = Z(\ker \Phi_\alpha)$ and $Z_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$.

Proof. If $\alpha = (\alpha_1, \cdots, \alpha_n) = (0, \cdots, 0)$, then $z_1 \in \ker \Phi_\alpha$ and $\ker \Phi_\alpha = \{f \in H^2(D^n)\ ;\ f(0, \cdots, 0) = 0\}$. Hence $Z(\ker \Phi_\alpha) = \{(0, \cdots, 0)\}$ and $Z_\partial(\ker \Phi_\alpha) = 0$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \neq (0, \cdots, 0)$, then there exists some $\alpha_j \neq 0, \alpha_j z_i - \alpha_i z_j \in \ker \Phi_\alpha$ for $i \neq j$, and $Z(\ker \Phi_\alpha) = \{(\alpha_1 \lambda, \cdots, \alpha_n \lambda) \in D^n\ ;\ \lambda \in \mathbf{C}\}$. Therefore for any $\alpha \in \bar{D}^n$, $Z_\partial(\ker \Phi_\alpha) = 0$ and the real $2n - 2$ dimensional Hausdorff measure of $Z(\ker \Phi_\alpha)$ is zero. Now a theorem of R.G.Douglas and K.Yan [1, Theorem 1] shows (1) and (2).

Theorem 4. *Suppose $n = 2$. Let $\alpha = (\alpha_1, \alpha_2) \in \overline{D^2}$ and M be an invariant subspace in $H^2(D^2)$.*

(1) If $M \supsetneq \ker \Phi_\alpha$, then $\mathcal{M}(M) = H^\infty(D^2)$.

(2) Let ℓ be a finite positive integer. Suppose there exists a function f in M such that $1 \leq O(f,p) \leq \ell$ for each p in $Z(M)$. When $Z(M) = Z(\ker \Phi_\alpha)$ and $Z_\partial(M) = 0$, $\mathcal{M}(M) = H^\infty(D^2)$ if and only if $|\alpha_1| = |\alpha_2|$.

Proof. (1) is proved in [8, (6) of Proposition 3].

(2) We have that $Z(\ker \Phi_\alpha) = \{(\alpha_1\lambda, \alpha_2\lambda) \in D^2 \ ; \ \lambda \in C\}$. If $\alpha = (\alpha_1, \alpha_2) = (0,0)$, then $Z(M) = \{(0,0)\}$ and so $\mathcal{M}(M) = H^\infty(D^2)$ by [1, Theorem 1]. We assume $\alpha = (\alpha_1, \alpha_2) \neq (0,0)$. Since $M \subseteq \ker \Phi_\alpha$, if $|\alpha_1| \neq |\alpha_2|$ then $\mathcal{M}(\ker \Phi_\alpha) \neq H^\infty(D^2)$ by [8, (4) of Proposition 3] and so $\mathcal{M}(M) \neq H^\infty(D^2)$. Assuming $|\alpha_1| = |\alpha_2| > 0$, we will show that $\mathcal{M}(M) = H^\infty(D^2)$. Note that

$$Z(M) = \cap \{Z(f_\beta) \ ; \ f_\beta \in M\}.$$

Since $Z(M) = Z(\alpha_1 z_2 - \alpha_2 z_1)$ and $f_\beta \in M, Z(f_\beta) \supseteq Z(\alpha_1 z_2 - \alpha_2 z_1)$. By Lemma 3,

$$f_\beta = (\alpha_1 z_2 - \alpha_2 z_1)^{\ell(\beta)} h_\beta$$

where $h_\beta \in N(D^2)$, $h_\beta(\alpha_1\lambda, \alpha_2\lambda) \not\equiv 0$ on D for each β and $\ell(\beta)$ is a positive integer. Since $Z(f_\beta) \supseteq Z(h_\beta)$, $Z(M) \supseteq \bigcap_\beta Z(h_\beta)$. If $\bigcap_\beta Z(h_\beta)$ is not discrete, then $h_\beta(\alpha_1\lambda, \alpha_2\lambda) \equiv 0$ on D because $Z(h_\beta) \supseteq \bigcap_\beta Z(h_\beta)$ and $Z(M) = Z(\alpha_1 z_2 - \alpha_2 z_1)$.

Suppose $\phi \in \mathcal{M}(M)$, then by definition $\phi f_\beta = (\alpha_1 z_2 - \alpha_2 z_1)^{\ell(\beta)} \phi h_\beta$ belongs to $H^2(D^2)$. Hence $(\alpha_1 z_2 - \alpha_2 z_1)^{\ell(\beta)}\phi$ is analytic on $D^2 \setminus Z(h_\beta)$ and $\ell(\beta) \leq \ell$. Since $\bigcap_\beta Z(h_\beta)$ is discrete, $\psi = (\alpha_1 z_2 - \alpha_2 z_1)^\ell \phi$ is analytic on D^2. For a nonzero function f in $M, \psi f = (\alpha_1 z_2 - \alpha_2 z_1)^\ell \phi f \in H^2(D^2)$. By the proof of [1, Theorem 1], $\psi \in N(D^2) \cap L^\infty(T^2)$ and by Lemma 6 $d\sigma_\psi \leq Z_\partial(M) = 0$. By [1, Proposition 2] ψ belongs to $H^\infty(D^2)$. Thus, since $F = (\alpha_1 z_2 - \alpha_2 z_1)^\ell$ is weakly outer and $\phi \in \mathcal{M}(M_F)$, ϕ belongs to $H^\infty(D^2)$ because $\mathcal{M}(M_F) = H^\infty(D^2)$ by Theorem 4.

In the previous paper, (2) of Theorem 4 was proved under the condition $\ell = 1$. When $Z(M) = \bigcap_{j=1}^N \{Z(f_j) : f_j \in M\}$ and $N < \infty$, it is clear that there exists a function f in M such that $1 \leq O(f,p) \leq \ell$ for each p in $Z(M)$.

§5. $\mathcal{M}(M) = H^\infty(D^n)$.

When F is a nonzero function in $H^2(D^n)$, it is interesting to study the set of multipliers of an invariant subspace M of $H^2(D^n)$ whose common zero set of M is just a zero set of F. In Section 3 and 4, we studied such a problem in very special cases. In Theorem 2, it was studied when M has a single generator. In (2) of Theorem 4, it was studied when M is finitely generated, $n = 2$ and F is a Weierstrass polynomial of degree 1 such that $F(z) = \alpha_2 z_1 - \alpha_1 z_2$. In this section, we are interested in when F is an arbitrary Weierstrass polynomial.

Theorem 5. *Suppose F is a Weierstrass polynomial of degree ℓ and M is an invariant subspace of $H^2(D^n)$ such that $Z(M) = Z(F)$ and $Z_\partial(M) = 0$. If for each p in $Z(M)$, there exists a function f in M such that $f_p(z_1, 0')$ has a zero of order ℓ at $z_1 = 0$, then the following (1) and (2) are true.*

(1) $F\mathcal{M}(M) \subseteq H^\infty(D^n)$.

(2) If $\mathcal{M}(M_F) = H^\infty(D^n)$, then $\mathcal{M}(M) = H^\infty(D^n)$.

Proof. It is necessary to show only (1). Suppose $F(z) = z_1^\ell + a_{\ell-1}(z_1')z_1^{\ell-1} + \cdots + a_1(z_1')z_1 + a_0(z_1')$ is a Weierstrass polynomial, then for each $z_1' \in D^{n-1}$,

$$F(z) = \prod_{j=1}^{\ell}(z_1 - \alpha_j(z_1')).$$

Let Δ_1 an Δ_1' be polydiscs in $\mathbf{C}$ and $\mathbf{C}^{n-1}$, respectively such that $\Delta = \Delta_1 \times \Delta_1'$. Suppose p is arbitrary point in $Z(M)$ and f is a function in M such that $f_p(z_1, 0, \cdots, 0)$ has a zero of order ℓ at $z_1 = 0$, by the Weierstrass preparation theorem, there exists a polydisc Δ in $\mathbf{C}^n$, center at $(0, \cdots, 0)$, such that $f_p(z) = W(z)h(z)$ for $z \in \Delta$, where h is analytic in Δ, h has no zero in Δ,

$$W(z_1, z_1') = z_1^\ell + b_{\ell-1}(z_1')z_1^{\ell-1} + \cdots + b_1(z_1')z_1 + b_0(z_1')$$

where $z = (z_1, z_1')$, $\{b_j\}_{j=0}^{\ell-1}$ are analytic on Δ' and $b_j(0, \cdots, 0) = 0$ for $0 \le j \le \ell - 1$. Since $F(p) = 0$, we may assume that $p = (p_1, p_1')$ and $p_1 = \alpha_1(p_1')$. Let $\beta_1(z_1'), \cdots, \beta_\ell(z_1')$ be the zeros of $f_p(\cdot, z_1')$ in Δ', counted according to multiplicities. Then $W(z) = \prod_{j=1}^{\ell}(z_1 - \beta_j(z_1'))$ $(z \in \mathbf{C} \times \Delta')$ (*see*[10, *p.*11]). If $z_1 + p_1 - \alpha_j(z_1' + p_1') = 0$ and $z \in \Delta$, then $f(z + p) = 0$. Hence we can assume that $\beta_j(z_1') = \alpha_j(z_1' + p_1') - p_1$. Thus $W(z) = F(z + p)$ on Δ because $O(W, 0) = O(f, p) = O(F, p)$. Suppose $\phi \in \mathcal{M}(M)$, then ϕf belongs to $H^2(D^n)$ and so

$$\phi(z + p)f(z + p) = \phi(z + p)F(z + p)h(z)$$

on $\tilde{\Delta}_p$ by what was just proved. Hence ϕF is analytic on $\tilde{\Delta}_p + p$. Since p is arbitrary point in $Z(M)$, ϕF is analytic on D^n. ϕF belongs to $\mathcal{M}(M)$ because $F \in H^\infty(D^n)$ and $\phi F \in N(D^n) \cap L^\infty(T^n)$. Now Lemma 6 and $Z_\partial(M) = 0$ imply (1).

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