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ON ALGEBRAICALLY TOTAL $*$–PARANORMALITY

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ABSTRACT. In this paper, we introduce the notion of algebraically $*$-TPN operators on a Hilbert space H as : An operator T is algebraically $*$-TPN if there exists a nonconstant complex polynomial p such that $p(T)$ is totally $*$-paranormal. In particular, we prove that this class of $*$-TPN (or equivalently, totally $*$-paranormal) operators forms a proper subclass of algebraically $*$-TPN operators. Also we prove that Weyl's theorem and the spectral mapping theorem hold for algebraically $* - TPN$ operators. Finally, we prove that if T is algebraically $*$-TPN, then $f(T)$ satisfies Weyl's theorem where f is analytic on an open neighborhood of $\sigma(T)$.

0. Introduction

Let H be an infinite dimensional complex Hilbert space and $\mathcal{L}(H)$ denote the space of all bounded linear operators from H to H. If $T \in \mathcal{L}(H)$, we write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T)$ for the spectrum of T and $\sigma_e(T)$ for the essential spectrum of T. Recall that an operator $T \in L(H)$ is *Fredholm* if its range $R(T)$ is closed and both the null spaces $N(T)$ and $N(T^*)$ are finite dimensional. The *index* of a Fredholm operator T, denoted by ind(T), is defined by

$$\text{ind}(T) = \dim N(T) - \dim N(T^*)(= \dim N(T) - \dim R(T)^{\perp}).$$

An operator $T \in \mathcal{L}(H)$ is called Weyl if T is a Fredholm operator of index zero. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by the formula

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

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Note that for any operator T, $\omega(T)$ is a nonempty compact subset of $\mathbb{C}$ ([3],[4]). We say that Weyl's theorem holds for T if

$$\sigma(T)\backslash\omega(T) = \pi_{00}(T)$$

where $\pi_{00}(T)$ denotes the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. An operator $T \in \mathcal{L}(H)$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T([3], [10]). An operator is called Browder if it is Fredholm of finite ascent and descent([6]).

H. Weyl examined the spectra of all compact perturbations $T + K$ of a single hermitian operator T and discovered that $\lambda \in \sigma(T + K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. Today this result is known as Weyl's theorem, and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [4], to several classes of operators including hyponormal operators by S. Berberian [2], [3].

In this paper, we introduce the notion of algebraically $*$-TPN operators on a Hilbert space H as follows: An operator T is algebraically $*$-TPN if there exists a nonconstant complex polynomial p such that $p(T)$ is totally $*$-paranormal. In particular, we prove that this class of $*$-TPN (or equivalently, totally $*$-paranormal) operators forms a proper subclass of algebraically $*$-TPN operators. Also we prove that Weyl's theorem and the spectral mapping theorem hold for algebraically $*$-TPN operators. Finally, we prove that if T is algebraically $*$-TPN, then $f(T)$ satisfies Weyl's theorem where f is analytic on an open neighborhood of $\sigma(T)$.

1. Weyl's theorem and Spectral mapping theorem

Recall that an operator $T \in \mathcal{L}(H)$ is said to be hyponormal if $TT^* \leq T^*T$, or equivalently, $\|T^*h\| \leq \|Th\|$ for every $h \in H$. A larger class of operators related to hyponormal operators is the following: An operator T is $*$-paranormal if $\|T^*h\|^2 \leq \|T^2h\|\|h\|$ for every $h \in H$. It is known in [1] that T is $*$-paranormal if and only if $T^{*2}T^2 - 2rTT^* + r^2 \geq 0$ for each $r > 0$. This class of operators was introduced and studied by S. M. Patel(cf. [1]) under the title 'Operators of class (M)'. An operator $T \in \mathcal{L}(H)$ is called totally $*$-paranormal(or shortly, $*$-TPN) if $T - \lambda I$ is $*$-paranormal for every

$\lambda \in \mathbb{C}$, or equivalently, $\|(T-\lambda I)^*h\|^2 \leq \|(T-\lambda I)^2h\|\|h\|$ for all $h \in H$ and all $\lambda \in \mathbb{C}$([8]). It was known ([8]) that this class forms a proper subclass of the $*$-paranormal operators and that the class of hyponormal operators forms a proper subclass of totally *-paranormal operators.

The following facts ([8]) follow from the above definition and the well-known facts of $*$-paranormal operators.

(a) If $T \in \mathcal{L}(H)$ is $*$-TPN, then so is $T-\lambda I$ for each $\lambda \in \mathbb{C}$.

(b) If $T \in \mathcal{L}(H)$ is $*$-TPN and $M \subseteq H$ is invariant under T, then $T|_M$ is $*$-TPN.

(c) If $T \in \mathcal{L}(H)$ is $*$-TPN and quasinilpotent, then T is zero.

(d) Let T be a weighted shift with weighs $\{\alpha_n\}_{n=0}^{\infty}$. If T is $*$-TPN, then $|\alpha_{n-1}|^2 \leq 2|\alpha_n|^2$ for each positive integer n.

We shall introduce the notion of an algebraically $*$-TPN operator:

Definition. An operator $T \in B(H)$ is called algebraically $*$-TPN if there exists a nonconstant complex polynomial p such that $p(T)$ is totally $*$-paranormal.

Evidently, $*$-$TPN \subseteq$ algebraically $*$-TPN, and the following example provides us with the class of $*$-TPN operators as the proper subclass of algebraically $*$-TPN.

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for l_2, let $\{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of nonnegative numbers and let W_α be the (unilateral) weighted shift with the weights $\alpha = \{\alpha_n\}$ defined by

$$W_\alpha e_n = \alpha_n e_{n+1} \quad (n \geq 0).$$

It is well-known that W_α is hyponormal if and only if the weight sequence $\{\alpha_n\}$ is monotonically increasing. A straightforward calculation shows that W_α^p is hyponormal for $p \in \mathbb{N}$ if and only if the weight sequence $\{\alpha_n\}$ satisfies that for each $m = 0, 1, \cdots, p-1$,

$$\prod_{j=m}^{p+m-1} \alpha_j \leq \prod_{j=p+m}^{2p+m-1} \alpha_j \leq \prod_{j=2p+m}^{3p+m-1} \alpha_j \leq \cdots \tag{2.1}$$

Example. Let $\alpha_0 = 1, \alpha_1 = \frac{1}{2}$, and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \cdots = 1$. Then W_α is not hyponormal since $\{\alpha_n\}$ is not monotonically increasing. Since

$|\alpha_0|^2 = 1 > 2|\alpha_1|^2 = \frac{1}{2}$, by the above remark (d), W_α is not $*$-TPN. But W_α^2 is hyponormal and so W_α^2 is $*$-TPN i.e., W_α is algebraically $*$-TPN. Hence the set of all totally $*$-paranormal operators is a proper subset of the set of all algebraically $*$-TPN operators.

The following facts follow from the above definition and the well-known facts of $*$-TPN operators.

(a) If $T \in B(H)$ is algebraically $*$-TPN and $M \subseteq H$ is invariant under T, then $T|_M$ is algebraically $*$-TPN.

(b) Unitary equivalence preserves algebraic $*-$TPN.

LEMMA 1. *If $T \in \mathcal{L}(H)$ is algebraically $*$-TPN and quasinilpotent, then T is nilpotent.*

Proof. Suppose $p(T)$ is $*$-TPN for some nonconstant polynomial p. Since total $*$-paranormality is translation-invariant, we may assume $p(0) = 0$. Thus we can write $p(\lambda) \equiv a_0\lambda^m(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$ ($m \neq 0, \lambda_i \neq 0$ for every $1 \leq i \leq n$). If T is quasinilpotent, then $\sigma(p(T)) = p(\sigma(T)) = p(\{0\}) = \{0\}$, so that $p(T)$ is also quasinilpotent. Since the only $*$-paranormal quasinilpotent operator is zero, it follows that

$$a_0T^m(T - \lambda_1 I)\cdots(T - \lambda_n I) = 0.$$

Since $T - \lambda_i I$ is invertible for every $1 \leq i \leq n$, we have $T^m = 0$. □

Note that if T is $*$-paranormal, then $N(T) = N(T^2)$.

LEMMA 2. *If T is algebraically $*$-TPN, then T has finite ascent.*

Proof. Suppose $p(T)$ is $*$-TPN for some nonconstant polynomial p. We may assume $p(0) = 0$. If $p(\lambda) \equiv a_0\lambda^m$, then $N(T^m) = N(T^{2m})$ because $*$-paranormal operators are of ascent 1. Thus we write

$$p(\lambda) \equiv a_0\lambda^m(\lambda - \lambda_1)\cdots(\lambda - \lambda_n) \quad (m \neq 0, \lambda_i \neq 0 \quad \text{for} \quad 1 \leq i \leq n).$$

We then claim that

$$N(T^m) = N(T^{m+1}) \tag{2.2}$$

To show (2.2), let $x(\neq 0) \in N(T^{m+1})$. Then we can write

$$p(T)x = (-1)^n a_0\lambda_1\cdots\lambda_n T^m x$$

Thus we have

$$\begin{aligned} \mid a_0\lambda_1\cdots\lambda_n\mid^2 \|T^m x\|^2 &= (p(T)x, p(T)x) = (p(T)^*p(T)x, x) \\ &\le \|p(T)^*p(T)x\|\|x\| \\ &\le \left(\|p(T)^3x\|\|p(T)x\|\right)^{1/2}\|x\| \\ &= 0 \end{aligned}$$

since $\|p(T)^3x\| = \|a_0^3(T-\lambda_1 I)^3\cdots(T-\lambda_n I)^3 T^{3m}x\| = 0$. Hence $x \in N(T^m)$ and so $N(T^{m+1}) \subseteq N(T^m)$. Also the reverse inclusion is evident. This completes the proof. □

LEMMA 3. *If $T \in B(H)$ is algebraically $*$-TPN, then T is isoloid.*

Proof. Suppose $p(T)$ is $*$-TPN for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso}\sigma(T)$. Then using the spectral decomposition, we can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$. Note that $T_1 - \lambda I$ is also algebraically $*$-TPN . Since $T_1 - \lambda I$ is quasinilpotent, by Lemma 1, $T_1 - \lambda I$ is nilpotent. Therefore $\lambda \in \pi_0(T_1)$ and hence $\lambda \in \pi_0(T)$. This shows that T is isoloid. □

THEOREM 4. *Weyl's theorem holds for every algebraically $*$-TPN operator.*

Proof. Suppose $p(T)$ is $*$-TPN for some nonconstant polynomial p. We first prove that $\pi_{00}(T) \subseteq \sigma(T)\backslash\omega(T)$. Since algebraically $*$-TPN operator is translation-invariant, it suffices to show that

$$0 \in \pi_{00}(T) \Longrightarrow T \text{ is Weyl but not invertible.}$$

Suppose $0 \in \pi_{00}(T)$. Now using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial B_0}(\lambda I - T)^{-1}d\lambda$, where B_0 is an open disk of center 0 which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \quad \text{where} \quad \sigma(T_1) = \{0\} \quad \text{and} \quad \sigma(T_2) = \sigma(T)\backslash\{0\}.$$

But then $T_1 (= T|_M = T|_{\operatorname{Im} P})$ is also algebraically $*$-TPN and quasinilpotent. Then by Lemma 1, T_1 is nilpotent. Thus we should have that $\dim R(P) < \infty$: if it were not so then $N(T_1)$ would be infinite dimensional so that

$0 \notin \pi_{00}(T)$, giving a contradiction. Therefore $T_1 = T|R(P)$ is a finite dimensional operator. Since finite dimensional operators are always Weyl, it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Therefore $\pi_{00}(T) \subseteq \sigma(T)\backslash\omega(T)$

For the reverse inclusion, suppose $\lambda \in \sigma(T)\backslash\omega(T)$. Thus $T - \lambda I$ is Weyl. Then by the "index product theorem",

$$\begin{aligned}\dim N((T-\lambda I)^n) - \dim R((T-\lambda I)^n)^{\perp} &= \operatorname{ind}((T-\lambda I)^n)\\ &= n\ \operatorname{ind}(T-\lambda I) = 0\end{aligned}$$

Thus if $\dim N((T-\lambda I)^n)$ is a constant, then so is $\dim R((T-\lambda I)^n)^{\perp}$ since $T-\lambda I$ is Fredholm. Consequently finite ascent forces finite descent. Therefore by Lemma 2, $T - \lambda I$ is Weyl of finite ascent and descent, and thus it is Browder. Therefore $\lambda \in \pi_{00}(T)$. This completes the proof. □

It was known that for hyponormal operators, the Weyl spectrum obeys the spectral mapping theorem.

THEOREM 5. *If $T \in \mathcal{L}(H)$ is algebraically $*$-TPN, then for every $f \in H(\sigma(T))$*

$$\omega(f(T)) = f(\omega(T)) \tag{2.3}$$

where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$.

Proof. First of all we prove the equality (2.3) when P is a polynomial. In view of ([7], Theorem 5]), it suffices to show that for each pair $\lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$,

$$\operatorname{ind}(T-\lambda I)\ \operatorname{ind}(T-\mu I) \geq 0 \tag{2.4}$$

By Lemma 2, $T - \lambda I$ has finite ascent for every $\lambda \in \mathbb{C}$. Observe that if $S \in \mathcal{L}(H)$ is Fredholm of finite ascent then $\operatorname{ind}(S) \leq 0$: Indeed, either if S has finite descent then S is Browder and hence $\operatorname{ind}(S) = 0$, or if S does not have finite descent then

$$n\ \operatorname{ind}(S) = \dim N(S^n) - \dim R(S^n)^{\perp} \longrightarrow -\infty \quad \text{as} \quad n \to \infty$$

which implies that $\operatorname{ind}(S) < 0$. Thus we can see that (2.4) holds for every algebraically $*$-TPN operator T. This proves that the equality $\omega(p(T)) = p(\omega(T))$ holds for every polynomial p.

If f is analytic on an open neighborhood of $\sigma(T)$, then, by Runge's theorem, there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [9], we have

$$\omega(f(T)) = \lim \omega(p_n(T)) = \lim p_n(\omega(T)) = f(\omega(T)).$$

□

COROLLARY 6. *If $T \in B(H)$ is algebraically $*$-TPN, then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$.*

Proof. Recall that if T is isoloid then

$$f(\sigma(T)\backslash\pi_{00}(T)) = \sigma(f(T))\backslash\pi_{00}(f(T)) \quad \text{for every} \quad f \in H(\sigma(T)).$$

Thus from Lemma 3, Theorem 4 and Theorem 5,

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T)) = f(\omega(T) = \omega(f(T))$$

which implies that Weyl's theorem holds for $f(T)$. □

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