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Real hypersurfaces in complex projective space satisfying a certain condition on Ricci tensors

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1 Introduction

Let $CP^n, n \geq 3$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface CP^n. Let ν be a unit local normal vector field on M and $\xi = -J\nu$, where J denotes the complex structure of CP^n. M has an almost contact metric structure (ϕ, ξ, η, g) induced from J. We denote R and S the curvature tensor and the Ricci tensor of M, respectively. Many differential geometeres have studied M (cf. [1], [5], [6], [8], [9], [10], [11] and [12]) by using the structure (ϕ, ξ, η, g).

Typical examples of real hypersurfaces in CP^n are homogeneous ones. Takagi [12] showed that all homogeneous real hypersurfaces in CP^n are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Namely, he showed the following: Let M be a homogeneous real hypersurface of CP^n. Then M is a tube of radius r over one of the following Kaehler submanifolds:
(A_1) hyperplane CP^{n-1}, where $0 < r < \frac{\pi}{2}$,
(A_2) totally geodesic $CP^k (1 \leq k \leq n-2)$,
(B) complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,
(C) $CP^1 \times CP^{\frac{n-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n (\geq 5)$ is odd,
(D) complex Grassmann $CG_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
(E) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.
Due to his classification, we find that the number of distinct constant principal curvatures of a homogeneous real hypersurface is 2, 3 or 5. Here note that the vector ξ of any homogeneous real hypersurface M (which is a tube of radius r) is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ with multiplicity 1 (See [1]) and that in the case of type A_1 M has two distinct principal curvatures and in the case of type A_2 (resp. B) M has three distinct principal curvatures $t, -\frac{1}{t}$ and $\alpha = t - \frac{1}{t}$ (resp. $\frac{1+t}{1-t}, \frac{t-1}{t+1}$ and $\alpha = t - \frac{1}{t}$).

The following result is well known ([3]) : There are no real hypersurfaces M with parallel Ricci tensor in $CP^n, n \geq 3$. Moreover, $CP^n, n \geq 3$, does not admit real hypersurfaces M with the weak condition $(R(X,Y)S)Z = 0$ for any $X, Y, Z \in TM$. With relation to this Gotoh [2] proved that if $n \geq 3$ and the shape operator A of a real hypersurface M satisfies $(R(X,Y)A)Z = 0$ for any $X, Y, Z \in \xi^{\perp}$, then M is locally congruent to a geodesic hypersphere.

The purpose of the present paper is to study more weaker condition

$$(R(X,Y)S)Z = 0 \text{ for any } X, Y, Z \in \xi^{\perp},$$

where $\xi^{\perp}$ denotes the orthogonal complement of ξ in TM. Specifically, we shall prove the following :

Theorem *Let M be a real hypersurface of $CP^n, n \geq 3$. Then M satisfies $(R(X,Y)S)Z = 0$ for any X, Y and Z in $\xi^{\perp}$ if and only if M is locally congruent to one of the following:*
(i) *a geodesic hypersurface,*
(ii) *a tube of radius $\frac{\pi}{4}$ over a totally geodesic $CP^{\frac{n-1}{2}}$,*
(iii) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = n - 2$.*

2 Preliminaries.

Let X be a tangent vector field on M. We write $JX = \phi X + \eta(X)\nu$, where ϕX is the tangent component of JX and $\eta(X) = g(X, \xi)$. As $J^2 = -Id$, where Id denotes the identity endomorphism on TCP^n, we get

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \quad \phi\xi = 0 \tag{1}$$

for any X tangent to M. It is also easy to see that for any X and Y tangent to M

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2}$$
$$\nabla_X \xi = \phi AX. \tag{3}$$

Finally, from the expression of the curvature tensor of CP^n, we see that the curvature tensor, Codazzi equation and the Ricci tensor of M are given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{4}$$
$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi, \tag{5}$$
$$SX = (2n+1)X - 3\eta(X)\xi + hAX - A^2 X, \tag{6}$$

where $h = \text{trace}A$ and S is the Ricci tensor of type (1.1) on M.

Now, we recall without proof the following result in order to prove our theorem:

Proposition (Maeda [7]) *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = rX$ for $X \perp \xi$, then we have $A\phi X = ((\alpha r + 2)/(2r - \alpha))\phi X$.*

3 Key lemma

Let M be a real hypersurface in $CP^n, n \geq 3$, whose the Ricci tensor S satisfies the identity $SX = aX + b\eta(X)\xi$ for some smooth functions a and b on M (, that is, M is η-Einstein). Then M is locally congruent to one of the following ([1], [5], [6]) :
(i) a geodesic hypersurface
(ii) a tube of radius r over a totally geodesic CP^k, $1 \leq k \leq n-2$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \dfrac{k}{n-k-1}$,
(iii) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = n-2$.
On the other hand, in [3] Ki, Nakagawa and Suh proved that $n \geq 3$ and the Ricci tensor S of a real hypersurface M satisfies $(R(X,Y)S)Z + (R(Y,Z)S)X + (R(Z,X)S)Y = 0$ for all tangent vectors X, Y and Z in TM if and only if M is η-Einstein.

Lemma *Let M be a real hypersurface of $CP^n, n \geq 3$. Then M satisfies*

$$(R(X,Y)S)Z + (R(Y,Z)S)X + (R(Z,X)S)Y = 0 \tag{7}$$

for any X, Y and Z in $\xi^\perp$ if and only if M is η-Einstein

From (4) and (6) we see that the assumption (7) of Lemma is equivalent to the equation

$$\begin{aligned} &g(\phi X, HZ)\phi Y + g(\phi Y, HX)\phi Z + g(\phi Z, HY)\phi X \\ &-g(\phi Y, HZ)\phi X - g(\phi Z, HX)\phi Y - g(\phi X, HY)\phi Z \\ &+2g(\phi X, Y)\phi HZ + 2g(\phi Y, Z)\phi HX + 2g(\phi Z, X)\phi HY = 0, \end{aligned} \tag{8}$$

where $H = hA - A^2, X, Y$ and $Z \in \xi^\perp$. Let $\{E_1, \dots, E_{2n-2}\}$ be an orthonormal basis of $\xi^\perp$ at any point of M. Putting $Z = E_i$ and taking the inner product

(8) by ϕE_i, we get

$$(9) \quad \begin{aligned} &g(\phi X, HE_i)g(\phi Y, \phi E_i) + g(\phi Y, HX)g(\phi E_i, \phi E_i) \\ &+g(\phi E_i, HY)g(\phi X, \phi E_i) - g(\phi Y, HE_i)g(\phi X, \phi E_i) \\ &-g(\phi E_i, HX)g(\phi Y, \phi E_i) - g(\phi X, HY)g(\phi E_i, \phi E_i) \\ &+2g(\phi X, Y)g(\phi HE_i, \phi E_i) + 2g(\phi Y, E_i)g(\phi HX, \phi E_i) \\ &+2g(\phi E_i, X)g(\phi HY, \phi E_i) = 0. \end{aligned}$$

Taking summation of (9) on i, we obtain

$$-(2n-2)g(\phi HX, Y) - (2n-2)g(H\phi X, Y) + 2(\text{trace}H - g(H\xi, \xi))g(\phi X, Y) = 0.$$

Hence we have

(10)
$$-(2n-2)\phi HX - (2n-2)H\phi X + 2(\text{trace}H - g(H\xi, \xi))\phi X = -(2n-2)g(H\phi X, \xi)\xi$$

If in (8) we take $X = E_j, Z = \phi E_j$ and take summation on j, we obtain for $Y \in \xi^\perp$

$$(11) \quad (-4n+6)\phi HY - 2H\phi Y + 2(\text{trace}H - g(H\xi, \xi))\phi Y = -2g(H\phi Y, \xi)\xi,$$

Taking the inner product (11) by any $X \in \xi^\perp$, we have

$$(12) \quad g((-4n+6)\phi HY - 2H\phi Y + 2(\text{trace}H - g(H\xi, \xi))\phi Y, X) = 0.$$

The equation (12) yields

$$g(Y, (4n-6)H\phi X + 2\phi HX - 2(\text{trace}H - g(H\xi, \xi))\phi X) = 0.$$

Then we can put

$$(4n-6)H\phi X + 2\phi HX - 2(\text{trace}H - g(H\xi, \xi))\phi X = c\xi$$

for some function c. Hence we get

(13)
$$(4n-6)H\phi Y + 2\phi HY - 2(\text{trace}H - g(H\xi, \xi))\phi Y = (4n-6)g(H\phi Y, \xi)\xi.$$

Combining (11) with (13) , we have

$$(14) \quad H\phi Y - \phi HY = g(H\phi Y, \xi)\xi,$$

since $n \geq 3$.
Now, define a unit cross section U of $\xi^\perp$ and smooth functions α, β on M by

$$(15) \quad H\xi = \alpha\xi + \beta U.$$

Hence we have

$$g(H\phi U,\xi)=g(\phi U,H\xi)=g(\phi U,\alpha\xi+\beta U)=0. \tag{16}$$

Thus from (14) and (16) we get

$$H\phi U=\phi HU. \tag{17}$$

Taking the inner product (8) by U, we have

$$\begin{aligned} & g(\phi X,HZ)g(\phi Y,U)+g(\phi Y,HX)g(\phi Z,U) \\ & +g(\phi Z,HY)g(\phi X,U)-g(\phi Y,HZ)g(\phi X,U) \\ & -g(\phi Z,HX)g(\phi Y,U)-g(\phi X,HY)g(\phi Z,U) \\ & +2g(\phi X,Y)g(\phi HZ,U)+2g(\phi Y,Z)g(\phi HX,U) \\ & +2g(\phi Z,X)g(\phi HY,U)=0. \end{aligned} \tag{18}$$

Taking the contraction of (18) on Z, we obtain

$$\begin{aligned} & g(\phi Y,U)H\phi X-g(\phi Y,HX)\phi U \\ & -g(\phi X,U)\phi HY-g(\phi X,U)H\phi Y \\ & +g(\phi Y,U)\phi HX+g(\phi X,HY)\phi U \\ & -2g(\phi X,Y)H\phi U+2g(\phi HX,U)\phi Y \\ & -2g(\phi HY,U)\phi X \\ & =g(g(\phi Y,U)H\phi X-g(\phi X,U)H\phi Y,\xi)\xi. \end{aligned} \tag{19}$$

Putting $X=U$ and $Y=\phi U$ in (19) , we get

$$H\phi U=g(U,HU)\phi U. \tag{20}$$

Assume that $\beta\neq 0$ at a point, say x. By (20) there exists a certain real number λ such that

$$H\phi U=\lambda\phi U. \tag{21}$$

Combining (17) with (20) , (21) we have

$$HU=\lambda U+\beta\xi. \tag{22}$$

On the other hand, (10) implies

$$2(2n-2)HU-2(\text{trace}H-\alpha)U=2(2n-2)\beta\xi.$$

Hence we obtain

$$(2n-2)\lambda=\text{trace}H-\alpha. \tag{23}$$

Putting $X = \phi Y$ in (19) , from (14) we obtain $\lambda = 0$. Combining $H = hA - A^2$ with (14) , (15) and (22) , we get

$$\beta = 0. \tag{24}$$

The equation (24) contradicts to $\beta \neq 0$. Therefore we get

$$H\xi = \alpha\xi \tag{25}$$

Also, from (10) , (14) , (21) , (23) and (25) we see that

$$HX = \lambda X \quad \text{for} \quad X \in \xi^{\perp}.$$

Therefore M is η-Einstein. Conversely, assume that M is η-Einstein. Then the result of Ki, Nakagawa and Suh ([3]) gurantees (7) . This proves Lemma.

4 Proof of Theorem

From the assumption of Theorem, i.e.,

$$(R(X,Y)S)Z = 0 \tag{26}$$

for any X, Y and $Z \in \xi^{\perp}$, we see that M satisfies (7) . Hence M is η-Einstein. Conversely, assume that M is η-Einstein. By (4) and (6) , we know that (26) is equal to the equation

$$\begin{aligned} g(Y,HZ)X - g(X,HZ)Y + g(\phi Y,HZ)\phi X - g(\phi X,HZ)\phi Y \\ -2g(\phi X,Y)\phi HZ + g(AY,HZ)AX - g(AX,HZ)AY \\ -g(Y,Z)HX + g(X,Z)HY - g(\phi Y,Z)H\phi X + g(\phi X,Z)H\phi Y \\ +2g(\phi X,Y)H\phi Z - g(AY,Z)HAX + g(AX,Z)HAY = 0. \end{aligned} \tag{27}$$

We shall determine η-Einstein real hypersurfaces M satisfying (27) .
Let M be of type A_1 (which is a tube of radius r). Let $t = \cot r$. Then the shape operator A of M is expressed as ([13]) :

$$AX = tX \tag{28}$$

for any $X \in \xi^{\perp}$. Substituting (28) into the left side of (27) , we get (26) .
Let M be of type A_2 (which is a tube of radius r, where $\cot^2 r = \frac{k}{n-k-1}, 0 < k < n-1$ and $0 < r < \frac{\pi}{2}$). Let $t = \cot r$. Then M has three distinct constant principal curvatures t with multiplicity $2k$, $-\frac{1}{t}$ with multiplicity $2n-2k-2$ and $t - \frac{1}{t}$ with multiplicity 1 ([13]). Let $X \in V_t, Y$ and $Z \in V_{-\frac{1}{t}}$, where V_t denote the eigenspace of A corresponding to the eigenvalue t. Then by Proposition we

know that $\phi X \in V_t, \phi Y$ and $\phi Z \in V_{-\frac{1}{t}}$. We put $Z = \phi Y$ and $|Y| = 1$ in (27) . Then the linear independence of the vectors $X, \phi X, Y$ and ϕY shows

$$\frac{(t^2+1)(t^2-ht-1)}{t^2}\phi X = 0,$$

i.e.,

(29) $$t^2 - ht - 1 = 0.$$

We now remark

(30) $$h = 2kt - \frac{2n-2k-2}{t} + t - \frac{1}{t}.$$

Substituting (30) into the left side of (29) , we have

$$t^2 = \frac{n-k-1}{k}.$$

Since M is η-Einstein, $t^2 = \frac{k}{n-k-1}$ (See Key lemma). Thus $t^2 = 1$. Hence M has three distinct constant principal curvatures ± 1 with multiplicity $n-1$ and 0 with multiplicity 1. This implies that $h = 0$ and $HX = -X$ for any $X \in \xi^{\perp}$. By easy computation we can verify the equation (27) for any $X, Y, Z \in \xi^{\perp}$, so that the manifold M of type A_2 of radius $\frac{\pi}{4}$ satisfies (26) .
Let M be of type B (which is a tube of radius r, where $\cot^2 2r = n-2$ and $0 < r < \frac{\pi}{4}$). Let $t = \cot r = \sqrt{n-1} + \sqrt{n-2}$. Then M has three distinct constant principal curvatures $r_1 = \frac{1+t}{1-t}$ with multiplicity $n-1$, $r_2 = \frac{t-1}{t+1}$ with multiplicity $n-1$ and $\alpha = t - \frac{1}{t}$ with multiplicity 1 ([13]). Note that the following :

$$r_1 + r_2 = -\frac{4}{\alpha},\ r_1 r_2 = -1,$$
$$h = \alpha - \frac{4(n-1)}{\alpha},$$
$$\alpha = 2\sqrt{n-2}.$$

Let $X \in V_{r_1}, Y$ and $Z \in V_{r_2}$. Then by Proposition we see that $\phi X \in V_{r_2}, \phi Y$ and $\phi Z \in V_{r_1}$. Putting $X = \phi Y, |X| = 1$ and $g(X, \phi Z) = 0$, we get

$$\begin{aligned}(R(X,Y)S)Z &= -2(r_1 - r_2)(h - r_1 - r_2)\phi Z\\ &= -2(r_1 - r_2)\frac{\alpha^2 - 4(n-2)}{\alpha}\phi Z.\end{aligned}$$

Moreover, we have

$(R(X,Y)S)Z = 0$ for $X, Y, Z \in V_{r_1}$ or $X, Y, Z \in V_{r_2}$,
$(R(X,Y)S)Z = 0$ for $X, Z \in V_{r_1}, Y(\perp \phi X, \phi Z) \in V_{r_2}$ or $X, Z \in V_{r_2}, Y(\perp \phi X, \phi Z) \in$

A similar computation asserts that the real hypersurface M of tye B in the case of $\alpha = 2\sqrt{n-2}$ satisfies (26) . This proves Theorem.

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