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SUBDIAGONAL ALGEBRAS IN NON-σ-FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}$ be a subdiagonal algebra of a von Neumann algebra $\mathcal{M}$, which is not σ-finite, with respect to a faithful normal expectation Φ. In this note we generalize some results of subdiagonal algebras in the σ-finite case to the non-σ-finite case. We prove that there is a unique maximal subdiagonal algebra $\mathfrak{A}_m$ with respect to Φ containing $\mathfrak{A}$. We show that if $\mathfrak{A}$ is maximal subdiagonal and φ is a faithful normal semi-finite weight on $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$, then $\mathfrak{A}$ is σ_t^{φ}-invariant ($\forall t \in \mathbb{R}$) , where $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ is the modular automorphism group associated with φ. As an application, we also give several characterizations of $\mathfrak{A}_m$.

1. INTRODUCTION

In [1], Arveson introduced the notion of subdiagonal algebras in a von Neumann algebra on a Hilbert space to study the analyticity in operator algebras. At first, we start by given the definition of subdiagonal algebras. Let $\mathcal{M}$ be a von Neumann algebra on a complex Hilbert space $\mathcal{H}$, and let Φ be a faithful normal positive idempotent linear map from $\mathcal{M}$ onto a von Neumann subalgebra $\mathfrak{D}$ of $\mathcal{M}$. A subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathfrak{D}$, is called a subdiagonal algebra in $\mathcal{M}$ with respect to Φ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{D}$,

(ii) Φ is multiplicative on $\mathfrak{A}$, and

(iii) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in $\mathcal{M}$.

The algebra $\mathfrak{D}$ is called the diagonal of $\mathfrak{A}$. Although subdiagonal algebras are not assumed to be σ-weakly closed in [1], the σ-weak closure of a subdiagonal algebra is again a subdiagonal algebra([1, Remark 2.1.2]). Thus we assume that our subdiagonal algebras are always σ-weakly closed. We say that $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to Φ in case $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to Φ. Put $\mathfrak{A}_0 = \{X \in \mathfrak{A} : \Phi(X) = 0\}$, and let $\mathfrak{A}_m$ be the set of all $A \in \mathcal{M}$ such that $\Phi(\mathfrak{A}A\mathfrak{A}_0) = \Phi(\mathfrak{A}_0A\mathfrak{A}) = 0$. Arveson has proved that $\mathfrak{A}_m$ is the unique maximal subdiagonal algebra with respect to Φ

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containing $\mathfrak{A}$ if $\mathcal{H}$ is separable. He also proved that if there is a faithful normal semi-finite trace τ on $\mathcal{M}$ satisfying $\tau \circ \Phi = \tau$, then $\mathfrak{A}_m = \{X \in \mathcal{M} : \Phi(X\mathfrak{A}_0) = 0\}$. The same result has been proved to be true for an arbitrary σ-finite von Neumann algebra in [3] by the author and Saito. In this note, we generalize some results on subdiagonal algebras in a σ-finite von Neumann algebra to the non-σ-finite case. We first prove that $\mathfrak{A}_m$ is also the unique maximal subdiagonal algebra with respect to Φ containing $\mathfrak{A}$. Let φ be a faithful normal semi-finite weight on $\mathcal{M}$ satisfying $\varphi \circ \Phi = \varphi$, we show that $\mathfrak{A}$ is invariant under the modular automorphism group $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ of φ if $\mathfrak{A}$ is maximal subdiagonal, which has been proved in [2] when φ is a faithful normal state by the author, Ohwada and Saito. As an application, we prove that $\mathfrak{A}_m = \{X \in \mathcal{M} : \Phi(X\mathfrak{A}_0) = 0\}$ in the non-σ-finite case.

2. Maximality for subdiagonal algebras

Let $\mathcal{M}$ be a non-σ-finite von Neumann algebra, and let $\mathfrak{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to Φ as defined in § 1. Then there always exists a faithful normal semi-finite weight φ on $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$. Put $\mathfrak{N}_\varphi = \{X \in \mathcal{M} : \varphi(X^*X) < \infty\}$, then $\mathfrak{N}_\varphi$ is a left ideal of $\mathcal{M}$ and σ-weakly dense in $\mathcal{M}$. Let $\mathcal{H}_\varphi$ be the Hilbert space associated with $\mathfrak{N}_\varphi$ with the scalar product

$$< a, b >_\varphi = \varphi(b^*a), \ \forall a, b \in \mathfrak{N}_\varphi.$$

Let $\pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\varphi)$ be the standard repersentation associated with φ(cf[5, Theorem 2.2]), then π_φ is a $*$-isomorphism. Since $\varphi \circ \Phi = \varphi$, we clearly have $\mathfrak{N}_\varphi \cap \mathfrak{D}$ is σ-weakly dense in $\mathfrak{D}$ and therefore there is an increasing net $\{u_i\}_{i\in\Lambda}$ in $\mathfrak{N}_\varphi \cap \mathfrak{D}_+$ such that $u_i \uparrow I$ (cf[5]). For $a \in \mathfrak{N}_\varphi$, by (2) in [5, $_p$20], we have

$$||a - u_i a||_\varphi^2 \to 0.$$

Moreover, for $\forall b \in \mathfrak{A}_0$, there is a net $\{b_i\}_{i\in\Lambda}$ in $\mathfrak{A}_0 \cap \mathfrak{N}_\varphi$ such that $b_i \to b$ σ-weakly. In fact, we have $bu_i \in \mathfrak{A}_0 \cap \mathfrak{N}_\varphi$ and $bu_i \to b$ σ-weakly. The same thing is true for $\mathfrak{A}_0^*$.

We define the closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ by $\mathcal{H}_1 = [\mathfrak{A}_0 \cap \mathfrak{N}_\varphi]$, $\mathcal{H}_2 = [\mathfrak{D} \cap \mathfrak{N}_\varphi]$ and $\mathcal{H}_3 = [\mathfrak{A}_0^* \cap \mathfrak{N}_\varphi]$ respectively, where $[S]$ is the closed linear span of a subset $\mathcal{S}$ of $\mathcal{H}_\varphi$.

Lemma 1. *Keep the notations as above. Then* $\mathcal{H}_\varphi = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$.

Proof. It is easy to check that $\{\mathcal{H}_i\}_{i=1}^3$is mutually orthogonal and $\mathcal{H}_k \subseteq \mathcal{H}_\varphi$ for k=1, 2, 3. Conversely, for $x \in \mathfrak{N}_\varphi \subseteq \mathcal{M}$, there are nets a_j and b_j in $\mathfrak{A}_0$ and d_j in $\mathfrak{D}$ such that $x_j = a_j + d_j + b_j^* \to x$ σ-strongly. It follows that $(x_j - x)^*(x_j - x) \to 0$ σ-weakly. By Proposition 1.14 in [5], we have

$$\|(x_j - x)u_i\|_\varphi^2 = \varphi(u_i^*(x_j - x)^*(x_j - x)u_i) \to 0$$

for every $i \in \Lambda$. Now $x_j u_i = a_j u_i + d_j u_i + b_j^* u_i$ and $a_j u_i \in \mathcal{H}_1$, $d_j u_i \in \mathcal{H}_2$ and $b_j^* u_i \in \mathcal{H}_3$ respectively. It follows that $x u_i \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ $(\forall i \in \Lambda)$. For every $i \in \Lambda$, we define

$$v_i = \sqrt{\frac{1}{\pi}} \int_{\mathbb{R}} e^{-t^2} \sigma_t(u_i) dt.$$

Then by Proposition 2.16 and Theorem 10.1 in [5], we have $v_i \in \mathfrak{T}_\varphi \cap \mathfrak{D}$, $v_i \to I$ σ-weakly and $\|\sigma_\alpha(v_i)\| \leqq \exp((\mathrm{Im}\alpha)^2)$ for every $\alpha \in \mathbb{C}$, where $\mathfrak{T}_\varphi$ $(\subset \mathfrak{N}_\varphi)$ is the Tomita algebra associated with φ (cf[4, 5]). Replacing u_i by v_i if necessary, we may assume that $u_i \in \mathfrak{T}_\varphi$. Thus by Propositions 2.14 and 2.16 in [5], we have $\|x u_i\| \leqq \exp(\frac{1}{4})\|x\|$, that is, $x u_i$ is a bounded net in $\mathcal{H}_\varphi$. Without loss of generality, we may assume that $x u_i \to \xi$ weakly in $\mathcal{H}_\varphi$. It is known that $x u_i \to x$ σ-weakly in $\mathcal{M}$, it follows that $\xi = x$ in $\mathcal{H}_\varphi$ from (13) and (14) in [5, $_p$28]. Thus $x \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and therefore $\mathcal{H}_\varphi = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. The proof is complete. □

Put $\mathfrak{A}_m$ as before. Then we have

Theorem 2. *$\mathfrak{A}_m$ is the unique maximal subdiagonal algebra with respect to Φ containing $\mathfrak{A}$.*

Proof. As in the proof of Theorem 2.2.1 in [1], we may prove that if $\tilde{\mathfrak{A}}$ is any subdiagonal algebra with respect to Φ containing $\mathfrak{A}$, then $\mathfrak{A}_m \supseteq \tilde{\mathfrak{A}}$. Thus we only need to prove that $\mathfrak{A}_m$ is a subdiagonal algebra with respect to Φ. According to the docomposition $\mathcal{H}_\varphi = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, it is easy to check that for $\forall A \in \mathfrak{A}$, $\forall B \in \mathfrak{A}_0$ and $\forall D \in \mathfrak{D}$,

$$\pi_\varphi(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix},$$

$$\pi_\varphi(B) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}$$

and

$$\pi_\varphi(D) = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix}$$

respectively. We also have

$$\mathfrak{D} = \left\{ D \in \mathcal{M} : \pi_\varphi(D) = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \right\}.$$

In fact, let $D \in \mathcal{M}$ such that

$$\pi_\varphi(D) = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix}.$$

Then $\Phi(D) \in \mathfrak{D}$ and so $\pi_\varphi((D))$ has the matrix form as follows:

$$\pi_\varphi(\Phi(D)) = \begin{pmatrix} V_{11} & 0 & 0 \\ 0 & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix}.$$

It follows that $\Phi(D) - D \in \mathrm{Ker}(\Phi)$. However we have

$$< (\Phi(D) - D)d_1, d_2 >_\varphi = \varphi(d_2^*(\Phi(D) - D)d_1) = \varphi \circ \Phi(d_2^*(\Phi(D) - D)d_1) = 0$$

for all d_1, d_2 are in $\mathfrak{D} \cap \mathfrak{N}_\varphi$. It follows that $\pi_\varphi(\Phi(D) - D)\mathcal{H}_2 \perp \mathcal{H}_2$. In particular, $(\Phi(D)-D)u_i \perp \mathcal{H}_2$. However, we also have $(\Phi(D)-D)u_i \in \mathcal{H}_2$, then $(\Phi(D)-D)u_i = 0$, which implies that $\Phi(D) = D$. Putting

$$\mathcal{A}_0 = \left\{ X \in \mathcal{M} : \pi_\varphi(X) = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\},$$

then we similarly have $\Phi(\mathcal{A}_0) = \{0\}$. It is trivial that $\mathcal{A}_0$ is a $\mathfrak{D}$ bimodule and $\mathfrak{A}_0 \subseteq \mathcal{A}_0$. Put

$$\mathfrak{A}_M = \left\{ X \in \mathcal{M} : \pi_\varphi(X) = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\}.$$

We can show that $\mathfrak{A}_M = \mathfrak{D} + \mathcal{A}_0$. Hence it is easy to check that $\mathfrak{A}_M$ is a subdiagonal algebra of $\mathcal{M}$ with respect to Φ containing $\mathfrak{A}$. Thus $\mathfrak{A}_m \supseteq \mathfrak{A}_M$. On the other hand, for $X \in \mathfrak{A}_m$, $a \in \mathfrak{A}_0 \cap \mathfrak{N}_\varphi$ and $b \in \mathfrak{A}^* \cap \mathfrak{N}_\varphi$,

$$< Xa, b >_\varphi = \varphi(b^* Xa) = \varphi \circ \Phi(b^* Xa) = 0.$$

It follows that $\pi_\varphi(X)\mathcal{H}_1 \subseteq \mathcal{H}_1$. We similarly have $\pi_\varphi(X)(\mathcal{H}_1 \oplus \mathcal{H}_2) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$, Thus $X \in \mathfrak{A}_M$ and therefore $\mathfrak{A}_m \subseteq \mathfrak{A}_M$, that is, $\mathfrak{A}_m(=\mathfrak{A}_M)$ is a subdiagonal algebra with respect to Φ. The proof is complete. □

3. σ_t^φ-INVARIANCE OF SUBDIAGONAL ALGEBRAS

From the Tomita-Takesaki theory, there is a σ-weakly continuous automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of $\mathcal{M}$ associated with φ. If φ is a faithful normal state, the author, Ohwada and Saito proved that $\mathfrak{A}$ is $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ -invariant if $\mathfrak{A}$ is maximal subdiagonal(cf[2, Theorem2.4]). Here we generalize this result to the non-σ-finite case.

Theorem 3. *Let $\mathfrak{A}$ be a maximal subdiagonal algebra of $\mathcal{M}$ with respect to Φ and let φ be a faithful normal semi-finite weight on $\mathcal{M}$ such that $\varphi\circ\Phi=\varphi$. Then $\mathfrak{A}$ is σ_t^φ-invariant, that is $\sigma_t^\varphi(\mathfrak{A})=\mathfrak{A}$, $\forall t\in\mathbb{R}$.*

Proof. ¿From the Tomita-Takesaki theory, we recall that the preclosed conjugate-linear operators S_0, with the σ-weakly dense domain $\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$, is defined by

$$S_0x=x^* \ (x\in\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*).$$

Denote by S the closure of S_0, then S has the following matrix decomposition with respect to the decomposition $\mathcal{H}_\varphi=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3$,

$$S=\begin{pmatrix} 0 & 0 & S_3\\ 0 & S_2 & 0\\ S_1 & 0 & 0\end{pmatrix},$$

where for i=1, 2, 3, S_i is a closed operator with domain $\mathfrak{F}_i$ in $\mathcal{H}_i$ such that $S_1\mathfrak{F}_1=\mathfrak{F}_3$, $S_2\mathfrak{F}_2=\mathfrak{F}_2$ and $S_3\mathfrak{F}_3=\mathfrak{F}_1$. In fact, we note that $\mathfrak{A}_0\cap\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$, $\mathfrak{D}\cap\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$ and $\mathfrak{A}_0^*\cap\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$ are σ-weakly dense in $\mathfrak{A}_0$, $\mathfrak{D}$ and $\mathfrak{A}_0^*$ respectively since $u_i\in\mathfrak{D}\cap\mathfrak{N}_\varphi\cap\mathfrak{N}_\varphi^*$ and $u_i\to I$ σ-weakly. Then with the similar calculation as in the proof of Lemma 2.3 in [2], we can obtain the desired form.

Put $\Delta=S^*S$, we recall that the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of $\mathcal{M}$ associated with φ has of the form as following:

$$\pi_\varphi(\sigma_t^\varphi(X))=\Delta^{it}X\Delta^{-it}(\ \forall t\in\mathbb{R}\ ,\ X\in\mathcal{M}\ ).$$

We note that Δ has of the matrix form

$$\Delta=\begin{pmatrix} S_1^*S_1 & 0 & 0\\ 0 & S_2^*S_2 & 0\\ 0 & 0 & S_3^*S_3\end{pmatrix}.$$

Thus by the proof of Theorem 2, it is easy to prove that, for every $t\in\mathbb{R}$,

$$\sigma_t^\varphi(\mathfrak{D})=\mathfrak{D} \text{ and } \sigma_t^\varphi(\mathfrak{A}_0)=\mathfrak{A}_0$$

since $\mathfrak{A}$ is maximal subdiagonal. The proof is complete. □

Let $S = J\Delta^{\frac{1}{2}}$ be the polar decomposition of S. From the Tomita's fundamental theorem (cf[6]) it follows that $J\pi_\varphi(\mathcal{M})J = (\pi_\varphi(\mathcal{M}))'$. It is easy to know that

$$J = \begin{pmatrix} 0 & 0 & J_3 \\ 0 & J_2 & 0 \\ J_1 & 0 & 0 \end{pmatrix}.$$

By considering $J\pi_\varphi(\mathfrak{A})J$ as a subdiagonal algebra in $(\pi_\varphi(\mathcal{M}))'$, we may show that $[J\pi_\varphi(\mathfrak{A}_0)J\mathcal{H}_3] = \mathcal{H}_3$, $[J\pi_\varphi(\mathfrak{D})J\mathcal{H}_2] = \mathcal{H}_2$ and $[J\pi_\varphi(\mathfrak{A}_0^*)J\mathcal{H}_1] = \mathcal{H}_1$ respectively. Thus we can obtain the following characterizations of $\mathfrak{A}_m$ as in Theorem 2.2 in [3]. Here we give the theorem without proof.

Theorem 4. *Keep the notions as above. Then*

$$\begin{aligned} \mathfrak{A}_m &= \{X \in \mathcal{M} : \Phi(XB) = 0,\ \forall B \in \mathfrak{A}_0\} \\ &= \{X \in \mathcal{M} : \Phi(BX) = 0,\ \forall B \in \mathfrak{A}_0\} \\ &= \{X \in \mathcal{M} : \pi_\varphi(X)\mathcal{H}_1 \subseteq \mathcal{H}_1\} \\ &= \{X \in \mathcal{M} : \pi_\varphi(X)(\mathcal{H}_1 \oplus \mathcal{H}_2) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2\}. \end{aligned}$$

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