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ANALYTIC CLUSTER SETS

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ABSTRACT. We study the cluster sets for analytic functions in the unit disk. Lindelöf and Meier types theorems are proved for analytic cluster sets.

1. INTRODUCTION

Let $D = \{z : |z| < 1\}$ be the unit disk in the finite complex plane $\mathbf{C}$ and $\Gamma = \{z : |z| = 1\}$. For each pair of points $a, b =\in D$ the hyperbolic distance between a and b is defined by

$$\sigma(a,b) = \frac{1}{2}\log\frac{|1-\bar{a}b| + |a-b|}{|1-\bar{a}b| - |a-b|}$$

and if L is any curve contained in D, we set

$$\sigma(a,L) = \inf_{b\in L}\sigma(a,b).$$

Let $h(\zeta,\alpha)$ denote the chord which is terminating at the point $\zeta = e^{i\theta} \in \Gamma$ and make up the angle of openning α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, with the radius of D at ζ. The subset bounded by the chords $h(\zeta,\alpha_1)$ and $h(\zeta,\alpha_2)$ and by the circle $|z - \frac{1}{2}\zeta| = \frac{1}{2}$ is denoted by $\Delta(\zeta,\alpha_1,\alpha_2)$ (or, simply, by $\Delta(\zeta)$ if we are not interested in the magnitude of angle $\Delta(\zeta,\alpha_1,\alpha_2)$).

Let f be an arbitrary real or complex-valued function defined on D. We denote by $C(f,\zeta,D)$, $C(f,\zeta,h(\zeta,\alpha))$ and $C(f,\zeta,\Delta(\zeta))$, respectively, the cluster set of f at the point $\zeta = e^{i\theta} \in \Gamma$ with respect to the disk D, the chord $h(\zeta,\alpha)$ and the angle $\Delta(\zeta)$.

A point $\zeta = e^{i\theta} \in \Gamma$ belongs to the set $K(f)$ if $C(f,\zeta,\Delta_1(\zeta)) = C(f,\zeta,\Delta_2(\zeta))$ for any two angles $\Delta_1(\zeta)$ and $\Delta_2(\zeta)$ with the vertix at the point ζ. A point $\zeta = e^{i\theta} \in \Gamma$ belongs to the set $C(f)$ if $\bigcap_\Delta C(f,\zeta,\Delta(\zeta)) = C(f,\zeta,D)$ (over all angles $\Delta(\zeta)$). By definition, $C(f) \subset K(f)$.

The structure of cluster sets of meromorphic functions in D was studied by many authors (see e.g. [CL], [G], [GH]). For example, by the strengthens version of Meier's theorem [G], for any meromorphic function f in D the unit circle Γ can be represented as union of disjoint sets of Meier points, precised Plessner points $I^*(f)$, set $P(f)$ and a set E of first Baire category and of type F_σ on Γ. The sets $I^*(f)$ and $P(f)$ are disjoint subsets of the set $I(f)$ of Plessner points for a meromorphic function f in D and a point $\zeta = e^{i\theta} \in \Gamma$ belongs to the set $I(f)$ if $\cap_\Delta C(f,\zeta,\Delta(\zeta)) = \Omega$, where Ω denotes the Reimann sphere. Moreover, by definition the sets $I^*(f)$ and $P(f)$ are connected with the concept of a P-sequence, related the property of

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normalacy for a meromorphic function f in D (see [G]). For a normal meromorphic function f in D (in particular, for an unbounded univalent function f in D) the set $P(f)$ is empty and $I^*(f) = I(f)$ (see [G]).

In this paper we study cluster sets of analytic functions defined in the unit disk D using results on analytically normal functions (Bloch functions) [ACP], [M], and prove the Lindelöf and Meier type theorems for analytic functions.

Let $d_f(z) = (1 - |z|^2)|f'(z)|$. An analytic function f in D satisfying the condition $\sup_{z\in D} d_f(z) < \infty$ is called Bloch function and the space of Bloch functions is denoted by $\mathcal{B}$ [ACP], [M]. In [M] the second author defined the concept of $\rho_{\mathcal{B}}$-sequences of points for analytic functions in the unit disk. A sequence $\{z_n\} \in D$, $\lim_{n\to\infty} |z_n| = 1$, is called $\rho_{\mathcal{B}}$-sequence for function f if for each sequence of positive numbers $\{\epsilon_n\}$, $\lim_{n\to\infty} |\epsilon_n| = 0$, there is a sequence of positive numbers $\{M_n\}$, $\lim_{n\to\infty} |M_n| = \infty$, such that

$$\operatorname{diam} f(D(z_n, \epsilon_n)) \geq M_n, \qquad n = 1, 2, \ldots .$$

According to Theorem 5.3 [M] an analytic function f in D is a Bloch function if and only if it doesn't have $\rho_{\mathcal{B}}$-sequences of points. Any Bloch function doesn't possess a P-sequence too, but on the other hand, there is an analytic function g in D that possesses a $\rho_{\mathcal{B}}$-sequence and doesn't have P-sequences; for example, the function $g(z) = (1 - z)^{-1}$.

2. Meier type theorem

Let f be an analytic function in D. We say that a point $\zeta = e^{i\theta} \in \Gamma$ belongs to the $M_{\mathcal{B}}(f)$ if $C(f, \zeta, D) = C(f, \zeta, h(\zeta, \varphi))$ for each chord $h(\zeta, \varphi)$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, and $\operatorname{diam} C(f, \zeta, D) < \infty$. We say that a point $\zeta = e^{i\theta} \in \Gamma$ belongs to the set $P_{\mathcal{B}}(f)$ if each chord $h(\zeta, \alpha)$ ending at ζ contains a $\rho_{\mathcal{B}}$-sequence of points for f. We say that a point $\zeta = e^{i\theta} \in \Gamma$ belongs to the set $I^*_{\mathcal{B}}(f)$ if

(1) $\bigcap_h C(f, \zeta, h(\zeta, \alpha)) = \bigcup_\Delta C(f, \zeta, \Delta(\zeta))$;
(2) $\operatorname{diam} \bigcap_h C(f, \zeta, h(\zeta, \alpha)) = \infty$;
(3) $\operatorname{diam} \bigcup_\Delta C(d_f, \zeta, \Delta(\zeta)) < \infty$.

It is easy to see (and it follows from the definitions) that sets $M_{\mathcal{B}}(f)$, $P_{\mathcal{B}}(f)$ and $I^*_{\mathcal{B}}(f)$ are mutually disjoint.

Theorem 1. *Let f be an analytic function in the unit disk D. Then*

$$\Gamma = M_{\mathcal{B}}(f) \cup P_{\mathcal{B}}(f) \cup I^*_{\mathcal{B}}(f) \cup E,$$

where E is a set on Γ of the first Baire category and of type F_σ on Γ.

The proof of Theorem 1 is based on Collingwood's Theorem on maximality, by analogy with the proof of Meier type Theorem in [G].

Lemma 1 **([CL], pp.382-395).** *If g is a continuos function in the unit disk D then the complement of $C(g)$ with respect to Γ is a set of first Baire category and of type F_σ.*

By applying Lemma 1 to functions f and d_f we obtain the following decompositions

$$\Gamma = C(f) \cup E_1 \tag{1}$$

$$\Gamma = C(d_f) \cup E_2 \tag{2}$$

where E_1 and E_2 are sets of first Baire category and of type F_σ. By taking intersection of (1) and (2) we obtain $\Gamma = M \bigcup E$ where $M = C(f) \bigcap C(d_f)$ and $E = E_1 \bigcup E_2$. It is clear that E is a set of first category and of type F_σ. It remains us to describe the set M.

For any point $\zeta = e^{i\theta} \in M$ there are four possibilities:

(I) $\mathrm{diam} C(f,\zeta,D) < \infty$ and $\limsup_{z\to\zeta} d_f(z) < \infty$;

(II) $\mathrm{diam} C(f,\zeta,D) = \infty$ and $\limsup_{z\to\zeta} d_f(z) < \infty$;

(III) $\mathrm{diam} C(f,\zeta,D) = \infty$ and $\limsup_{z\to\zeta} d_f(z) = \infty$;

(IV) $\mathrm{diam} C(f,\zeta,D) < \infty$ and $\limsup_{z\to\zeta} d_f(z) = \infty$.

In fact, case (IV) cannot happen since the condition $\limsup_{z\to\zeta} d_f(z) = \infty$ implies, by Theorem 5.3 in [M], the existence of a $\rho_{\mathcal{B}}$-sequence for f tending to $\zeta \in \Gamma$, and hence, $\mathrm{diam} C(f,\zeta,D)$ must be unbounded.

Lemma 2. *A chord $h(\zeta,\alpha)$ doesn't contain $\rho_{\mathcal{B}}$-sequence of points for analytic function f in D if and only if there exists some angle $\Delta(\zeta,\alpha_1,\alpha_2)$ containing the chord $h(\zeta,\alpha)$ for which $C(d_f,\zeta,\Delta(\zeta,\alpha_1,\alpha_2))$ is bounded.*

Proof. The necessity of the conditions of Lemma 2 were proved in [M], Theorem 5.3. In order to prove the sufficiency, we assume that, for some angle $\Delta(\zeta,\alpha_1,\alpha_2)$ containing the chord $h(\zeta,\alpha)$ the cluster set $C(d_f,\zeta,\Delta(\zeta,\alpha_1,\alpha_2))$ is bounded and the chord $h(\zeta,\alpha)$ contains a $\rho_{\mathcal{B}}$-sequence of points $\{z_n\}$ for f. By Theorem 5.4 [M], there exists a sequence $\{z'_n\}$, $\lim_{n\to\infty} \sigma(z_n,z'_n) = 0$, for which $\lim_{n\to\infty} d_f(z'_n) = \infty$. Since the condition $\lim_{n\to\infty} \sigma(z'_n, h(\zeta,\alpha)) = 0$, beginning with some index N all the points z'_n get into the angle $\Delta(\zeta,\alpha_1,\alpha_2)$. This contradicts our assumption that $\Delta(\zeta,\alpha_1,\alpha_2)$ doesn't contain a $\rho_{\mathcal{B}}$-sequence for f. □

Lemma 2 implies that if assertion (III) is realized then every angle $\Delta(\zeta,\alpha_1,\alpha_2)$ with vertix at ζ contains a $\rho_{\mathcal{B}}$-sequence for f and, consequently, $\zeta \in P_{\mathcal{B}}(f)$.

Lemma 3. *Let f be an analytic function in D and $\zeta = e^{i\theta} \in K(f)$. If $C(d_f,\zeta,\Delta(\zeta,\alpha_1,\alpha_2))$ is bounded for any angle $\Delta(\zeta,\alpha_1,\alpha_2)$ with vertix at ζ then for any chord $h(\zeta,\alpha)$ the set $C(f,\zeta,h(\zeta,\alpha))$ coincides with $C(f,\zeta,\Delta(\zeta,\alpha_1,\alpha_2))$. In particular, if the set $C(d_f,\zeta,D)$ is bounded at the point $\zeta \in C(f)$ then $\cap_h C(f,\zeta,h(\zeta,\alpha)) = C(f,\zeta,D)$.*

Proof. Assume that there exists a chord $h(\zeta,\alpha_0)$ and value $a \in \overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ such that $a \notin C(f,\zeta,h(\zeta,\alpha_0))$ and also that in each angle $\Delta(\zeta,\alpha_1,\alpha_2)$ covering the chord $h(\zeta,\alpha_0)$ there exists a sequence of points $\{z_n^{(\Delta)}\}$, $\lim_{n\to\infty} z_n^{(\Delta)} = \zeta$, for which $\lim_{n\to\infty} f(z_n^{(\Delta)}) = a$. By shrinking the angle $\Delta(\zeta,\alpha_1,\alpha_2)$ to the chord $h(\zeta,\alpha_0)$ we choose a subsequence $\{z_k\}$ such that $\lim_{k\to\infty} z_k = \zeta$, $\lim_{k\to\infty} f(z_k) = a$ and $\lim_{k\to\infty} \sigma(z_k, h(\zeta,\alpha_0)) = 0$. We also take on the chord $h(\zeta,\alpha_0)$ a sequence of points $\{z'_k\}$ such that $\lim_{k\to\infty} \sigma(z_k,z'_k) = 0$. By assumption, $\lim_{k\to\infty} f(z'_k) \neq a$. According to Theorem 5.4 [M], each of the sequence $\{z_k\}$ and $\{z'_k\}$ is a $\rho_{\mathcal{B}}$-sequence for f. By Lemma 2, the set $C(d_f,\zeta,\Delta(\zeta,\alpha_1,\alpha_2))$ is unbounded for some angle $\Delta(\zeta,\alpha_1,\alpha_2)$ covering the chord $h(\zeta,\alpha_0)$. It contrudicts our assumption. □

Lemma 3 implies that if the possibility (I) is realized then $\zeta = e^{i\theta} \in M_{\mathcal{B}}(f)$ and if the possibility (II) is realized then $\zeta = e^{i\theta} \in I^*_{\mathcal{B}}(f)$ and hence Theorem 1 is proved.

3. Lindelöf type theorem

We say that $\zeta = e^{i\theta} \in \Gamma$ is an analytic Lindelöf point for analytic function f in D if $C(f,\zeta,h(\zeta,\alpha_1)) = C(f,\zeta,h(\zeta,\alpha_2))$ for any two chords $h(\zeta,\alpha_1)$ and $h(\zeta,\alpha_2)$ and $\operatorname{diam} C(f,\zeta,h(\zeta,\alpha)) < \infty$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. The set of analytic Lindelöf points for a function f is denoted by $L_B(f)$.

We define the notion of σ-porous set introduced by E.P.Dolzhenko [D]. Let E be a set on Γ, a point $\zeta = e^{i\theta} \in \Gamma$ and a real $\epsilon > 0$. We denote by $r(\zeta,E,\epsilon)$ the length of the largest open arc which belongs to the arc $\gamma_{\zeta,\epsilon} = \{\xi = e^{i\varphi} : \quad |\varphi - \theta| < \epsilon\}$ and doesn't intersect E (if there is no such an arc, we put $r(\zeta,E,\epsilon) = 0$). The point $\zeta = e^{i\theta}$ is called a point of porosity of the set E if

$$r(\zeta,E) = \limsup_{\epsilon\to 0} \frac{r(\zeta,E,\epsilon)}{\epsilon} > 0.$$

The set E is called *porous* on Γ if every point of the set E is a point of porosity for E. A set on Γ is called a *σ-porous set* if it is the union of not more than a countable collection of porous sets.

It follows from the definition that any porous set, and therefore, any σ-porous set is a set of the first Baire category and of linear Lebesgue measure zero on Γ. The converse assertions are not, in general, true (see also [R], [Y]).

Denote by $p(E)$ the collection of all points of a set E such that any point of $p(E)$ is non-isolated point of the set E and it is a point of porosity for E. A set E on Γ is called a perfect σ-porous set if there exists a finite or countable collection of closed sets $\{F_n\}$ on Γ such that $E = \bigcup_{n=1}^{\infty} p(F_n)$.

Lemma 4 [K]. *For an arbitrary mapping $f : D \to \overline{\mathbf{C}}$ the set $\Gamma \setminus K(f)$ is a perfect σ-porous set on Γ. Converse, for any perfect σ-porous set E on Γ there exists an analytic and bounded function g in D such that $K(g) = \Gamma \setminus E$.*

Theorem 2. *Let f be an analytic function in D. Then $\Gamma = L_B(f) \cup I_B^*(f) \cup P_B(f) \cup E$ where E is a perfect σ-porous set on Γ.*

Proof. By analogy with the proof of Theorem 1, we apply Lemma 4 to the functions f and d_f and obtain $\Gamma = M \cup E$ where $M = K(f) \cap K(d_f)$ and $E = E_1 \cup E_2$. It is clear that E is a perfect σ-porous set on Γ. It remains to describe the set M.

For any point $\zeta = e^{i\theta} \in M$ there are four possibilities:

(I') $\operatorname{diam} C(f,\zeta,\Delta(\zeta)) < \infty$ and $\limsup_{z\to\zeta, z\in\Delta(\zeta)} d_f(z) < \infty$ for any $\Delta(\zeta)$;

(II') $\operatorname{diam} C(f,\zeta,\Delta(\zeta)) = \infty$ and $\limsup_{z\to\zeta, z\in\Delta(\zeta)} d_f(z) < \infty$ for any $\Delta(\zeta)$;

(III') $\operatorname{diam} C(f,\zeta,\Delta(z)) = \infty$ and $\limsup_{z\to\zeta, z\in\Delta(\zeta)} d_f(z) = \infty$ for any $\Delta(\zeta)$;

(IV') $\operatorname{diam} C(f,\zeta,\Delta(\zeta)) < \infty$ and $\limsup_{z\to\zeta, z\in\Delta(\zeta)} d_f(z) = \infty$ for any $\Delta(\zeta)$.

As in the proof of Theorem 1, an analogical argument shows that the case (IV') cannot happen. By Lemma 2, if case (III') is realized then $\zeta = e^{i\theta} \in P_B$. Lemma 3 implies that if case (I') holds then $\zeta = e^{i\theta} \in L_B$, and if case (II') is realized then $\zeta = e^{i\theta} \in I_B^*$, and hence Theorem 2 is proved. □

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