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ON PSEUDO-UMBILICAL SURFACES WITH NONZERO PARALLEL MEAN CURVATURE VECTOR IN $\mathbb{C}P^3(\tilde{c})$ II

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Abstract. In this paper, we classify pseudo-umbilical surfaces in a complex 3-dimensional complex projective space under some additional condition.

1. INTRODUCTION

Let $\mathbb{C}P^m(\tilde{c})$ be a complex m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $\tilde{c}$. The class of totally umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$ was completely classified by Chen and Ogiue [1]. However, it is well known that the class of pseudo-umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$ is too wide to classify. Thus, it is reasonable to study pseudo-umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$ under some additional condition.

Recently, the author [5] proved the following Theorem.

Theorem A. *Let M be an $n(\geq 2)$-dimensional pseudo-umbilical submanifold with nonzero parallel mean curvature vector in $\mathbb{C}P^m(\tilde{c})$. If $2m-n \geq 2$, then $m > n$ and M^n is immersed in $\mathbb{C}P^m(\tilde{c})$ as a totally real submanifold.*

Immediately, we see that $\mathbb{C}P^2(\tilde{c})$ admits no pseudo-umbilical surfaces with nonzero parallel mean curvature vector. The aim of this paper is to classify pseudo-umbilical surfaces with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Now we get the following Theorem.

Theorem 1.1. *Let $M(K)$ be a complete pseudo-umbilical surface of constant Gauss curvature K with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^3(\tilde{c})$. Then $M(K)$ is one of the following:*

(1) *$M(K)$ is an extrinsic hypersphere in a 3-dimensional real projective space $\mathbb{R}P^3(\tilde{c}/4)$ of $\mathbb{C}P^3(\tilde{c})$.*
(2) *$M(K)$ is a constant isotropic totally real surface in $\mathbb{C}P^3(\tilde{c})$ and the covariant derivative $\bar{\nabla}\sigma$ of the second fundamental form σ is proportional to $J\zeta$.*

Remark 1.1. By Proposition 4.1, we can descrive the covariant derivative $\bar{\nabla}\sigma$ of the second fundamental form σ of the surface (2) in Theorem 1.1 explicitly.

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2. PRELIMINARIES

Let M be an n-dimensional submanifold of a complex m-dimensional Kaehler manifold $\tilde{M}$ with complex structure J and Kaehler metric g. A submanifold M of a Kaehler manifold $\tilde{M}$ is said to be ***totally real*** if each tangent space of M is mapped into the normal space by the complex structure of $\tilde{M}$.

Let ∇(resp.$\tilde{\nabla}$) be the covariant differentiation on M(resp.$\tilde{M}$). We denote by σ the second fundamental form of M in $\tilde{M}$. Then the Gauss formula and the Weigarten formula are given respectively by

$$\sigma(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y, \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for vector fields X, Y tangent to M and a vector field ξ normal to M, where $-A_\xi X$(resp.$D_X\xi$) denotes the tangential(resp. normal) component of $\tilde{\nabla}_X\xi$. A normal vector field ξ is said to be *parallel* if $D_X\xi = 0$ for any vector field X tangent to M. The covariant derivative $\bar{\nabla}\sigma$ of the second fundamental form σ is defined by

$$(\bar{\nabla}_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{2.1}$$

for all vector fields X, Y and Z tangent to M. The second fundamental form σ is said to be *parallel* if $\bar{\nabla}_X\sigma = 0$.

Let $\zeta = (1/n)\text{trace } \sigma$ and $H = |\zeta|$ denote the mean curvature vector and the mean curvature of M in $\tilde{M}$, respectively. If the second fundamental form σ satisfies $\sigma(X,Y) = g(X,Y)\zeta$, then M is said to be *totally umbilical* submanifold in $\tilde{M}$. By *extrinsic sphere*, we mean a totally umbilical submanifold with nonzero parallel mean curvature vector. If the second fundamental form σ satisfies $g(\sigma(X,Y),\zeta) = g(X,Y)g(\zeta,\zeta)$, then M is said to be *pseudo-umbilical* submanifold in $\tilde{M}$.

The submanifold M of $\tilde{M}$ is said to be a λ-*isotropic* submanifold if $|\sigma(X,X)| = \lambda$ for all unit tangent vectors X at each point. In particular, if the function is constant, then M is said to be a *constant isotropic* submanifold in $\tilde{M}$. The first normal space at x, $N_x^1(M)$ is defined to be the vector space spanned by all vectors $\sigma(X,Y)$.

Let R (resp.$\tilde{R}$) be the Riemannian curvature for ∇(resp.$\tilde{\nabla}$). Then the Gauss equation is given by

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) =& g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ & - g(\sigma(Y,Z),\sigma(X,W)) \end{aligned} \tag{2.2}$$

for all vector fields X, Y, Z and W tangent to M. The Codazzi equation is given by

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X\sigma)(Y,Z) - (\bar{\nabla}_Y\sigma)(X,Z) \tag{2.3}$$

for all vector fields X, Y and Z tangent to M.

3. Lemmas

Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^m(\tilde{c})$. By Theorem A, we see that M is a totally real submanifold in $\mathbb{C}P^m(\tilde{c})$. Thus the normal space $T_x^{\perp}(M)$ is decomposed in the following way; $T_x^{\perp}(M) = JT_x(M) \oplus \nu_x$ at each point x of M, where ν_x denotes the orthogonal complement of $JT_x(M)$ in $T_x^{\perp}(M)$. We prepare the following Lemma.

Lemma 3.1. *Let M be a pseudo-umbilical submanifold with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^m(\tilde{c})$. Then we have*

(1) $\zeta \in \nu_x$
(2) $g(\sigma(X,Y), J\zeta) = 0$
(3) $g((\bar{\nabla}_X\sigma)(Y,Z), \zeta) = 0$
(4) $g((\bar{\nabla}_X\sigma)(Y,Z), J\zeta) = H^2 g(\sigma(Y,Z), JX)$

for all vector fields X, Y and Z tangent to M.

Proof. Lemma 3.1(1),(2) and (3) has been proved in [7]. By Lemma 3.1(2), we get

$$\begin{aligned} g((\bar{\nabla}_X\sigma)(Y,Z), J\zeta) &= g(D_X(\sigma(Y,Z)), J\zeta) \\ &= g(\tilde{\nabla}_X(\sigma(Y,Z)), J\zeta) \\ &= g(J\sigma(Y,Z), \tilde{\nabla}_X\zeta) \\ &= g(J\sigma(Y,Z), -A_\zeta X) \\ &= g(J\sigma(Y,Z), -H^2 X) \\ &= H^2 g(\sigma(Y,Z), JX) \end{aligned}$$

for all vector fields X, Y and Z tangent to M. □

Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^3(\tilde{c})$. We choose a local orthonormal frame field

$$e_1, e_2, e_3, e_4 = Je_1, e_5 = Je_2, e_6 = Je_3$$

of $\mathbb{C}P^3(\tilde{c})$ such that e_1, e_2 are tangent to M. By Lemma 3.1(1), we choose e_3 in such a way that its direction coincides with that of the mean curvature vector ζ. Since M is a pseudo-umbilical surface, it is umbilic with respect to the direction of the mean curvature vector ζ. In [6], we showed the followings

Proposition 3.1. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Then the surface satisfies*

$$\begin{cases} \sigma(e_1,e_1) = He_3 + ae_4 + be_5 \\ \sigma(e_1,e_2) = \qquad\quad be_4 - ae_5 \\ \sigma(e_2,e_2) = He_3 - ae_4 - be_5 \end{cases}$$

for some functions a, b with respect to the orthonormal local frame field $\{e_i\}$.

Proposition 3.2. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Then M is an isotropic totally real surface in $\mathbb{C}P^3(\tilde{c})$.*

Proposition 3.3. *Let M be a complete pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. If M is not totally umbilical, then the surface is an isotropic totally real surface in $\mathbb{C}P^3(\tilde{c})$ whose second fundamental form is not parallel.*

4. Proof of Theorem 1.1

The following is a key Lemma for Theorem 1.1. By the similar calculation as in Maeda [2], we obtain

Lemma 4.1. *Let $M(K)$ be a pseudo-umbilical surface of constant Gauss curvature K with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Then we have*

$$g((\bar{\nabla}_X\sigma)(Y,Z),\sigma(S,T)) = 0 \tag{4.1}$$

for all vector fields X, Y, Z, S and T tangent to M.

Proof. By (2.2) and Proposition 3.1, we get the Gauss curvature $K = \tilde{c}/4 + H^2 - 2a^2 - 2b^2$. By assumption, both the Gauss curvature K and the mean curvature H are constant. So we see that a^2+b^2 is constant. By Proposition 3.1 and Proposition 3.2, we get $\lambda^2 = H^2 + a^2 + b^2$. Immediately, we see that the surface is a constant λ-isotropic surface in $\mathbb{C}P^3(\tilde{c})$. Now we have (see [3]).

$$g(\sigma(X,X),\sigma(X,X)) = \lambda^2 g(X,X)g(X,X)$$

which is equivalent to

$$\begin{aligned} &g(\sigma(X,Y),\sigma(Z,W)) + g(\sigma(X,Z),\sigma(Y,W)) + g(\sigma(X,W),\sigma(Y,Z)) \\ &= \lambda^2(g(X,Y)g(Z,W) + g(X,Z)g(Y,W) + g(X,W)g(Y,Z)) \end{aligned} \tag{4.2}$$

By Theorem A, we see that the surface is immersed in $\mathbb{C}P^3(\tilde{c})$ as a totally real submanifold. Thus, the equations (2.2) and (2.3) are reduced to (4.3) and (4.4), respectively.

$$\begin{aligned} &g(\sigma(X,Y),\sigma(Z,W)) - g(\sigma(Z,Y),\sigma(X,W)) \\ &= (K - \tilde{c}/4)(g(X,Y)g(Z,W) - g(Z,Y)g(X,W)) \end{aligned} \tag{4.3}$$

$$(\bar{\nabla}_X\sigma)(Y,Z) = (\bar{\nabla}_Y\sigma)(X,Z) \tag{4.4}$$

Exchanging X and Y in (4.3), we get

$$\begin{aligned} &g(\sigma(Y,X),\sigma(Z,W)) - g(\sigma(Z,X),\sigma(Y,W)) \\ &= (K - \tilde{c}/4)(g(Y,X)g(Z,W) - g(Z,X)g(Y,W)) \end{aligned} \tag{4.5}$$

Summing up (4.2),(4.3) and (4.5), we have

$$\begin{aligned} &3g(\sigma(X,Y),\sigma(Z,W)) \\ &= (\lambda^2 + 2(K - \tilde{c}/4))g(X,Y)g(Z,W) \\ &+ (\lambda^2 - (K - \tilde{c}/4))(g(X,Z)g(Y,W) + g(X,W)g(Y,Z)) \end{aligned} \tag{4.6}$$

Differentiating (4.6) with respect to any tangent vector field T, we get

(4.7) $$g((\bar{\nabla}_T\sigma)(X,Y),\sigma(Z,W)) = -g(\sigma(X,Y),(\bar{\nabla}_T\sigma)(Z,W))$$

By (4.4) and (4.7) we have

$$\begin{aligned} g((\bar{\nabla}_T\sigma)(X,Y),\sigma(Z,W)) &= -g(\sigma(X,Y),(\bar{\nabla}_Z\sigma)(T,W)) \\ &= g((\bar{\nabla}_X\sigma)(Z,Y),\sigma(T,W)) \\ &= -g(\sigma(Z,Y),(\bar{\nabla}_W\sigma)(X,T)) \\ &= g((\bar{\nabla}_Y\sigma)(Z,W),\sigma(X,T)) \\ &= -g(\sigma(Z,W),(\bar{\nabla}_T\sigma)(X,Y)) \end{aligned}$$

for all vector fields X, Y, Z, W and T tangent to M. □

If $a^2+b^2=0$ in Proposition 3.1 (i.e., M is totally umbilical), then we have the case (1) by Naitoh's work [4](for details, see [6]). If $a^2+b^2 \neq 0$ in Proposition 3.1 (i.e., M is not totally umbilical), then we get $\bar{\nabla}\sigma \neq 0$ by proposition 3.3. Thus immediately by Lemma 3.1(3), Lemma 4.1 and Proposition 3.1, we get

(4.8) $$(\bar{\nabla}_X\sigma)(Y,Z) = fJ\zeta$$

for some function $f \neq 0$ with respect to orthonormal local frame field $\{e_i\}$. This completes the proof of Theorem 1.1.

Immediately, by Lemma 3.1(4), Proposition 3.1 and (4.8), we have

Proposition 4.1. *Let $M(K)$ be a complete pseudo-umbilical surface of constant Gauss curvature K with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^3(\bar{c})$. Then the surface satisfies*

$$\begin{cases} (\bar{\nabla}_{e_1}\sigma)(e_1,e_1) = aJ\zeta \\ (\bar{\nabla}_{e_1}\sigma)(e_1,e_2) = (\bar{\nabla}_{e_2}\sigma)(e_1,e_1) = bJ\zeta \\ (\bar{\nabla}_{e_2}\sigma)(e_1,e_2) = (\bar{\nabla}_{e_1}\sigma)(e_2,e_2) = -aJ\zeta \\ (\bar{\nabla}_{e_2}\sigma)(e_2,e_2) = -bJ\zeta \end{cases}$$

for some functions a, b in Proposition 3.1.

REFERENCES

1. B.Y.Chen and K.Ogiue, *Two theorems on Kaehler manifolds*, Michigan Math. J. **21** (1974), 225–229.
2. S.Maeda, *Isotropic immersions*, Canad.J. Math **38** (1986), 416–430.
3. S.Maeda and N.Sato, *On submanifolds all of whose geodesics are circles in a complex space form*, Kodai Math. J. **6** (1983), 157–166.
4. H.Naitoh, *Isotropic submanifolds with parallel second fundamental form in $P^m(c)$*, Osaka J. Math. **18** (1981), 427–464.
5. N.Sato, *Totally real submaifolds of a complex space form with nonzero parallel mean curvature vector*, Yokohama Math. J. **44** (1997), 1–4.
6. ———, *On pseudo-umbilical surfaces with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\bar{c})$*, Nihonkai Math. J. **9** (1998), 91–96.
7. ———, *Pseudo-umbilical surfaces with nonzero parallel mean curvature vector in $\mathbb{C}P^4$*, Yokohama Math. J. **45** (1998), 97–104.

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