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ON THE SPECTRAL GEOMETRY FOR THE JACOBI OPERATORS OF HARMONIC MAPS INTO PRODUCT MANIFOLDS OF QUATERNIONIC PROJECTIVE SPACES

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ABSTRACT. We study the spectral characterization of harmonic submersions when the target manifold is $QP^n \times QP^n$.

1. INTRODUCTION

The inverse eigenvalue problem of the second order operators arising in Riemannian geometry has been studied by many authors. Among them, the Jacobi operator for a harmonic map was studied in [8,11,12,13], and that for the area functional was studied in [1,5,9]. The Jacobi operator of a harmonic map f arises in the second variation formula of the energy of the harmonic map f. This formula can be expressed in terms of an elliptic differential operator J_f(called the *Jacobi operator*) defined on the space of sections of the bundle induced from the tangent bundle of the target manifold.

The spectral charaterization of harmonic Riemannian submersions among the set of all harmonic morphisms has been studied in the cases

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when the target manifolds are the standard sphere S^n, complex projective space CP^n([13]), the quaternionic projective space QP^n([11]) and product manifolds ([8]).

In this paper, we shall prove the following

Main Theorem. *Let f and f' be harmonic morphisms of a compact Riemannian manifold (M,g) with constant scalar curvature into $QP^n \times QP^n$, where QP^n is the quaternionic projective space of real $4n$-dimension. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a Riemannian submersion, then so is f'.*

2. Preliminaries

Let (M,g) be an m-dimensional connected, closed(i.e., compact without boundary) Riemannian manifold with metric g and (N,h) an n-dimensional Riemannian manifold with metric h. A smooth map $f : (M,g) \longrightarrow (N,h)$ is said to be *harmonic* if it is a critical point of the energy functional

$$E(f) := \int_M e(f)\, dv_g,$$

where the energy density $e(f)$ of f is defined to be $e(f) := \frac{1}{2}\sum_{i=1}^m h(f_*e_i, f_*e_i)$, f_* is the differential of f, $\{e_1, \cdots, e_m\}$ is a local orthonormal frame field on M, and dv_g is the volume element with respect to g.

Let us consider the Jacobi operator J_f for a harmonic map f defined by

$$J_f V := \tilde{\Delta}_f V - \mathcal{R}_f V$$

for $V \in \Gamma(E)$(the space of smooth sections of E), where $\tilde{\Delta}$ is the rough Laplacian associated to the induced connection $\tilde{\nabla}$ of the induced bundle $E := f^*TN$ defined by $\tilde{\nabla}_X V := \nabla^h_{f_*X} V$ (for X a tangent vector of M, ∇^h the Levi-Civita connection of the metric h), and $\mathcal{R}_f V := \sum_{i=1}^m R_h(V, f_*e_i)f_*e_i$($R_h$ is the Riemannian curvature tensor of (N,h)). In this paper, we take the convention

$$R_h(\tilde{X},\tilde{Y}) := [\nabla^h_{\tilde{X}}, \nabla^h_{\tilde{Y}}] - \nabla^h_{[\tilde{X},\tilde{Y}]},$$

where $\tilde{X}$ and $\tilde{Y}$ are tangent vector fields on N. Then J_f is self-adjoint, elliptic of second order and has a discrete spectrum as a consequence of the compactness of M.

Consider the semigroup e^{-tJ_f} given by

$$e^{-tJ_f}V(x) = \int_M K(t,x,y,J_f)V(y)\,dv_g(y),$$

where $K(t,x,y,J_f) \in \mathrm{Hom}(E_y, E_x)$ is the kernel function ($x, y \in M$, E_x is the fibre of E over x). Then we have asymptotic expansions for the L^2-trace

$$(2.1) \qquad Tr(e^{-tJ_f}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(J_f) \qquad (t \downarrow 0^+),$$

where each $a_n(J_f)$ is the spectral invariant of J_f, which depends only on the discrete spectrum ;

$$Spec(J_f) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}.$$

Applying Gilkey's results in [4,p.327] to the Jacobi operator J_f of a harmonic map f, we obtain

Theorem 2.1 [cf.13]. *For a harmonic map* $f : (M,g) \longrightarrow (N,h)$,

$$(2.2) \qquad a_0(J_f) = n\,Vol(M,g),$$

$$(2.3) \qquad a_1(J_f) = \frac{n}{6}\int_M \tau_g\,dv_g + \int_M Tr(\mathcal{R}_f)\,dv_g,$$

(2.4)

$$a_2(J_f) = \frac{n}{360}\int_M [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2]\,dv_g + \frac{1}{360}\int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(\mathcal{R}_f) + 180 Tr({\mathcal{R}_f}^2)]\,dv_g,$$

where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ on E,which is defined by $R^{\tilde{\nabla}} := f^*R_h$, and R_g, ρ_g, τ_g are the curvature tensor, Ricci tensor, scalar curvature of M, respectively.

Remark 2.2. *$\dim(M) = m$ is determined by $Spec(J_f)$ (i.e., dim (M) is a spectral invariant of J_f) because of the asymptotic expansion* (2.1).

3. Product manifolds of quaternionic Kaehler manifolds

To begin with we define an almost product manifold. Let N be a smooth manifold with a tensor F of type (1,1) such that

$$F^2 = I,$$

where I denotes the identity transformation. Then we say that N is an *almost product manifold* with *almost product structure* F. If an almost product manifold N admits a Riemannian metric h such that

$$h(F\tilde{X}, F\tilde{Y}) = h(\tilde{X}, \tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N, then N is called to be an *almost product Riemannian manifold.* Let N_1 be a quaternionic Kaehler manifold with metric h_1. Then there exists a 3-dimensional vector bundle E_1 of tensors of type $(1,1)$ such that in any coordinate neighborhood U_1 of N_1, there exists a local basis of almost Hermitian structures ϕ_1, ϕ_2, ϕ_3 of E_1 satisfying

$$\begin{aligned} &\phi_s^2 = -I(\text{the identity transformation})(s = 1, 2, 3), \\ &\phi_1 \circ \phi_2 = -\phi_2 \circ \phi_1 = \phi_3, \phi_2 \circ \phi_3 = -\phi_3 \circ \phi_2 = \phi_1, \\ &\phi_3 \circ \phi_1 = -\phi_1 \circ \phi_3 = \phi_2, \end{aligned} \tag{3.1}$$

and local 1-forms a_1, a_2 and a_3 on U_1 satisfying

$$\begin{aligned} {}^1\nabla_X \phi_1 &= a_3(X)\phi_2 - a_2(X)\phi_3 \\ {}^1\nabla_X \phi_2 &= -a_3(X)\phi_1 + a_1(X)\phi_3 \\ {}^1\nabla_X \phi_3 &= a_2(X)\phi_1 - a_1(X)\phi_2 \end{aligned} \tag{3.2}$$

for any vector field X on N_1, where ${}^1\nabla$ is the Levi-Civita connection of N_1. The bundle E_1 satisfying (3.1) and (3.2) is called a *quaternionic Kaehler structure* in N_1. The bundle E_1 satisfying the algebraic relation (3.1) is called an *almost quaternionic structure.* A manifold with an almost quaternionic structure is called an *almost quaternionic manifold*(cf. [6,14]).

Let N_2 be another quaternionic Kaehler manifold with metric h_2. Note that a local basis of almost Hermitian structures ψ_1, ψ_2, ψ_3 of the quaternionic Kaehler structure E_2 of N_2 satisfy the above algebraic relation (3.1), and there exist local 1-forms b_1, b_2 and b_3 in a coordinate neighborhood U_2 of N_2 satisfying the relation (3.2) in covariant differentiation.

Now we consider a product manifold $N := N_1 \times N_2$ of two quaternionic Kaehler manifolds N_1 and N_2([10]). We denote by P and Q the projection operators of the tangent space of N to that of N_1 and N_2, respectively. Then we have

$$P^2 = P,\ Q^2 = Q,\ PQ = 0 = QP.$$

Setting $F = P - Q$, then we obtain $F^2 = I$, i.e., F is an almost product structure on N. Moreover, we define a Riemannian metric h on N by

$$h(\tilde{X}, \tilde{Y}) = h_1(P\tilde{X}, P\tilde{Y}) + h_2(Q\tilde{X}, Q\tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ of N. Then we have

$$h(F\tilde{X}, \tilde{Y}) = h(F\tilde{Y}, \tilde{X}).$$

For any vector field $\tilde{X}$ on N we put

$$\theta_s\tilde{X} = \phi_s P\tilde{X} + \psi_s Q\tilde{X},\ s = 1, 2, 3. \tag{3.3}$$

Now we consider the vector bundle E over N generated by $\{\theta_s = \phi_s \oplus \psi_s : s = 1, 2, 3\}$, where $\{\phi_s : s = 1, 2, 3\}$ and $\{\psi_s : s = 1, 2, 3\}$ are local bases of quaternionic Kaehler structures E_1 and E_2 respectively. Then, for any local coordinate neighborhood $U_1 \times U_2$, we see that the local basis of almost Hermitian structures $\theta_1, \theta_2, \theta_3$ satisfies the algebraic relation (3.1). Moreover we know from (3.3) that

$$P\theta_s = \phi_s P,\ Q\theta_s = \psi_s Q. \tag{3.4}$$

Let N be a Riemannian product manifold of quaternionic Kaehler manifolds with the almost product structure F, and $\{\theta_s : s = 1, 2, 3\}$ a canonical local basis of the almost quaternionic structure. Let f :

$M \longrightarrow N$ be an isometric immersion of a Riemannian manifold M into N. If $Ff_*(T_xM) \subset f_*(T_xM)$ (resp. $Ff_*(T_xM) \subset f_*(T_xM)^\perp$) for each $x \in M$, then f is said to be an *F-invariant* (resp. *F-anti-invariant*) immersion. f is called an *invariant*(resp. *totally real*) immersion if for each $x \in M$, f_*T_xM is invariant(resp. totally real) subspace under $\{\theta_s; s = 1,2,3\}$ (cf. [3,10,11]).

Let N_1 be a quaternionic Kaehler manifold with a local basis $\{\phi_1, \phi_2, \phi_3\}$ of E_1. Let $Q(X)$ be the so-called *quaternionic section* determined by X, which is a 4-plane spanned by $\{X, \phi_s X : s = 1,2,3\}$, where X is a unit vector on N_1. Any 2-plane in a quaternionic section is called a *quaternionic plane.* The sectional curvature of a quaternionic plane π is called the *quaternionic sectional curvature* of π. A quaternionic Kaehler manifold is a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant.

It is well known that a quaternionic Kaehler manifold N_1 is a quaternionic space form with constant quaternionic sectional curvature λ_1 if and only if its curvature tensor R_1 is of the form (cf. [6], [14]) :

$$\begin{aligned} R_1(X,Y)Z = \frac{\lambda_1}{4}\big[&h_1(Y,Z)X - h_1(X,Z)Y \\ &+ \sum_s \{h_1(\phi_s Y, Z)\phi_s X - h_1(\phi_s X, Z)\phi_s Y \\ &- 2h_1(\phi_s X, Y)\phi_s Z\}\big], \end{aligned}$$

where X, Y and Z are vector fields on N_1.

Here and in the sequel, we denote by ${N_1}^{n_1}(\lambda_1)$ the real $4n_1$-dimensional quaternionic space form of constant quaternionic sectional curvature λ_1.

Let ${N_2}^{n_2}(\lambda_2)$ be a real $4n_2$-dimensional quaternionic space form with constant quaternionic sectional curvature λ_2 and a local basis $\{\psi_1, \psi_2, \psi_3\}$ of E_2. Then the curvature tensor R_2 of N_2 is given by

$$\begin{aligned} R_2(X,Y)Z = \frac{\lambda_2}{4}\big[&h_2(Y,Z)X - h_2(X,Z)Y \\ &+ \sum_s \{h_2(\psi_s Y, Z)\psi_s X - h_2(\psi_s X, Z)\psi_s Y \\ &- 2h_2(\psi_s X, Y)\psi_s Z\}\big], \end{aligned}$$

where X, Y and Z are vector fields on N_2.

Now we consider the product manifold $N = N_1^{n_1}(\lambda_1) \times N_2^{n_2}(\lambda_2)$ of quaternionic space forms $N_1^{n_1}(\lambda_1)$ and $N_2^{n_2}(\lambda_2)$. Then the curvature tensor R_h of $N = N_1{}^{n_1}(\lambda_1) \times N_2{}^{n_2}(\lambda_2)$ is given by

(3.5)

$$\begin{aligned}
R_h(\tilde{X},\tilde{Y})\tilde{Z} = \alpha\big[&h(\tilde{Y},\tilde{Z})\tilde{X} - h(\tilde{X},\tilde{Z})\tilde{Y} + h(F\tilde{Y},\tilde{Z})F\tilde{X} - h(F\tilde{X},\tilde{Z})F\tilde{Y} \\
&+ \sum_s \{h(\theta_s\tilde{Y},\tilde{Z})\theta_s\tilde{X} - h(\theta_s\tilde{X},\tilde{Z})\theta_s\tilde{Y} - 2h(\theta_s\tilde{X},\tilde{Y})\theta_s\tilde{Z}\} \\
&+ \sum_s \{h(F\theta_s\tilde{Y},\tilde{Z})F\theta_s\tilde{X} - h(F\theta_s\tilde{X},\tilde{Z})F\theta_s\tilde{Y} - 2h(F\theta_s\tilde{X},\tilde{Y})F\theta_s\tilde{Z}\}\big] \\
&+ \beta\big[h(F\tilde{Y},\tilde{Z})\tilde{X} - h(F\tilde{X},\tilde{Z})\tilde{Y} + h(\tilde{Y},\tilde{Z})F\tilde{X} - h(\tilde{X},\tilde{Z})F\tilde{Y} \\
&+ \sum_s \{h(F\theta_s\tilde{Y},\tilde{Z})\theta_s\tilde{X} - h(F\theta_s\tilde{X},\tilde{Z})\theta_s\tilde{Y} + h(\theta_s\tilde{Y},\tilde{Z})F\theta_s\tilde{X} \\
&- h(\theta_s\tilde{X},\tilde{Z})F\theta_s\tilde{Y} - 2h(F\theta_s\tilde{X},\tilde{Y})\theta_s\tilde{Z} - 2h(\theta_s\tilde{X},\tilde{Y})F\theta_s\tilde{Z}\}\big]
\end{aligned}$$

for any vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ on N, where $F := P - Q$ is an almost product structure on N, $\alpha := \dfrac{\lambda_1 + \lambda_2}{16}$ and $\beta := \dfrac{\lambda_1 - \lambda_2}{16}$ ([10]).

Remark 3.1. *In the product manifold $N = N_1{}^{n_1}(\lambda_1) \times N_2{}^{n_2}(\lambda_2)$, if $n_1 = n_2$ and $\lambda_1 = \lambda_2$, then N is an Einstein manifold. In fact, The Ricci tensor ρ_h of N is given by*

$$\begin{aligned}
\rho_h(\tilde{X},\tilde{Y}) = \alpha\{&(4(n+4)h(\tilde{X},\tilde{Y}) + (Tr_h F)h(F\tilde{X},\tilde{Y})\} \\
&+ \beta\{(4(n+4)h(F\tilde{X},\tilde{Y}) + (Tr_h F)h(\tilde{X},\tilde{Y})\}
\end{aligned}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N. If $\lambda_1 = \lambda_2$ (i.e., $\beta = 0$) and $n_1 = n_2 = n$ (i.e., $Tr_h F = 0$), then $\rho_h = 4(n+4)\alpha h$. Hence N is an Einstein manifold.

4. Spectral invariants for J_f of a harmonic map f

In this section we consider the target manifold N as $N = N_1^{n_1}(\lambda)$

$\times\ N_2^{n_2}\ (\lambda)$. In this case $\beta = 0$. We adopt the following notations :

$$\Omega(\tilde{X}, \tilde{Y}) := h(\tilde{X}, F\tilde{Y}),$$
$$\Omega_s(\tilde{X}, \tilde{Y}) := h(\tilde{X}, \theta_s\tilde{Y}),$$
$$\Theta_s(\tilde{X}, \tilde{Y}) := h(\tilde{X}, F\theta_s\tilde{Y}),$$
$$\Omega_s \boxtimes \Theta_s(\tilde{X}, \tilde{Y}) := \Omega_s(\tilde{X}, \tilde{Y})\Theta_s(\tilde{X}, \tilde{Y}), s = 1, 2, 3$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N. Then for a harmonic map $f : (M, g) \longrightarrow (N, h)$ we obtain from (3.4) and (3.5)

(4.1) $$Tr(\mathcal{R}_f) = 2(n + 16)\alpha[e(f) + (Tr_h F)(Tr_g f^*\Omega)],$$

(4.2)
$$\begin{aligned} Tr(\mathcal{R}_f^2) &= \sum_{i,j=1}^{m} \sum_{a=1}^{n} h(R_h(f_*e_i, v_a)f_*e_i, R_h(f_*e_j, v_a)f_*e_j) \\ &= \alpha^2\big[4(n+32)e(f)^2 + 56\|f^*h\|^2 + 24\sum_s \|f^*\Omega_s\|^2 + 56\|f^*\Omega\|^2 \\ &+ (n+32)(Tr_g f^*\Omega)^2 + 24\|f^*\Theta_s\|^2 + 4e(f)(Tr_g f^*\Omega)(Tr_h F)\big], \end{aligned}$$

(4.3)
$$\begin{aligned} \|R^{\tilde{\nabla}}\|^2 &= \sum_{i,j=1}^{m} \sum_{a,b=1}^{n} h(R_h(f_*e_i, f_*e_j)v_a, v_b)h(R_h(f_*e_i, f_*e_j)v_a, v_b) \\ &= \alpha^2\big[64e(f)^2 - 16\|f^*h\|^2 + (4n+18)\sum_s \|f^*\Omega_s\|^2 \\ &+ 16(Tr_g f^*\Omega)^2 - 16\|f^*\Omega\|^2 + (4n+18)\sum_s \|f^*\Theta_s\|^2 \\ &+ 8\sum_{i,j,s} f^*(\Omega_s \boxtimes \Theta_s)(e_i, e_j)(Tr_h F)\big], \end{aligned}$$

where $\{v_a : a = 1, \cdots, n := n_1 + n_2\}$ is a local orthonormal frame field on N and $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M.

Substituting (4.1) $\sim$ (4.3) into Theorem 2.1, we get

Proposition 4.1. *Let $f : (M, g) \longrightarrow N = N^{n_1}(\lambda) \times N^{n_2}(\lambda)$ be a harmonic map of an m-dimensional compact Riemannian manifold (M, g) into an $n(= n_1 + n_2)$-dimensional product manifold N. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

(4.4)
$$a_0(J_f) = nVol(M, g),$$

(4.5)
$$\begin{aligned} a_1(J_f) = &\frac{n}{6}\int_M \tau_g dv_g \\ &+ \alpha \int_M \left[2(n+16)e(f) + (Tr_h F)(Tr_g f^*\Omega)\right] dv_g, \end{aligned}$$

(4.6)
$$\begin{aligned} a_2(J_f) = &\frac{n}{360}\int_M [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2] dv_g \\ &+ \frac{\alpha^2}{12}\int_M \Big[8(3n+88)e(f)^2 + 352\|f^*h\|^2 \\ &+ 2(3n+88)(Tr_g f^*\Omega)^2 + 2(63-2n)\sum_s \|f^*\Theta_s\|^2 \\ &+ 352\|f^*\Omega\|^2 - 8\sum_{i,j,s} f^*(\Omega_s \boxtimes \Theta_s)(e_i, e_j)(Tr_h F) \\ &+ 24e(f)(Tr_g f^*\Omega)(Tr_h F) + 2(63-2n)\sum_s \|f^*\Omega_s\|^2\Big] dv_g \\ &+ \frac{1}{6}\int_M \left[2(n+16)e(f) + (Tr_h F)(Tr_g f^*\Omega)\right]\tau_g\, dv_g. \end{aligned}$$

Corollary 4.2. *Let f and f' be harmonic maps of a Riemannian manifold (M, g) of constant scalar curvature into $N = N^{n_1}(\lambda) \times N^{n_2}(\lambda)$ with $n_1 = n_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we obtain*
(i) $E(f) = E(f')$.

(ii) $\int_M [8(3n+88)e(f)^2 + 352\|f^*h\|^2 + 2(3n+88)(Tr_g f^*\Omega)^2$
$+2(63-2n)\sum_s \|f^*\Theta_s\|^2 + 352\|f^*\Omega\|^2 + 2(63-2n)\sum_s \|f^*\Omega_s\|^2]\, dv_g$
$= \int_M [8(3n+88)e(f')^2 + 352\|f'^*h\|^2 + 2(3n+88)(Tr_g f'^*\Omega)^2$
$+2(63-2n)\sum_s \|f'^*\Theta_s\|^2 + 352\|f'^*\Omega\|^2 + 2(63-2n)\sum_s \|f'^*\Omega_s\|^2]\, dv_g$

Proof. Since $n_1 = n_2$, $Tr_h F = 0$. Hence (i) follows from (4.5) and (ii) follows from (i) and (4.6), respectively.

Corollary 4.3. *Let f and f' be isometric minimal immersions of (M,g) into $N = N^{n_1}(\lambda)\times N^{n_2}(\lambda)$ with $n_1 = n_2$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then we have*

(4.7)
$$\int_M [2(3n+88)(Tr_g f^*\Omega)^2 + 2(63-2n)\sum_s \|f^*\Theta_s\|^2 + 352\|f^*\Omega\|^2 + 2(63-2n)\sum_s \|f^*\Omega_s\|^2]\, dv_g$$
$$= \int_M [2(3n+88)(Tr_g f'^*\Omega)^2 + 2(63-2n)\sum_s \|f'^*\Theta_s\|^2 + 352\|f'^*\Omega\|^2 + 2(63-2n)\sum_s \|f'^*\Omega_s\|^2]\, dv_g.$$

Proof. Note that $Tr_h F = 0$, $e(f) = \frac{m}{2}$ and $\|f^*h\|^2 = m$. Then (4.7) follows from (ii).

Now we prepare the following lemma for later use.

Lemma 4.4. *Let f be an isometric immersion of a compact Riemannian manifold (M,g) into an almost quaternionic manifold (N,h). Then we have the inequality*

(4.8)
$$0 \leq \int_M \sum_s \|f^*\Omega_s\|^2\, dv_g \leq 3dim(M)\, Vol(M,g).$$

Moreover,

(i) the equality $\int_M \sum_s \|f^*\Omega_s\|^2\, dv_g = 0$ *holds if and only if the immersion* f *is totally real, and*

(ii) the equality $\int_M \sum_s \|f^*\Omega_s\|^2\, dv_g = 3\dim(M)Vol(M,g)$ *holds if and only if the immersion* f *is invariant.*

Proof. The proof is similar to that of Lemma 6.4([13]).

Proposition 4.5. *Let* f *and* f' *be* F*-anti-invariant minimal immersions of* (M,g) *into* $QP^n \times QP^n$*. Assume that* $Spec(J_f) = Spec(J_{f'})$*. Then*

(i) if f *is a totally real immersion, then so is* f'*, and*

(ii) if f *is an invariant immersion, then so is* f'*.*

Proof. Note that $\|f^*\Omega\|^2 = m = \|f^*h\|^2$ and $\|f'^*\Omega\|^2 = m = \|f'^*h\|^2$. Since f and f' are F-anti-invariant immersions, $\|f^*\Omega\| = 0 = Tr_g f^*\Omega$. From this and (4.7), we get

$$\text{(4.9)} \qquad \int_M [\sum_s \|f^*\Theta_s\|^2 + \sum_s \|f^*\Omega_s\|^2]\, dv_g = \int_M [\sum_s \|f'^*\Theta_s\|^2 + \sum_s \|f'^*\Omega_s\|^2]\, dv_g.$$

Assume that f is a totally real immersion. Then we have $\sum_s \|f^*\Theta_s\|^2 = 0 = \sum_s \|f^*\Omega_s\|^2$. Hence the equation (4.9) implies that $\sum_s \|f'^*\Theta_s\|^2 = 0 = \sum_s \|f'^*\Omega_s\|^2$. Then Lemma 4.4 implies that f' is a totally real immersion.

Next, assume that f is an invariant immersion. Then we have $\sum_s \|f^*\Theta_s\|^2 = 3m = \|f^*\Omega\|^2$. From (4.9) we obtain

$$0 = \int_M [(3m - \sum_s \|f'^*\Omega_s\|^2) + (3m - \sum_s \|f'^*\Theta_s\|^2)]\, dv_g.$$

This and (4.8) give $\sum_s \|f'^*\Omega_s\|^2 = 3m$. Hence Lemma 4.4 shows that f' is also an invariant immersion.

5. Proof of Main Theorem

To proceed to the proof of main theorem we need the notion of harmonic morphisms (for details, see [2,7]).

A smooth map $f : (M, g) \longrightarrow (N, h)$ is a *harmonic morphism* if $\nu \circ f$ is a harmonic function in $f^{-1}(V)$ for every function ν which is harmonic in an open set $V \subset N$ such that $f^{-1}(V) \neq \phi$.

A smooth map $f : (M, g) \longrightarrow (N, h)$ is *horizontally weakly conformal* if (i) $f_{*x} : T_x M \longrightarrow T_{f(x)} N$ is surjective at each point x with $e(f)(x) \neq 0$, and (ii) there exists a smooth function λ on M such that for each $x \in M$ with $e(f)(x) \neq 0$, $f^*h(X, Y) = \lambda^2(x) g(X, Y)$ for $X, Y \in H_x$, where H_x is the orthogonal complement of $\mathrm{Ker} f_*$ with respect to $g_x, x \in M$.

Lemma 5.1 [2,7]. *(i) if* $\dim(M) < \dim(N)$*, then every harmonic morphism is constant.*

(ii) If $\dim(M) \geq \dim(N)$*, then a smooth map* $f : (M, g) \longrightarrow (N, h)$ *is a harmonic morphism if and only if* f *is horizontally weakly conformal and harmonic.*

It is known (cf.[2]) that the set $M^* := \{x \in M : e(f)(x) \neq 0\}$ is open and dense in M, the function λ^2 is given by $\lambda^2 = 2e(f)\dim(N)^{-1}$, and $\|f^*h\|^2 = \dim(N)\lambda^4$. A smooth map $f : (M, g) \longrightarrow (N, h)$ is a *Riemannian submersion* if it is horizontally weakly conformal with $\lambda = 1$ on M.

Proof of Main Theorem. It is sufficient to show that the function λ^2 for f' satisfies $\lambda^2 = 1$ everywhere on M. Note that $e(f') = n\lambda^2$ and $\|f'^*h\|^2 = 2n\lambda^4$, where n is of quaternionic dimension.

First of all, we show that if f is a harmonic morphism of (M, g) into $(QP^n \times QP^n, h)$, then

$$\|Tr_g f^*\Omega\|^2 = \lambda^4 (Tr\tilde{F})^2, \qquad (5.1)$$

$$\|f^*\Omega\|^2 = \|f^*h\|^2 \text{ on } M^*.$$

In fact, at each point $x \in M^*$, we can define a linear transformation $\tilde{F}$ of H_x into itself such that $F \circ f_* = f_* \circ \tilde{F}$. Then

$$\tilde{F}^2 = I, g(\tilde{F}X, \tilde{F}Y) = g(X, Y), \; X, Y \in H_x.$$

Taking an orthonormal basis $\{e_a; a = 1, ..., 2n\}$ of (H_x, g_x), we obtain

$$
\begin{aligned}
(Tr_g f^*\Omega)^2 &= \Big[\sum_{a=1}^{2n} h(f_*e_a, Ff_*e_a)\Big]^2 = \Big[\sum_{a=1}^{2n} h(f_*e_a, f_*\tilde{F}e_a)\Big]^2 \\
&= \Big[\sum_{a=1}^{2n} \lambda^2 g(e_a, \tilde{F}e_a)\Big]^2 =: \lambda^4 (Tr\tilde{F})^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|f^*\Omega\|^2 &= \sum_{a,b=1}^{2n} h(f_*e_a, Ff_*e_b)^2 = \sum_{a,b=1}^{2n} h(f_*e_a, f_*\tilde{F}e_b)^2 \\
&= 2n\lambda^4 = \|f^*h\|^2,
\end{aligned}
$$

where $Tr\tilde{F}$ is constant on M^*.

Next, at each point $x \in M^*$, we define a linear transformation $\tilde{\theta}_s$ of H_x into itself such that $\theta_s \circ f_* = f_* \circ \tilde{\theta}_s$. Then we obtain

$$
\begin{aligned}
&\tilde{\theta}_s^2 = -I (s = 1, 2, 3), \tilde{\theta}_1 \circ \tilde{\theta}_2 = -\tilde{\theta}_2 \circ \tilde{\theta}_1 = \tilde{\theta}_3 \\
&\tilde{\theta}_2 \circ \tilde{\theta}_3 = -\tilde{\theta}_3 \circ \tilde{\theta}_2 = \tilde{\theta}_1, \tilde{\theta}_3 \circ \tilde{\theta}_1 = -\tilde{\theta}_1 \circ \tilde{\theta}_3 = \tilde{\theta}_2, \\
&g(\tilde{\theta}_s X, \tilde{\theta}_s Y) = g(X, Y) \text{ and } g(\tilde{\theta}_s X, X) = 0, X, Y \in H_x.
\end{aligned}
$$

Also we get

$$
(5.2) \qquad \|f^*\Theta_s\|^2 = \|f^*\Omega_s\|^2 = \|f^*h\|^2 = 2n\lambda^4.
$$

Now, let f and f' be harmonic morphisms (M, g) into $(QP^n \times QP^n, h)$ with $Spec(J_f) = Spec(J_{f'})$. Using (5.1) and (5.2), we have from Corollary 4.2

(i′) $E(f) = E(f')$

and

(ii′) $\int_M \{8(3n+88)e(f)^2 + 4(365-6n)\|f^*h\|^2 + 2(3n+88)(Tr_g f^*\Omega)^2\}\, dv_g$
$= \int_M \{8(3n+88)e(f')^2 + 4(365-6n)\|f'^*h\|^2 + 2(3n+88)(Tr_g f'^*\Omega)^2\}\, dv_g$.

If f is a Riemannian submersion, then $e(f) = n$ and $\|f^*h\|^2 = 2n$. Hence (i′) is equivalent to $\int_M \lambda^2\, dv_g = \int_M dv_g$, and (ii′) is equivalent to $\int_M \lambda^4 \, dv_g = \int_M dv_g$. Therefore we get $\lambda^2 = 1$ everywhere on M by the Cauchy-Schwarz inequality. Thus we complete the proof.

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