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Leaf space of a certain Hopf r-foliation

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Abstract

The Hopf r-foliation $\mathcal{F}^r$ on S^3 is a generalization of the classical Hopf fibration of S^3. When r is an integer and is greater than 1, we describe the leaf space $S^3/\mathcal{F}^r$ of the Hopf r-foliation as a surface of revolution $(\mathbf{S}_r, ds^2_{\mathbf{S}_r})$ in $(R^3, ds^2_{R^3})$. Then the natural projection $\tilde{p} : (S^3, ds^2_{S^3}) \longrightarrow (\mathbf{S}_r, ds^2_{\mathbf{S}_r})$ becomes a C^∞ Riemannian V-submersion.

1 Introduction

For a given positive number r, we consider a foliation $\mathcal{F}^r$ defined on the unit 3-sphere S^3 whose leaves are given by the flow

$$\gamma_t^r(z, w) = (e^{irt}z, e^{it}w), \qquad (z, w) \in S^3, \quad t \in R$$

on $S^3 \subset \mathbf{C}^2$([1,10]). We call $\mathcal{F}^r$ the Hopf r-foliation on S^3 ([10]). It should be remark that the Hopf 1-foliation $\mathcal{F}^1$ on S^3 is the one given by the classical Hopf fiberation of S^3. If r is a rational number, then each leaf of $\mathcal{F}^r$ is closed and the canonical metric $ds^2_{S^3}$ on S^3 is a bundle-like metric with respect to $\mathcal{F}^r$. Thus the leaf space $S^3/\mathcal{F}^r$ becomes a C^∞ Riemannian V-manifold([7,8]). See Satake[9] for the notion of V-manifolds. When r is an integer and is greater than 1, we can realize the leaf space $S^3/\mathcal{F}^r$ as a surface of revolution $(\mathbf{S}_r, ds^2_{\mathbf{S}_r})$ in a Euclidean 3-space R^3,where $ds^2_{\mathbf{S}_r}$ is the metric induced from the canonical metric $ds^2_{R^3}$ on R^3. A parametrization of the surface $\mathbf{S}_r$ is given explicitly in section 3. Consequently, the natural projection $p : S^3 \longrightarrow S^3/\mathcal{F}^r$ induces a mapping $\tilde{p} : S^3 \longrightarrow \mathbf{S}_r$. Then our main theorem in this paper is

Theorem. *Let r be an integer and suppose $r > 1$. Let $\mathcal{F}^r$ be the Hopf r-foliation on S^3. Then the leaf space $S^3/\mathcal{F}^r$ is homeomorphic to the surface of revolution $\mathbf{S}_r$ in R^3, and the mapping $\tilde{p} : (S^3, ds^2_{S^3}) \longrightarrow (\mathbf{S}_r, ds^2_{\mathbf{S}_r})$ is a C^∞ Riemannian V-submersion.*

When r is a positive rational number and is not an integer, we can also construct a surface of revolution $(\hat{\mathbf{S}}_r, ds^2_{\hat{\mathbf{S}}_r})$ and obtain a C^∞ V-submersion $\hat{p} : S^3 \longrightarrow \hat{\mathbf{S}}_r$. However, $\hat{p} : (S^3, ds^2_{S^3}) \longrightarrow (\hat{\mathbf{S}}_r, ds^2_{\hat{\mathbf{S}}_r})$ is not a C^∞ Riemannian V-submersion (Remark in section 4).

We shall work in C^∞ category. The author would like to thank Professor S. Nishikawa for his helpful remarks. The author would like to thank the referee for the elimination for errors.

2 Hopf r-foliation

The unit 3-sphere S^3 in R^4 is regarded as

$$S^3 = \{(z,w) \in \mathrm{C}^2 \mid |z|^2 + |w|^2 = 1\},$$

where $|z|^2 = z \cdot \bar{z}$, $\bar{z}$ beimg the complex conjugate of z. For a fixed positive number r, a one-dimensional foliation $\mathcal{F}^r$ on S^3 is defined by the flow

$$\gamma_t^r(z,w) = (e^{irt}z, e^{it}w), \qquad (z,w) \in S^3, \quad t \in R,$$

that is, the leaf of $\mathcal{F}^r$ through the point $(z,w) \in S^3$ is the orbit $\{\gamma_t^r(z,w) \mid t \in R\}$ of γ_t^r. Since the classical Hopf fibration of S^3 is regarded as the foliation $\mathcal{F}^1$ (the case of $r = 1$), we call $\mathcal{F}^r$ the Hopf r-foliation on S^3([10]). Each foliation $\mathcal{F}^r$ has two special leaves $T_0 = \{\gamma_t^r(0,1) \mid t \in R\}$ and $T_1 = \{\gamma_t^r(1,0) \mid t \in R\}$, which are great circles in S^3. Regarding the stucture of $\mathcal{F}^r$, we have the following facts:

(F.1) If $r \neq 1$, then the foliation $\mathcal{F}^r$ is not regular([1,5,7,8,10]).

(F.2) With respect to the canonical metric $ds^2_{S^3}$ on S^3, the vector field Z^r generating the flow γ_t^r is a Killing vector field on S^3([1,10]).

(F.3) The foliation $\mathcal{F}^r$ is a Riemannian foliation, and the metric $ds^2_{S^3}$ is a bundle-like metric with respect to $\mathcal{F}^r$([3,7]).

(F.4) If r is a rational number, then the leaves of $\mathcal{F}^r$ are closed, and the leaf space $S^3/\mathcal{F}^r$ is a C^∞ Riemannian V-manifold([4,8]).

The vector field Z^r generating the flow γ_t^r is given by

$$Z^r_{(z,w)} = (irz, iw), \qquad (z,w) \in S^3.$$

We consider two vector fields X and Y on S^3 defined by

$$\begin{aligned} X_{(z,w)} &= (|w|^2 z, -|z|^2 w), \\ Y_{(z,w)} &= (i|w|^2 z, -ir|z|^2 w). \end{aligned}$$

Remark that X and Y vanish on T_0 and T_1 and that ,for example, the vector field Y has the expression in the natural coordinates (x_1, x_2, x_3, x_4) of R^4 as follows:

$$\begin{aligned} Y_{(z,w)} &= Y_{(x_1,x_2,x_3,x_4)} \\ &= \left((x_3)^2 + (x_4)^2\right)\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) - r\left((x_1)^2 + (x_2)^2\right)\left(-x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}\right). \end{aligned}$$

The following lemmas are easily proved.

Lemma 1 *It holds that*

$$\|X_{(z,w)}\|^2 = |z|^2|w|^2,$$
$$\|Y_{(z,w)}\|^2 = |z|^2|w|^2(r^2|z|^2 + |w|^2).$$

Lemma 2 *The vector fields X and Y on S^3 are infinisetimal automorphisms of $\mathcal{F}^r$.*

Lemma 3 *The vector fields X, Y and Z^r are orthogonal to each other on $S^3 \setminus \{T_0, T_1\}$.*

The sectional curvature for the plane spanned by X and Y is called the basic Riemannian sectional curvature of $\mathcal{F}^r$, which is regarded as a real-valued function K_r on $S^3 \setminus \{T_0, T_1\}$, since $\mathcal{F}^r$ is of codimension 2. The following lemma was proved in [1].

Lemma 4 *For any $(z, w) \in S^3 \setminus \{T_0, T_1\}$, it holds*

$$K_r(z,w) = 1 + \frac{3r^2}{(r^2|z|^2 + |w|^2)^2}.$$

We regard the unit circle S^1 as the quotient set $R/2\pi Z$. A parametrization of S^3 is then given by the mapping

$$\mathbf{x} : (0,1) \times S^1 \times S^1 \longrightarrow S^3 \subset \mathbf{C}^2,$$

where $\mathbf{x}(u, \theta_1, \theta_2)$ is defined by

$$\mathbf{x}(u, \theta_1, \theta_2) = (ue^{i\theta_1}, \sqrt{1-u^2}e^{i\theta_2}). \tag{1}$$

We set $\mathbf{x}(0, \theta_1, \theta_2) = (0, e^{i\theta_2})$ and $\mathbf{x}(1, \theta_1, \theta_2) = (e^{i\theta_1}, 0)$. Then we have a mapping $\mathbf{x} : [0,1] \times S^1 \times S^1 \longrightarrow S^3$. We notice that

$$\mathbf{x}(\{0\} \times S^1 \times S^1) = T_0,$$
$$\mathbf{x}(\{1\} \times S^1 \times S^1) = T_1,$$

and the mapping $\mathbf{x} \mid_{(0,1)\times S^1 \times S^1}$ restricted on $(0,1) \times S^1 \times S^1$ is a diffeomorphism from $(0,1) \times S^1 \times S^1$ to $S^3 \setminus \{T_0, T_1\}$. The foliation $\mathcal{F}^r$ on S^3 induces a foliation F^r on $(0,1) \times S^1 \times S^1$ via the mapping $\mathbf{x} \mid_{(0,1)\times S^1 \times S^1}$. The metric $ds^2_{S^3}$ is given by

$$ds^2_{S^3} = (1-u^2)^{-1}(du)^2 + u^2(d\theta_1)^2 + (1-u^2)(d\theta_2)^2 \tag{2}$$

on $\mathbf{x}((0,1) \times S^1 \times S^1) = S^3 \setminus \{T_0, T_1\}$.

In terms of the parametrization (1) of S^3, we have the following expression of the vector fields X, Y and Z^r :

$$X_{\mathbf{x}(u,\theta_1,\theta_2)} = u(1-u^2)\mathbf{x}_*\left(\frac{\partial}{\partial u}\right),$$
$$Y_{\mathbf{x}(u,\theta_1,\theta_2)} = (1-u^2)\mathbf{x}_*\left(\frac{\partial}{\partial \theta_1}\right) - ru^2\mathbf{x}_*\left(\frac{\partial}{\partial \theta_2}\right),$$
$$Z^r_{\mathbf{x}(u,\theta_1,\theta_2)} = r\mathbf{x}_*\left(\frac{\partial}{\partial \theta_1}\right) + \mathbf{x}_*\left(\frac{\partial}{\partial \theta_2}\right)$$

on $\mathbf{x}((0,1)\times S^1\times S^1)=S^3\setminus\{T_0,T_1\}$, where $\mathbf{x}_*$ denotes the differential of the mapping $\mathbf{x}$.

Let L^r denote the tangent bundle and Q^r the normal bundle of $\mathcal{F}^r$. Let $V(\mathcal{F}^r)$ denote the set of all infinitesimal automorphisms of $\mathcal{F}^r$. By the fact (F.3) , the normal bundle Q^r of $\mathcal{F}^r$ is identified with the orthogonal complement $(L^r)^\perp$ of the tangent bundle L^r of $\mathcal{F}^r$, and the Riemannian metric induces the holonomy invariant metric g_{Q^r} on the normal bundle Q^r . Thus we can define the following notion. Let $\Pi:\Gamma(TS^3)\longrightarrow\Gamma(Q^r)$ be a projection, where $\Gamma(TS^3)$ denotes the set of all sections of the tangent bundle of S^3, and $\Gamma(Q^r)$ the set of all sections of the normal bundle of $\mathcal{F}^r$. Then the set

$$\overline{V}(\mathcal{F}^r)=\{\Pi(W)\in\Gamma(Q^r)\mid W\in V(\mathcal{F}^r)\subset\Gamma(TS^3)\}$$

gives rise to the set of all transversal infinitesimal automorphisms of $\mathcal{F}^r$. Among $\overline{V}(\mathcal{F}^r)$ we have the set of transversal Killing field of $\mathcal{F}^r$, that is , $\Pi(W)\in\overline{V}(\mathcal{F}^r)$ is a transversal Killing field of $\mathcal{F}^r$ if $\Pi(W)$ satisfies $\Theta(W)g_{Q^r}=0$. Here $\Theta(W)$ denotes the transversal Lie derivative operator with respect to $\Pi(W)$ (See [2,3,5] for details). The following theorem was proved in [6].

Theorem 5 *For the vector field Y on S^3, a transversal infinitesimal automorphism*

$$\Pi\left(\frac{1}{r^2u^2+(1-u^2)}Y_{(z,w)}\right)$$

of $\mathcal{F}^r$ is a transversal Killing field of $\mathcal{F}^r$.

This is proved by direct calculation of $\Theta\left(\frac{1}{r^2u^2+(1-u^2)}Y_{(z,w)}\right)g_{Q^r}$.

3 A surface of revolution

Roughly speaking, the basic Riemannian sectional curvature of $\mathcal{F}^r$ corresponds with the "Gaussian curvature" of the leaf space $S^3/\mathcal{F}^r$ (This leaf space is a Riemannian V-manifold. See (F.4) in section 2) . Thus, if we can construct a surface with corresponding Gaussian curvature to the curvature in Lemm 4, we may describe the leaf space $S^3/\mathcal{F}^r$ as the surface. We construct the surface as a surface of revolution.

Let r be a fixed real number and suppose $r\geq 1$. We define a function f on $[0,1]$ by

$$f(u)=u(1-u^2)^{1/2}(r^2u^2+(1-u^2))^{-1/2}. \tag{3}$$

Then f is of class C^∞ on $(0,1)$, and the first derivative f' of f on $(0,1)$ is given by

$$f'(u)=(1-2u^2-(r^2-1)u^4)(r^2u^2+(1-u^2))^{-3/2}(1-u^2)^{-1/2}.$$

We notice that $f(0)=f(1)=0$,f has the maximum value $(r+1)^{-1}$ at $u=(r+1)^{-1/2}$, and

$$\lim_{h\to+0}\frac{f(h)-f(0)}{h}=1,$$

$$\lim_{k\to -0}\frac{f(1+k)-f(1)}{k}=-\infty.$$

Then we have

$$0<(1-u^2)^{-1}-(f'(u))^2<(1-u^2)^{-1}, \qquad u\in(0,1)$$

and

$$\lim_{u\to +0}((1-u^2)^{-1}-(f'(u))^2)=0,$$
$$\lim_{u\to 1-0}((1-u^2)^{-1}-(f'(u))^2)=+\infty.$$

Since the improper integral

$$\int_0^1\sqrt{\frac{1}{1-u^2}}\,du$$

converges, so does the improper integral

$$\int_0^1\sqrt{\frac{1}{1-u^2}-(f'(u))^2}\,du.$$

Thus we can define a function g on $[0,1]$ by

$$g(u)=\int_0^u\sqrt{\frac{1}{1-s^2}-(f'(s))^2}\,ds. \tag{4}$$

Then we have

$$0=g(0)<g(u)<g(1), \qquad u\in(0,1).$$

We set $g_*=g(1)$, the maximum value of g. The function g is of class C^∞ on $(0,1)$ and the first derivative g' of g on $(0,1)$ is given by

$$g'(u)=((1-u^2)^{-1}-(f'(u))^2)^{1/2}.$$

Now, we construct a surface of revolution S_r in the Euclidean (x_1,x_2,x_3)-space R^3. The profile curve C of S_r in (x_1,x_3)-plane is defined by

$$\begin{cases} x_1=f(u)\\ x_3=g(u)\end{cases}$$

for $u\in[0,1]$,where f and g are functions defined by (3) and (4), respectively. Since we have

$$\lim_{u\to +0}\frac{g'(u)}{f'(u)}=\lim_{u\to +0}\left(\frac{1}{(1-u^2)(f'(u))^2}-1\right)^{1/2}=0,$$

the profile curve C is perpendicular to the x_3-axis at the origin in (x_1,x_3)-plane. We also have

$$\lim_{u\to 1-0}(1-u^2)(f'(u))^2=\lim_{u\to 1-0}\frac{(1-2u^2-(r^2-1)u^4)^2}{(r^2u^2+(1-u^2))^3}=r^{-2}.$$

By the above facts and

$$\lim_{u\to 1-0} f'(u) = -\infty,$$

we have

$$\lim_{u\to 1-0} \frac{g'(u)}{f'(u)} = \lim_{u\to 1-0} (-1)\left(\frac{1}{(1-u^2)(f'(u))^2} - 1\right)^{1/2} = -(r^2-1)^{1/2}.$$

Thus the angle θ between the curve C and the x_3-axis at the point $(0, g_*)$ is given by

$$\tan\theta = (r^2-1)^{-1/2}.$$

Remark. If $r = 1$, then we have

$$\begin{cases} x_1 = f(u) = u(1-u^2)^{1/2} \\ x_3 = g(u) = u^2 \end{cases}$$

for $u \in [0,1]$. Thus the profile curve C is a half circle ($(x_1)^2 + (x_3 - 1/2)^2 = 1/4$ and $x_1 \geq 0$) so that $\mathbf{S}_1$ is a sphere of radius 1/2.

A parametrization of $\mathbf{S}_r$ is given by the mapping

$$\mathbf{y} : (0,1) \times S^1 \longrightarrow \mathbf{S}_r \subset R^3,$$

where $\mathbf{y}(u,\tau)$ is defined by

$$\mathbf{y}(u,\tau) = (f(u)\cos\tau, f(u)\sin\tau, g(u)). \tag{5}$$

Setting $\mathbf{y}(0,\tau) = (0,0,0)$ and $\mathbf{y}(1,\tau) = (0,0,g_*)$, we have a mapping $\mathbf{y} : [0,1] \times S^1 \longrightarrow \mathbf{S}_r$. The mapping $\mathbf{y}\ |_{(0,1)\times S^1}$ restricted on $(0,1) \times S^1$ is a diffeomorphism from $(0,1) \times S^1$ to $\mathbf{S}_r \setminus \{(0,0,0),(0,0,g_*)\}$. We notice that $\mathbf{y}(\{0\} \times S^1) = (0,0,0)$ and $\mathbf{y}(\{1\} \times S^1) = (0,0,g_*)$. It follows from the above facts that $\mathbf{S}_r$ is a surface of class C^0 and $\mathbf{S}_r \setminus \{(0,0,g_*)\}$ is of class C^∞. The metric $ds^2_{\mathbf{S}_r}$ on $\mathbf{S}_r$ induced from the canonical metric $ds^2_{R^3}$ on R^3 is given by

$$ds^2_{\mathbf{S}_r} = (1-u^2)^{-1}(du)^2 + u^2(1-u^2)\{r^2u^2 + (1-u^2)\}^{-1}(d\tau)^2 \tag{6}$$

on $\mathbf{y}((0,1) \times S^1) = \mathbf{S}_r \setminus \{(0,0,0),(0,0,g_*)\}$.

Lemma 6 *The Gaussian curvature K of $\mathbf{S}_r$ is given by*

$$K(u,\tau) = 1 + \frac{3r^2}{(r^2u^2 + (1-u^2))^2}$$

on $\mathbf{y}((0,1) \times S^1) = \mathbf{S}_r \setminus \{(0,0,0),(0,0,g_*)\}$.

Proof. Since the surface of revolution S_r defined by

$$\mathbf{y}(u,\tau) = (f(u)\cos\tau, f(u)\sin\tau, g(u)),$$

the Gaussian curvature K of S_r has the following expression:

$$K(u,\tau) = \frac{(f'(u)g''(u) - f''(u)g'(u))g'(u)}{f(u)((f'(u))^2 + (g'(u))^2)^2}.$$

From the equality $g'(u) = ((1-u^2)^{-1} - (f'(u))^2)^{1/2}$, we have

$$(f'(u)g''(u) - f''(u)g'(u))g'(u) = u(1-u^2)^{-2}f'(u) - (1-u^2)^{-1}f''(u).$$

Thus we see that $K(u,\tau) = u(f(u))^{-1}f'(u) - (1-u^2)(f(u))^{-1}f''(u)$. Now, by the definition of f, we have

$$\begin{aligned}
&u(f(u))^{-1}f'(u) - (1-u^2)(f(u))^{-1}f''(u)\\
&= (r^2u^2 + (1-u^2))^{-2}(1-u^2)^{-1}\\
&\quad \times\{(r^2u^2 + (1-u^2))(1-2u^2-(r^2-1)u^4) - r^2(-3+2u^2-(r^2-1)u^4)\}\\
&= (r^2u^2 + (1-u^2))^{-2}(1-u^2)^{-1}\{(r^2u^2+(1-u^2))^2(1-u^2) + 3r^2(1-u^2)\}.
\end{aligned}$$

Hence we have $K(u,\tau) = 1 + \dfrac{3r^2}{(r^2u^2 + (1-u^2))^2}$. ■

4 Leaf space

In this section, we assume that r is an integer and is greater than 1. We fix r and the Hopf r-foliation $\mathcal{F}^r$on S^3. By identifying each leaf of $\mathcal{F}^r$ to a point, we then obtain the quotient space $S^3/\mathcal{F}^r$ formed from S^3 , which is called the leaf space of the foliation $\mathcal{F}^r$ on S^3. Let $p: S^3 \longrightarrow S^3/\mathcal{F}^r$ be the identification mapping. Since all leaves of $\mathcal{F}^r$ are closed and $ds^2_{S^3}$ is a bundle-like metric with respect to $\mathcal{F}^r$, the holonomy group $H(L)$ of any leaf L of $\mathcal{F}^r$ is a finite group and $S^3/\mathcal{F}^r$ is a connected metric space([7,8]). Then $S^3/\mathcal{F}^r$ is a C^∞ Riemannian V-manifold and the mapping $p : S^3 \longrightarrow S^3/\mathcal{F}^r$ is a C^∞ Riemannian V-submersion. The notion of Riemannian V-submersion is a version of Riemannian submersion in the theory of V-manifold ([4,7,8], see [9] for the V-manifold category).

The holonomy group $H(T_1)$ of the leaf T_1 is a cyclic group of order r, and $H(T_0)$ is trivial.

The action of $H(T_1)$ on a flat neighborhood([7,8]) of $(1,0) \in T_1 \subset S^3$ induces the action of a finite group of rotations

$$G = \left\{ \begin{pmatrix} \cos 2\pi/r & -\sin 2\pi/r \\ \sin 2\pi/r & \cos 2\pi/r \end{pmatrix}^m \mid m = 0,1,2,\ldots,r-1 \right\}$$

on

$$U_\varepsilon = \{(x_3, x_4) \in R^2 \mid (x_3)^2 + (x_4)^2 < \varepsilon^2\}.$$

Thus an open neighborhood U of $p(T_1)$ in $S^3/\mathcal{F}^r$ is homeomorphic to the quotient space U_ε/G of the open disk in R^2 by G. The space U_ε/G is a cone with the angle θ between the axis and the generating line. Here θ satisfies the equation: $\sin\theta = r^{-1}$, that is, $\tan\theta = (r^2-1)^{-1/2}$. Since the action of $H(T_0)$ on a neighborhood of $(0,1) \in T_0 \subset S^3$ is trivial, an open neighborhood V of $p(T_0)$ in $S^3/\mathcal{F}^r$ is homeomorphic to an open disk

$$V_\varepsilon = \{(x_1, x_2) \in R^2 \mid (x_1)^2 + (x_2)^2 < \varepsilon^2\}.$$

Now we consider a mapping

$$\mathbf{j} : (0,1) \times S^1 \times S^1 \longrightarrow (0,1) \times S^1,$$

where $\mathbf{j}(u, \theta_1, \theta_2)$ is defined by

(7) $$\mathbf{j}(u, \theta_1, \theta_2) = (u, \theta_1 - r\theta_2).$$

Lemma 7 *The mapping* $\mathbf{j}$ *is surjective.*

Proof. Take an element θ_1 of S^1. For any $(u,\tau) \in (0,1) \times S^1$, we have a real number $(\theta_1 - \tau)/r$. Then there exists an element θ_2 of S^1 satisfying

$$\theta_2 \equiv (\theta_1 - \tau)/r \qquad (mod\, 2\pi).$$

Thus, there exists an element (u, θ_1, θ_2) of $(0,1) \times S^1 \times S^1$ satisfying $\mathbf{j}(u, \theta_1, \theta_2) = (u, \tau)$.

Next, if we take another element $\theta_1' \in S^1$, then we have an element $\theta_2' \in S^1$ satisfying

$$\theta_2' \equiv (\theta_1' - \tau)/r \qquad (mod\, 2\pi),$$

that is, for an integer ℓ

$$\theta_2' - (\theta_1' - \tau)/r = 2\ell\pi.$$

Put $t_0 = \theta_1' - \theta_1$. Then we have

$$\theta_1' - r\theta_2' = (\theta_1 + t_0) - r\left\{\frac{1}{r}(\theta_1 + t_0 - \tau) + 2\ell\pi\right\} = \tau - 2r\ell\pi.$$

Thus we have that $\mathbf{j}(u, \theta_1', \theta_2') = (u, \tau)$. ■

Lemma 8 *If two elements* (u, θ_1, θ_2) *and* $(u, \hat\theta_1, \hat\theta_2)$ *of* $(0,1) \times S^1 \times S^1$ *satisfy*

$$\hat\theta_1 \equiv \theta_1 + rt \qquad (mod\, 2\pi)$$

$$\hat\theta_2 \equiv \theta_2 + t \qquad (mod\, 2\pi)$$

for $t \in R$, *then it holds that*

$$\mathbf{j}(u, \hat\theta_1, \hat\theta_2) = \mathbf{j}(u, \theta_1, \theta_2).$$

Proof. By the assumption, there exist two integers ℓ, k such that

$$\hat{\theta}_1 - (\theta_1 + rt) = 2\ell\pi, \qquad \hat{\theta}_2 - (\theta_2 + t) = 2k\pi.$$

Since r is an integer, we have

$$\hat{\theta}_1 - r\hat{\theta}_2 \equiv \theta_1 - r\theta_2 \qquad (mod\, 2\pi),$$

which implies that $\mathbf{j}(u, \hat{\theta}_1, \hat{\theta}_2) = \mathbf{j}(u, \theta_1, \theta_2)$. ■

By Lemma 8, we have

Lemma 9 *The mapping* $\mathbf{j}$ *maps each leaf of the foliation* F^r *on* $(0,1) \times S^1 \times S^1$ *to a point of* $(0,1) \times S^1$.

Lemma 10 *The mapping* $\mathbf{j}$ *is a submersion.*

Proof. It is obvious that the mapping $\mathbf{j}$ is of class C^∞. The Jacobi matrix of $\mathbf{j}$ at any point $(u, \theta_1, \theta_2) \in (0,1) \times S^1 \times S^1$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \end{bmatrix}.$$

Thus the mapping $\mathbf{j}$ is a submersion. ■

We set $\mathbf{j}(0, \theta_1, \theta_2) = (0, \theta_1 - r\theta_2)$ and $\mathbf{j}(1, \theta_1, \theta_2) = (1, \theta_1 - r\theta_2)$.

Let $[(z,w)]$ denote the image of $(z,w) \in S^3$ by the mapping $p : S^3 \longrightarrow S^3/\mathcal{F}^r$, that is, $p((z,w)) = [(z,w)]$. For any $[(z,w)] \in (S^3/\mathcal{F}^r) \setminus \{p(T_0), p(T_1)\}$, we have the following expression of (z,w) :

$$(z,w) = \left(ue^{i\theta_1}, \sqrt{1-u^2}e^{i\theta_2}\right) \qquad (u \neq 0, 1).$$

Thus, by (1),(5) and (7), we have

$$\begin{aligned} \mathbf{x}^{-1}(z,w) &= (u, \theta_1, \theta_2) \in (0,1) \times S^1 \times S^1, \\ \mathbf{j}\left(\mathbf{x}^{-1}(z,w)\right) &= (u, \theta_1 - r\theta_2) \in (0,1) \times S^1, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{y}\left(\mathbf{j}\left(\mathbf{x}^{-1}(z,w)\right)\right) \\ &= (f(u)\cos(\theta_1 - r\theta_2), f(u)\sin(\theta_1 - r\theta_2), g(u)) \in \mathbf{S}_r \setminus \{(0,0,0),(0,0,g_*)\}. \end{aligned}$$

If we take another element $(\hat{z}, \hat{w}) \in S^3 \setminus \{T_0, T_1\}$ satisfying $p((\hat{z}, \hat{w})) = [(z,w)]$, then there exists a real number t such that

$$\gamma_t^r(z,w) = (\hat{z}, \hat{w}),$$

that is,

$$\begin{aligned} \hat{\theta}_1 &\equiv \theta_1 + rt \qquad (mod\, 2\pi), \\ \hat{\theta}_2 &\equiv \theta_2 + t \qquad (mod\, 2\pi), \\ \hat{u} &= u, \end{aligned}$$

where $\hat{z} = \hat{u}e^{i\hat{\theta}_1}$ and $\hat{w} = \sqrt{1-u^2}e^{i\hat{\theta}_2}$. Thus we have

$$\begin{aligned} f(\hat{u}) &= f(u), \\ g(\hat{u}) &= g(u), \\ e^{i(\hat{\theta}_1 - r\hat{\theta}_2)} &= e^{i(\theta_1 - r\theta_2)}. \end{aligned}$$

Therefore, $\mathbf{y}\,(\mathbf{j}\,(\mathbf{x}^{-1}\,(p^{-1}([(z,w)]))))$ is independent of the choice of an element (z,w) in $p^{-1}([(z,w)])$.

We set that

$$\begin{aligned} \mathbf{y}\,(\mathbf{j}\,(\mathbf{x}^{-1}\,(p^{-1}([(0,w)])))) &= (0,0,0), \\ \mathbf{y}\,(\mathbf{j}\,(\mathbf{x}^{-1}\,(p^{-1}([(z,0)])))) &= (0,0,g_*) \end{aligned}$$

for any $(0,w) \in T_0$ and $(z,0) \in T_1$.

Lemma 11 *There exists a homeomorphism* $\varphi : S^3/\mathcal{F}^r \longrightarrow \mathbf{S}_r$.

Proof. We remark that $p(T_0) = [(0,1)]$ and $p(T_1) = [(1,0)]$.
For any $[(z,w)] \in (S^3/\mathcal{F}^r) \setminus \{[(0,1)],[(1,0)]\}$, we define $\varphi([(z,w)])$ by

$$\varphi([(z,w)]) = \mathbf{y}\left(\mathbf{j}\left(\mathbf{x}^{-1}\left(p^{-1}([(z,w)])\right)\right)\right).$$

Also we define $\varphi([(0,1)])$ and $\varphi([(1,0)])$ by

$$\varphi([(0,1)]) = (0,0,0),\ \varphi([(1,0)]) = (0,0,g_*).$$

Thus we have a mapping $\varphi : S^3/\mathcal{F}^r \longrightarrow \mathbf{S}_r$. By the above lemmas, it is obvious that φ is a homeomorphism. ■

Since $S^3/\mathcal{F}^r$ and $\mathbf{S}_r$ are C^∞ V-manifolds, we have the V-manifold version of the above lemma. In fact, by the notion of V-manifold mapping ([9]), we have the following

Lemma 12 *The mapping* $\varphi : S^3/\mathcal{F}^r \longrightarrow \mathbf{S}_r$ *is a bijective V-manifold mapping.*

We consider a mapping

$$\tilde{p} : S^3 \longrightarrow \mathbf{S}_r,$$

where $\tilde{p}(z,w)$ is defined by

$$\tilde{p}(z,w) = \mathbf{y}(\mathbf{j}(\mathbf{x}^{-1}(z,w))).$$

The mapping $p : S^3 \longrightarrow S^3/\mathcal{F}^r$ is a C^∞ V-submersion, and so is the mapping $\tilde{p}$. Namely, we have

Lemma 13 *The mapping $\tilde{p} : S^3 \longrightarrow \mathrm{S}_r$ is a C^∞ V-submersion.*

Now, by the parametrization (5) of S_r, we have, on $\mathbf{y}((0,1)\times S^1) = \mathrm{S}_r\backslash\{(0,0,0),(0,0,g_*)\}$,

$$\mathbf{y}_u = \mathbf{y}_*\left(\frac{\partial}{\partial u}\right) = (f'(u)\cos\tau, f'(u)\sin\tau, g'(u)),$$

$$\mathbf{y}_\tau = \mathbf{y}_*\left(\frac{\partial}{\partial \tau}\right) = (-f(u)\sin\tau, f(u)\cos\tau, 0).$$

By the proof of Lemma 10, we have

$$\mathbf{j}_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}, \quad \mathbf{j}_*\left(\frac{\partial}{\partial \theta_1}\right) = \frac{\partial}{\partial \tau}, \quad \mathbf{j}_*\left(\frac{\partial}{\partial \theta_2}\right) = -r\frac{\partial}{\partial \tau}.$$

Thus we have

$$\begin{aligned}
\tilde{p}_*(X_{(z,w)}) &= u(1-u^2)\cdot \mathbf{y}_u,\\
\tilde{p}_*(Y_{(z,w)}) &= \{r^2u^2 + (1-u^2)\}\cdot \mathbf{y}_\tau,\\
\tilde{p}_*(Z^r_{(z,w)}) &= \mathbf{o}
\end{aligned}$$

for $(z,w) = (ue^{i\theta_1}, \sqrt{1-u^2}e^{i\theta_2}) \in S^3 \quad (u \neq 0,1)$, where $\mathbf{o}$ denotes the zero vector.

Let $\|\bullet\|_{S^3}$ (resp. $\|\bullet\|_{\mathrm{S}_r}$) be the norm with respect to the metric $ds^2_{S^3}$ (resp. $ds^2_{\mathrm{S}_r}$) on S^3 (resp. S_r). We have

Lemma 14 *For infinitesimal automorphisms X and Y of $\mathcal{F}^r$ on $S^3 \setminus \{T_1\}$, it holds that*

$$\begin{aligned}
\|\tilde{p}_*(X)\|_{\mathrm{S}_r} &= \|X\|_{S^3},\\
\|\tilde{p}_*(Y)\|_{\mathrm{S}_r} &= \|Y\|_{S^3}.
\end{aligned}$$

Remark. For a transversal Killing field $\Pi\left(\frac{1}{r^2u^2+(1-u^2)}Y_{(z,w)}\right)$ of $\mathcal{F}^r$, the vector field $\tilde{p}_*\left(\frac{1}{r^2u^2+(1-u^2)}Y_{(z,w)}\right)$ on $\mathrm{S}_r \setminus \{(0,0,g_*)\}$ is a Killing vector field with respect to $ds^2_{\mathrm{S}_r}$.

By Lemmas 13 and 14, we have

Lemma 15 *The mapping $\tilde{p} : S^3 \longrightarrow \mathrm{S}_r$ is a C^∞ Riemannian V-submersion.*

Therefore, we have

Theorem 16 *Let r be an integer and greater than 1. Let $\mathcal{F}^r$ be the Hopf r-foliation on S^3. Then the leaf space $S^3/\mathcal{F}^r$ is homeomorphic to the surface of revolution S_r in R^3 given in the previous section, and the mapping $\tilde{p} : (S^3, ds^2_{S^3}) \longrightarrow (\mathrm{S}_r, ds^2_{\mathrm{S}_r})$ is a C^∞ Riemannian V-submersion.*

Remark. We suppose that r is a positive rational number q/p, where two positive integers p and q are relatively prime and $p \neq 1$. Then the action of $H(T_0)$ induces the action of group of rotations of order p on an open disk V_ϵ, and the action of $H(T_1)$ induces the action of group of rotations of order q on an open disk U_ϵ. We construct a surface of revolution $\hat{\mathrm{S}}_r$ with profile curve $\hat{C}$

$$\begin{cases} x_1 = f(u) \\ x_3 = \hat{g}(u), \end{cases}$$

where $\hat{g}$ is given by

$$\hat{g}(u) = \int_0^u \sqrt{\frac{p^2}{1-s^2} - (f'(s))^2}\, ds$$

for any $u \in [0,1]$. Then the angle θ_0 between the curve $\hat{C}$ and the x_3-axis at the point $(0,0)$ is given by

$$\tan\theta_0 = (p^2-1)^{-1/2},$$

and the angle θ_1 between the curve $\hat{C}$ and the x_3-axis at the point $(0, \hat{g}(1))$ is given by

$$\tan\theta_1 = (q^2-1)^{-1/2}.$$

For r is a positive rational number q/p (two positive integers p and q are relatively prime and $p \neq 1$), we consider a mapping

$$\hat{\mathbf{j}} : (0,1) \times S^1 \times S^1 \longrightarrow (0,1) \times S^1,$$

where $\hat{\mathbf{j}}(u, \theta_1, \theta_2)$ is defined by

$$\hat{\mathbf{j}}(u, \theta_1, \theta_2) = (u, p\theta_1 - q\theta_2).$$

And we set $\hat{\mathbf{j}}(0, \theta_1, \theta_2) = (0, p\theta_1 - q\theta_2)$ and $\hat{\mathbf{j}}(1, \theta_1, \theta_2) = (1, p\theta_1 - q\theta_2)$. Then $\hat{\mathbf{j}}$ maps each leaf of the foliation F^r on $(0,1) \times S^1 \times S^1$ to a point of $(0,1) \times S^1$ (See Lemmas 7,8,9). The Jacobi matrix of $\hat{\mathbf{j}}$ at any point $(u, \theta_1, \theta_2) \in (0,1) \times S^1 \times S^1$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & p & -q \end{bmatrix}.$$

Thus the mapping $\hat{\mathbf{j}}$ is a submersion, and we have

$$\hat{\mathbf{j}}_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}, \quad \hat{\mathbf{j}}_*\left(\frac{\partial}{\partial \theta_1}\right) = p\frac{\partial}{\partial \tau}, \quad \hat{\mathbf{j}}_*\left(\frac{\partial}{\partial \theta_2}\right) = -q\frac{\partial}{\partial \tau}.$$

A parametrization of $\hat{\mathrm{S}}_r$ is given by the mapping

$$\hat{\mathbf{y}} : (0,1) \times S^1 \longrightarrow \hat{\mathrm{S}}_r \subset R^3,$$

where $\hat{\mathbf{y}}(u, \tau)$ is defined by

$$\hat{\mathbf{y}}(u, \tau) = (f(u)\cos\tau, f(u)\sin\tau, \hat{g}(u)).$$

And we set $\hat{\mathbf{y}}(0,\tau) = (0,0,0)$ and $\hat{\mathbf{y}}(1,\tau) = (0,0,\hat{g}(1))$.

Then we consider a mapping $\hat{p} : S^3 \longrightarrow \hat{\mathrm{S}}_r$ defined by

$$\hat{p} = \hat{\mathbf{y}} \circ \hat{\mathbf{j}} \circ \mathbf{x}^{-1},$$

where $\mathbf{x}$ is a parametrization of S^3 defined in section 2. For infinitesimal automorphisms X and Y of $\mathcal{F}^r$ on $S^3 \setminus \{T_0, T_1\}$, we have

$$\hat{p}_*(X_{(z,w)}) = u(1-u^2) \cdot \hat{\mathbf{y}}_* \left(\frac{\partial}{\partial u}\right)$$
$$\hat{p}_*(Y_{(z,w)}) = \{p(1-u^2) + rqu^2\} \cdot \hat{\mathbf{y}}_* \left(\frac{\partial}{\partial \tau}\right),$$

and

$$\|\hat{p}_*(X)\|_{\hat{\mathrm{S}}_r} = \|X\|_{S^3}$$
$$\|\hat{p}_*(Y)\|_{\hat{\mathrm{S}}_r} = p\|Y\|_{S^3},$$

for $(z,w) = (ue^{i\theta_1}, \sqrt{1-u^2}e^{i\theta_2}) \in S^3 \quad (u \neq 0,1)$.

Therefore, we have a C^∞ V-submersion $\hat{p} : S^3 \longrightarrow \hat{\mathrm{S}}_r$. But, $\hat{p} : (S^3, ds^2_{S^3}) \longrightarrow (\hat{\mathrm{S}}_r, ds^2_{\hat{\mathrm{S}}_r})$ is not a C^∞ Riemannian V-submersion.

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