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ON SOME OPERATORS WHOSE PRODUCTS ARE POSITIVE

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ABSTRACT. Let A, B be bounded linear operators on $\mathcal{H}$ satisfying

$$AB \geq 0,\ A^2B \geq 0,\ AB^2 \geq 0.$$

We study the positivity of A and B under the condition $\mathrm{Ker}AB = \{0\}$ and the representation for contractions A, B using positive operators.

It is known that a bounded linear operator T on a Hilbert space $\mathcal{H}$ which satisfies $T^n \geq 0$ $(n \geq 2)$ is not necessarily positive. In fact, if T satisfies that T^2, T^3 are positive, then T can be decomposed into a direct sum of operators N and S such that $N^2 = 0, S \geq 0$ (cf. [2]). So it is clear that $T^n \geq 0$ $(n \geq 2)$ and $\mathrm{Ker}T = \{0\}$ imply the positivity of T. This result motivates the following conjecture:

For bounded linear operators A and B on $\mathcal{H}$ satisfying

$$(*) \qquad AB \geq 0,\ A^2B \geq 0 \text{ and } AB^2 \geq 0,$$

if it holds $\mathrm{Ker}AB = \{0\}$, then both A and B are positive.

We can easily see that this conjecture fails without the assumption $\mathrm{Ker}AB = \{0\}$ (see Example). As stated in the following, in many cases the above conjecture is true. But, in general, we do not know whether the assumption $(*)$ and $\mathrm{Ker}AB = \{0\}$ imply the positivity of A and B or not. So our aim is, under these assumptions, to give a sufficient condition which implies their positivity.

Throughout this paper, we assume that bounded linear operators A and B satisfy the condition $(*)$.

Lemma 1. *If $\overline{\mathrm{Ran}B} = \mathcal{H}$, then $A \geq 0$. Similarly, if $\overline{\mathrm{Ran}A^*} = \mathcal{H}$, then $B \geq 0$.*

Proof. By the assumption, we get

$$AB^2 = (AB^2)^* = B^*(AB)^* = B^*AB.$$

So we have,

$$\langle ABx|Bx\rangle = \langle B^*ABx|x\rangle = \langle AB^2x|x\rangle \geq 0,$$

for all $x \in \mathcal{H}$. Thus the condition $\overline{\mathrm{Ran}B} = \mathcal{H}$ implies $A \geq 0$.

Since $AB=(AB)^*=B^*A^*$, if we consider A and B instead of B^* and A^*, then the condition $\overline{\mathrm{Ran}A^*} = \mathcal{H}$ implies $B \geq 0$. □

We remark that the assumption $(*)$ implies the positivity of A^nB and AB^n $(n = 1, 2, \dots)$ by the similar argument in the above proof.

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Proposition 2. *If the operator AB has its bounded inverse, then A, B are positive.*

Proof. From $\mathrm{Ker}B \subset \mathrm{Ker}AB = \{0\}$, we have the injectivity of B. The relation

$$B = (AB)^{-1}AB^2 = (AB)^{-1}B^*AB,$$

says that B is similar to B^*, so we have

$$\overline{\mathrm{Ran}B} = (\mathrm{Ker}B^*)^\perp = \mathcal{H}.$$

By Lemma 1, we have $A \geq 0$.

Applying the same argument for $B^*A^* = AB$, we also have the positivity of B^*, that is, $B \geq 0$. □

We consider the case that $\mathrm{Ker}AB = \{0\}$. The positivity and commutativity of A and B follows from Lemma 1 if either A^* or B has a dense range. If $\mathcal{H}$ is finite-dimensional, then we can get $A \geq 0$ and $B \geq 0$ by the invertibility of AB. If $AB = BA$, then we can also get the positivity of A and B by the relation

$$\mathcal{H} = \overline{\mathrm{Ran}(AB)^*} = \overline{\mathrm{Ran}(BA)^*} = \overline{\mathrm{Ran}BA} = \overline{\mathrm{Ran}A^*B^*}.$$

In the case that A is hyponormal, then we have $(AB)A = A^*(AB)$ by $A(AB) = (AB)A^*$ and the Fuglede-Putnam theorem [3]. This shows that A^* has a dense range. So we have that if either A or B^* is hyponormal, then both A and B are positive.

We have seen as above, if the operators A, B hold a property related to normality or invertibility, then we can get their positivity. From this point of view, we will treat an accretive operator or a semi-Fredholm operator. We call a bounded linear operator T is accretive if $T + T^* \geq 0$. It is clear that T^* is also accretive if T is accretive.

Proposition 3. *If either A or B is accretive and* $\mathrm{Ker}AB = \{0\}$, *then both A and B are positive.*

Proof. We may assume that A is accretive. Let $f \in \mathcal{H}$ satisfy $(A + A^2B)f = 0$. By the assumption, we have

$$0 \leq \mathrm{Re}\langle Af|f\rangle = -\langle A^2Bf|f\rangle \leq 0.$$

This means $\langle A^2Bf|f\rangle = 0$, and we have $f = 0$ since $\mathrm{Ker}A^2B = \mathrm{Ker}ABA^* = \{0\}$. So $A + A^2B$ is injective. From the assumption $AB \geq 0$, we have that $A = (A + A^2B)(I + AB)^{-1}$ is injective, so we get the positivity of A and B by Lemma 1. □

We call a bounded linear operator T left (resp. right) semi-Fredholm if $\mathrm{Ran}T$ is closed and $\mathrm{Ker}T$ (resp. $\mathrm{Ker}T^*$) is finite-dimensional. We call an operator T Fredholm if T is left semi-Fredholm and right semi-Fredholm. It is known that the adjoint of a semi-Fredholm operator is also semi-Fredholm. Then we have the following result:

Theorem 4. *If A, B are semi-Fredholm operators and* $\mathrm{Ker}AB = \{0\}$, *then A, B are positive.*

Proof. By Lemma 1, it suffices to show that $\mathrm{Ran}B = \mathcal{H}$. For any $g \in (\mathrm{Ran}B)^\perp$ there exists $h \in \mathrm{Ran}B$ such that $g = B^*h$, since $\mathrm{Ran}B$ is closed and $\mathrm{Ker}B = \{0\}$. Then we have, for any $f \in \mathcal{H}$,

$$0 = \langle Bf|g\rangle = \langle Bf|B^*h\rangle = \langle B^2f|h\rangle,$$

so we get $h \in \mathrm{Ran}B \cap (\mathrm{Ran}B^2)^\perp$. Since $\mathrm{Ran}B \cap \mathrm{Ker}A = \{0\}$, we have $\mathrm{Ran}B \subset \mathrm{Ran}A^*$. So there exists $k \in \mathcal{H}$ such that $h = A^*k$. Then we have

$$0 = \langle B^2f|h\rangle = \langle B^2f|A^*k\rangle = \langle AB^2f|k\rangle.$$

From the fact

$$\overline{\mathrm{Ran}AB^2} = (\mathrm{Ker}AB^2)^\perp = \mathcal{H},$$

we get $k = 0$, $h = 0$ and $g = 0$. This means $\mathrm{Ran}B = \mathcal{H}$. $\square$

Let P be the orthogonal projection onto $(\mathrm{Ker}AB)^\perp$. In the rest of this paper, we only compute A^nB and AB^n $(n = 1, 2, \dots)$, so we may assume that A and B have the following form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & 0 \end{pmatrix}$$

with respect to the decomposition $(\mathrm{Ker}AB)^\perp \oplus (\mathrm{Ker}AB)$. Clearly we have $AB = ABP = PAB$. Then we can show that A_{11}^* and B_{11} are positive with respect to the inner product induced by AB.

Theorem 5. *For bounded linear operators A, B, there exist a pair of positive operators D_1, D_2 on a Hilbert space $\mathcal{K}$ and a bounded linear operator V from $\mathcal{H}$ to $\mathcal{K}$ such that*

$$A^{n+1}B^{m+1} = V^*D_1^nD_2^mV \text{ for all } n, m \in \mathbb{N} \cup \{0\}.$$

Moreover, if $A^2B^2 \geq 0$, then we have $D_1D_2 = D_2D_1$.

Proof. We define a new inner product $(\,\cdot\,|\,\cdot\,)$ on the closed subspace $(\mathrm{Ker}AB)^\perp$ by

$$(x|y) = \langle ABx|y\rangle$$

for all $x, y \in (\mathrm{Ker}AB)^\perp$. We denote by $\mathcal{K}$ the completion of $(\mathrm{Ker}AB)^\perp$ by this inner product.

Let P be the orthogonal projection from $\mathcal{H}$ onto $(\mathrm{Ker}AB)^\perp$. We define two linear operators D_1, D_2 on $(\mathrm{Ker}AB)^\perp$ by $x \mapsto PA^*x$, $x \mapsto PBx$ respectively. It follows from $(*)$ that $A^3B \geq 0$ and

$$\begin{aligned}(D_1x|D_1x) &= (PA^*x|PA^*x) = \langle ABPA^*x|PA^*x\rangle \\ &= \langle A^2BA^*x|x\rangle = \langle A^3Bx|x\rangle \\ &= \langle (A^3BA^3B)^{\frac{1}{2}}x|x\rangle = \langle (ABA^{*2}A^2AB)^{\frac{1}{2}}x|x\rangle \\ &\leq \|A^2\|\langle (ABAB)^{\frac{1}{2}}x|x\rangle = \|A^2\|\langle ABx|x\rangle \\ &= \|A^2\|(x|x),\end{aligned}$$

which show that D_1 can be extended to a bounded linear operator on $\mathcal{K}$. Since we have

$$(D_1x|x) = \langle ABPA^*x|x\rangle = \langle ABA^*x|x\rangle = \langle A^2Bx|x\rangle \geq 0$$

for all $x \in (\mathrm{Ker}AB)^{\perp}$, D_1 is a positive operator on $\mathcal{K}$. In a similar fashion, D_2 can be extended to a positive operator on $\mathcal{K}$.

Since $PA^* = PA^*P$ and $PB = PBP$, we have $(PA^*)^n = PA^{*n}$ and $(PB)^n = PB^n$. If we define a bounded linear operator V from $\mathcal{H}$ to $\mathcal{K}$ by $x \mapsto Px$ for any $x \in \mathcal{H}$, then we can get the required identity as follows:

$$\begin{aligned}\langle A^{n+1}B^{m+1}x|x\rangle &= \langle A^nABB^mPx|Px\rangle \\ &= \langle ABPA^{*n}PB^mPx|Px\rangle \\ &= (D_1^nD_2^mVx|Vx) \\ &= \langle V^*D_1^nD_2^mVx|x\rangle.\end{aligned}$$

The assumption $A^2B^2 \geq 0$ implies the following relation:

$$(D_1D_2x|x) = \langle ABPA^*PBx|x\rangle = \langle A^2B^2x|x\rangle \geq 0,$$

so we have $D_1D_2 = (D_1D_2)^* = D_2D_1$. This completes the proof. $\square$

With related to the above result, we have some examples as follows:

Example. We put

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -10 & 1 & 0 & 0 \\ 4 & -1 & 0 & 0 \end{pmatrix},$$

then A, B satisfy $(*)$ but A^2B^2 is not positive.

We put

$$A = \begin{pmatrix} 1 & 1 & -1 & -4 \\ 2 & 2 & -4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then A, B satisfy $(*)$ and $A^2B^2 \geq 0$ but PAP and PBP are not positive, where P is the orthogonal projection onto $(\mathrm{Ker}AB)^{\perp}$.

We consider the case that A, B are contractions and $A^2B^2 \geq 0$. In the above proof, we see that D_1 and D_2 become commuting positive contractions. So we can get some results related to commuting positive contractions. For example, we have the following result:

Corollary 6. *Let A, B be contractions which satisfy $A^2B^2 \geq 0$. Let f, g be holomorphic functions on an open neighborhood of $\{z : |z| \leq 1\}$. If $f(0) = g(0) = 0$, then*

$$\|f(A)g(B)\| \leq \sup\{|f^{[1]}(z)g^{[1]}(w)| : z, w \in [0,1]\},$$

where

$$f^{[1]}(t) = \begin{cases} \frac{f(t)}{t}, & (1 \geq t > 0) \\ f'(0), & (t = 0), \end{cases} \quad g^{[1]}(t) = \begin{cases} \frac{g(t)}{t}, & (1 \geq t > 0) \\ g'(0), & (t = 0). \end{cases}$$

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