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EXISTENCE OF NONEXPANSIVE RETRACTIONS AND MEAN ERGODIC THEOREMS IN HILBERT SPACES

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Abstract

Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C such that the set $F(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is nonempty. We consider the existence of an ergodic retraction and prove that if $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to an element of $F(\mathcal{S})$.

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Then, a mapping $T : C \to C$ is said to be *Lipschitzian* if there exists a nonnegative real number k such that

$$\|Tx - Ty\| \le k\|x - y\| \quad \text{for every} \quad x, y \in C.$$

T is said to be *nonexpansive* if $k = 1$. Let S be a semigroup. Then, a family $\mathcal{S} = \{T_t : t \in S\}$ of mappings from C into itself is said to be a *Lipschitzian semigroup* on C with Lipschitz constants $\{k_t : t \in S\}$ if it satisfies the following:

(1) for each $t \in S$, there exists a nonnegative real number k_t such that

$$\|T_t x - T_t y\| \le k_t\|x - y\| \quad \text{for every} \quad x, y \in C;$$

(2) $T_{st}x = T_s T_t x$ for every $s, t \in S$ and $x \in C$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$. $\mathcal{S}$ is said to be a *nonexpansive semigroup* on C if $k_t = 1$ for every $t \in S$. $\mathcal{S}$ is also said to be an *asymptotically nonexpansive semigroup* on C if $\inf_s \sup_t k_{ts} \le 1$ and $\sup_t k_t < \infty$. In particular, $\mathcal{S}$ is said to be a *one-parameter asymptotically nonexpansive semigroup* on C if $S = [0, \infty)$ and for each $x \in C$, the mapping $t \mapsto T_t x$ from S into C is continuous.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{co}\{T^n x : n = 1, 2, \ldots\}$ for every $x \in C$, where $\overline{co}A$ denotes the closure of the convex hull of A. Such a retraction is said to be an *ergodic retraction.* Hirano and Takahashi [3] provided nonlinear ergodic theorems for a one-parameter asymptotically nonexpansive semigroup in a Hilbert space. In [8], Takahashi proved the existence of an ergodic retraction for an amenable semigroup of nonexpansive mappings in a Hilbert space: If S is an amenable semigroup, C is a nonempty closed convex subset of a Hilbert space H and $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty, then there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Further, Takahashi [9] provided a necessary and sufficient condition for the existence of an ergodic retraction in a Hilbert space: If S is a right reversible semigroup and C, H and $\mathcal{S}$ are as above, then $\bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\} \cap F(\mathcal{S})$ is nonempty for every $x \in C$ if and only if there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Mizoguchi and Takahashi [6] extended this result to the case when $\mathcal{S}$ is an asymptotically nonexpansive semigroup. Takahashi's result was also extended to the case when S is not a directed system by Lau, Nishiura and Takahashi [4]. Further, Lau, Shioji and Takahashi [5] extended this result to a uniformly convex Banach space whose norm is Fréchet differentiable. Rodé [7] also found a sequence of means on a semigroup, generalizing the Cesàro means and extended Baillon's theorem: If S, C, H and $\mathcal{S}$ are as in Takahashi [8] and $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to an element of $F(\mathcal{S})$.

In this paper, we prove the existence of an ergodic retraction for an asymptotically nonexpansive semigroup in a Hilbert space and then establish a mean convergence theorem of Rodé's type. These results are generalizations of Takahashi [8] and Rodé [7]. We also provide a necessary and sufficient condition for the existence of an ergodic retraction in a Hilbert space. This result is a generalization of Lau, Nishiura and Takahashi [4].

2 Preliminaries

Throughout this paper, we assume that a Hilbert space is real. Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ of $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing constants and let X^* be its dual. Then, an element μ of X^* is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. Occasionally, we use $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be l_s- and r_s-invariant, i.e., $l_s(X) \subset X$ and $r_s(X) \subset X$ for every $s \in S$.

Then, a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for every $f \in X$ and $s \in S$. A mean μ on X is said to be *invariant* if it is both right and left invariant. X is said to be *amenable* if there exists an invariant mean on X. A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically invariant* if for each $f \in X$ and $s \in S$,

$$\lim_\alpha (\mu_\alpha(l_s f) - \mu_\alpha(f)) = 0 \quad \text{and} \quad \lim_\alpha (\mu_\alpha(r_s f) - \mu_\alpha(f)) = 0.$$

Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$. For each $x \in C$, define the set

$$Q(x) = \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x : t \in S\}.$$

Let X be a subspace of $B(S)$ such that $1 \in X$ and the function $t \mapsto \|T_t x - y\|^2$ is an element of X for every $x \in C$ and $y \in H$. Then, by the Riesz representation theorem, for any $\mu \in X^*$ and $x \in C$ there exists a unique element x_0 of H such that

$$\mu_t \langle T_t x, y \rangle = \langle x_0, y \rangle$$

for every $y \in H$. We write such x_0 by $T_\mu x$. See [8] for more details.

3 Lemmas

In this section, we prove two lemmas which are crucial in the proofs of our theorems.

LEMMA 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, the function $t \mapsto \|T_t x - y\|^2$ is an element of X for every $x \in C$ and $y \in H$ and X is l_s-invariant for every $s \in S$. If μ is a left invariant mean on X, then for each $x \in C$, $T_\mu x \in F(\mathcal{S})$.*

Proof. Let μ be a left invariant mean on X. Let $\varepsilon > 0$ and $x \in C$. From $\inf_s \sup_t k_{ts} \le 1$, there exists $s_0 \in S$ such that $\sup_t k_{ts_0}^2 < 1 + \varepsilon^2$. For each $y \in H$ and $s, t \in S$, since

$$2\langle T_s x - y, T_t x - y \rangle = \|T_s x - y\|^2 + \|T_t x - y\|^2 - \|T_s x - T_t x\|^2,$$

we have

$$\begin{aligned} \|T_\mu x - y\|^2 &= \mu_s \langle T_s x - y, T_\mu x - y \rangle \\ &= \mu_s(\mu_t \langle T_s x - y, T_t x - y \rangle) \\ &= \frac{1}{2}\mu_s(\|T_s x - y\|^2 + \mu_t\|T_t x - y\|^2 - \mu_t\|T_s x - T_t x\|^2) \\ &= \mu_s\|T_s x - y\|^2 - \frac{1}{2}\mu_s(\mu_t\|T_s x - T_t x\|^2). \end{aligned}$$

Putting $y = T_\mu x$, we have

$$\frac{1}{2}\mu_s(\mu_t\|T_s x - T_t x\|^2) = \mu_s\|T_s x - T_\mu x\|^2.$$

Since μ is left invariant, we have, for each $t \in S$,

$$\begin{aligned}\|T_\mu x - T_{ts_0}T_\mu x\|^2 &= \mu_s\|T_s x - T_{ts_0}T_\mu x\|^2 - \frac{1}{2}\mu_s(\mu_w\|T_s x - T_w x\|^2) \\ &= \mu_s\|T_{ts_0s}x - T_{ts_0}T_\mu x\|^2 - \mu_s\|T_s x - T_\mu x\|^2 \\ &\leq (k_{ts_0}^2 - 1)\mu_s\|T_s x - T_\mu x\|^2 \\ &\leq 4d^2\varepsilon^2,\end{aligned}$$

where $d = \sup\{\|T_t x\| : t \in S\}$, and hence

$$\begin{aligned}\|T_\mu x - T_t T_\mu x\| &\leq \|T_\mu x - T_{tts_0}T_\mu x\| + \|T_{tts_0}T_\mu x - T_t T_\mu x\| \\ &\leq 2d\varepsilon + 2k_t d\varepsilon \\ &= 2(1 + k_t)d\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\|T_\mu x - T_t T_\mu x\| = 0$$

for every $t \in S$. This implies that $T_\mu x \in F(\mathcal{S})$. □

LEMMA 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$. Then for each $x \in C$, $y \in Q(x) \cap F(\mathcal{S})$ and $z \in F(\mathcal{S})$,*

$$\sup_{s\in S}\inf_{t\in S}\langle T_{ts}x - y, y - z\rangle \geq 0.$$

Proof. Let $x \in C$, $y \in Q(x) \cap F(\mathcal{S})$ and $z \in F(\mathcal{S})$. Let $\varepsilon > 0$. We choose $\delta > 0$ so small that

$$((1+\delta)^4 - 1)d^2 + \frac{1}{4}(1+\delta)^2\delta < \varepsilon^2 \quad \text{and} \quad \delta < \varepsilon,$$

where $d = \sup\{\|T_t x\| : t \in S\}$. From $\inf_s \sup_t k_{ts} \leq 1$, there exists $s_0 \in S$ such that $\sup_t k_{ts_0} < 1 + \delta$. Since $\inf_s \sup_t \|T_{ts_0s}x - y\|^2 \leq (1+\delta)^2 \inf_s \|T_s x - y\|^2$, there also exists $s_1 \in S$ such that

$$\|T_{ts_0s_1}x - y\|^2 < (1+\delta)^2\|T_s x - y\|^2 + \delta$$

for every $s, t \in S$. Then, we have, for each λ with $0 \leq \lambda \leq 1$ and $s, t \in S$,

$$\begin{aligned}&\|\lambda T_{ts_0ss_0s_1}x + (1-\lambda)y - T_{ts_0}(\lambda T_{ss_0s_1}x + (1-\lambda)y)\|^2 \\ &= (1-\lambda)\|T_{ts_0}(\lambda T_{ss_0s_1}x + (1-\lambda)y) - y\|^2 \\ &\quad + \lambda\|T_{ts_0}(\lambda T_{ss_0s_1}x + (1-\lambda)y) - T_{ts_0ss_0s_1}x\|^2 - \lambda(1-\lambda)\|T_{ts_0ss_0s_1}x - y\|^2 \\ &\leq \lambda(1-\lambda)(1+\delta)^2\|T_{ss_0s_1}x - y\|^2 - \lambda(1-\lambda)\|T_{ts_0ss_0s_1}x - y\|^2 \\ &\leq \frac{1}{4}\left((1+\delta)^2((1+\delta)^2\|T_{ts_0ss_0s_1}x - y\|^2 + \delta) - \|T_{ts_0ss_0s_1}x - y\|^2\right) \\ &\leq ((1+\delta)^4 - 1)d^2 + \frac{1}{4}(1+\delta)^2\delta < \varepsilon^2\end{aligned}$$

and hence

$$\begin{aligned}
&\|\lambda T_{ts_0ss_0s_1}x + (1-\lambda)y - z\| \\
&\quad \le \|\lambda T_{ts_0ss_0s_1}x + (1-\lambda)y - T_{ts_0}(\lambda T_{ss_0s_1}x + (1-\lambda)y)\| \\
&\qquad + \|T_{ts_0}(\lambda T_{ss_0s_1}x + (1-\lambda)y) - z\| \\
&\quad < \varepsilon + (1+\varepsilon)\|\lambda T_{ss_0s_1}x + (1-\lambda)y - z\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\| \\
&\quad \le \inf_s \sup_t \|\lambda T_{ts_0ss_0s_1}x + (1-\lambda)y - z\| \\
&\quad \le (1+\varepsilon)\inf_s \|\lambda T_{ss_0s_1}x + (1-\lambda)y - z\| + \varepsilon \\
&\quad \le (1+\varepsilon)\sup_s \inf_t \|\lambda T_{ts}x + (1-\lambda)y - z\| + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\| \le \sup_s \inf_t \|\lambda T_{ts}x + (1-\lambda)y - z\|$$

for every λ with $0 \le \lambda \le 1$. Let $\varepsilon > 0$ and $0 \le \lambda \le 1$. Then, there exists $s_2 \in S$ such that

$$\|\lambda T_{ws_2}x + (1-\lambda)y - z\| < \inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\| + \varepsilon$$

for every $w \in S$. From $y \in \overline{\text{co}}\{T_{ws_2}x : w \in S\}$, we have

$$\|y - z\| \le \inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\|y - z\| \le \inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\|$$

for every λ with $0 \le \lambda \le 1$. For each λ with $0 < \lambda \le 1$ and $t \in S$, since

$$\|y - z + \lambda(T_t x - y)\|^2 = \|y - z\|^2 + 2\lambda\langle T_t x - y, y - z\rangle + \lambda^2\|T_t x - y\|^2,$$

we obtain

$$\begin{aligned}
0 &\le \inf_s \sup_t \|\lambda T_{ts}x + (1-\lambda)y - z\|^2 - \|y - z\|^2 \\
&\le \sup_s \inf_t \|\lambda T_{ts}x + (1-\lambda)y - z\|^2 - \|y - z\|^2 \\
&\le 2\lambda \sup_s \inf_t \langle T_{ts}x - y, y - z\rangle + 4\lambda^2 d^2
\end{aligned}$$

and hence

$$0 \le \sup_s \inf_t \langle T_{ts}x - y, y - z\rangle + 2\lambda d^2.$$

It follows that

$$0 \le \sup_s \inf_t \langle T_{ts}x - y, y - z\rangle \quad \text{as} \quad \lambda \downarrow 0. \ \square$$

4 Ergodic Theorems

In this section, we establish our nonlinear ergodic theorems.

THEOREM 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$. Let $x \in C$. Then $Q(x) \cap F(\mathcal{S})$ contains at most one point.*

Proof. Let $y, z \in Q(x) \cap F(\mathcal{S})$ and $\varepsilon > 0$. By Lemma 3.2, there exists $s_0 \in S$ such that

$$\langle T_{ts_0}x - y, y - z\rangle > -\varepsilon$$

for every $t \in S$. Since $z \in \overline{co}\{T_{ts_0}x : t \in S\}$, it follows that

$$\langle z - y, y - z\rangle \geq -\varepsilon$$

and, hence $\|y - z\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $y = z$. □

Using Lemma 3.1 and Theorem 4.1, we show a nonlinear ergodic theorem for an asymptotically nonexpansive semigroup in a Hilbert space. This generalizes the results of Takahashi [8] and Rodé [7].

THEOREM 4.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, the function $t \mapsto \|T_t x - y\|^2$ is an element of X for every $x \in C$ and $y \in H$ and X is l_s- and r_s-invariant for every $s \in S$. If X is amenable, then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.*

Proof. Assume that X is amenable. Then, there exists an invariant mean μ on X. Since μ is a mean on X, it follows from the separation theorem that $T_\mu x \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$; see [8] for details. Since μ is right invariant, we have, for each $x, y \in C$

$$\begin{aligned}
\|T_\mu x - T_\mu y\|^2 &= \langle T_\mu x - T_\mu y, T_\mu x - T_\mu y\rangle \\
&= \mu_t\langle T_t x - T_t y, T_\mu x - T_\mu y\rangle \\
&= \mu_t\langle T_{ts} x - T_{ts} y, T_\mu x - T_\mu y\rangle \\
&\leq \sup_t \|T_{ts} x - T_{ts} y\| \|T_\mu x - T_\mu y\| \\
&\leq (\sup_t k_{ts}) \|x - y\| \|T_\mu x - T_\mu y\|
\end{aligned}$$

for every $s \in S$, and hence

$$\begin{aligned}
\|T_\mu x - T_\mu y\|^2 &\leq (\inf_s \sup_t k_{ts}) \|x - y\| \|T_\mu x - T_\mu y\| \\
&\leq \|x - y\| \|T_\mu x - T_\mu y\|.
\end{aligned}$$

So, T_μ is nonexpansive. Let $x \in C$. Since μ is right invariant, we also have

$$\begin{aligned}\langle T_\mu x, y\rangle &= \mu_t\langle T_t x, y\rangle \\ &= \mu_t\langle T_{ts} x, y\rangle \\ &= \mu_t\langle T_t T_s x, y\rangle \\ &= \langle T_\mu T_s x, y\rangle\end{aligned}$$

for every $y \in H$ and $s \in S$. So, $T_\mu T_s = T_\mu$ for every $s \in S$. By Lemma 3.1, $T_s T_\mu = T_\mu$ for every $s \in S$. Therefore, we have

$$\begin{aligned}\langle {T_\mu}^2 x, y\rangle &= \mu_t\langle T_t T_\mu x, y\rangle \\ &= \mu_t\langle T_\mu x, y\rangle \\ &= \langle T_\mu x, y\rangle\end{aligned}$$

for every $y \in H$. So, ${T_\mu}^2 = T_\mu$. Putting $P = T_\mu$, we have that P is a nonexpansive retraction of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. For each $x \in C$ and $s \in S$, we have

$$Px = PT_s x \in \overline{co}\{T_{ts} x : t \in S\}.$$

So by Theorem 4.1, we obtain

$$\{Px\} = Q(x) \cap F(\mathcal{S})$$

for every $x \in C$. Hence such P is unique.

Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Let μ be a cluster point of $\{\mu_\alpha\}$ in the weak* topology. It is obvious that μ is a mean. We show that μ is invariant. Let $\varepsilon > 0, f \in X$ and $s \in S$. Then, there exists α_0 such that

$$|\mu_\alpha(f) - \mu_\alpha(l_s f)| < \frac{\varepsilon}{3}$$

for every $\alpha \geq \alpha_0$. Since μ is a cluster point of $\{\mu_\alpha\}$, we can choose $\alpha_1 \geq \alpha_0$ such that

$$|\mu_{\alpha_1}(f) - \mu(f)| < \frac{\varepsilon}{3}$$

and

$$|\mu_{\alpha_1}(l_s f) - \mu(l_s f)| < \frac{\varepsilon}{3}.$$

So, we have

$$\begin{aligned}&|\mu(f) - \mu(l_s f)| \\ &\leq |\mu(f) - \mu_{\alpha_1}(f)| + |\mu_{\alpha_1}(f) - \mu_{\alpha_1}(l_s f)| + |\mu_{\alpha_1}(l_s f) - \mu(l_s f)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mu(f) = \mu(l_s f).$$

This implies that μ is left invariant. Similarly, μ is right invariant. Let $\{T_{\mu_{\alpha_\beta}}x\}$ be a subnet of $\{T_{\mu_\alpha}x\}$ such that $\{T_{\mu_{\alpha_\beta}}x\}$ converges weakly to some $z \in C$ and let λ be a cluster point of $\{\mu_{\alpha_\beta}\}$ in the weak* topology. Since λ is also a cluster point of $\{\mu_\alpha\}$, λ is an invariant mean. So, we obtain

$$z = T_\lambda x = Px.$$

This implies that $\{T_{\mu_\alpha}x\}$ converges weakly to Px. □

Finally, we show a necessary and sufficient condition for the existence of an ergodic retraction for an asymptotically nonexpansive semigroup in a Hilbert space.

THEOREM 4.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty. Then the following are equivalent:*

(1) for each $x \in C$, the set $Q(x) \cap F(\mathcal{S})$ is nonempty;

(2) there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. (1) $\Rightarrow$ (2). If for each $x \in C$, the set $Q(x) \cap F(\mathcal{S}) \neq \emptyset$, then by Theorem 4.1, $Q(x) \cap F(\mathcal{S})$ contains exactly one point Px. Then, clearly, P is a retraction of C onto $F(\mathcal{S})$ such that $T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$. Also, if $u \in S$ and $x \in C$, we have

$$\bigcap_{s\in S} \overline{\text{co}}\{T_{ts}x : t \in S\} \subset \bigcap_{s\in S} \overline{\text{co}}\{T_{tsu}x : t \in S\}$$

and hence

$$Q(x) \cap F(\mathcal{S}) = Q(T_ux) \cap F(\mathcal{S}).$$

This implies $PT_t = P$ for every $t \in S$. Finally we show that P is nonexpansive. Let $x, y \in C$ and $\varepsilon > 0$. From $\inf_s \sup_t k_{ts} \leq 1$, there exists $s_0 \in S$ such that $\sup_t k_{ts_0} < 1 + \varepsilon$. By Lemma 3.2, we have

$$\sup_s \inf_t \langle T_{ts}T_{s_0}x - Px, Px - Py\rangle \geq 0.$$

Then, there exists $u \in S$ such that

$$\langle T_{tu}T_{s_0}x - Px, Px - Py\rangle > -\varepsilon$$

for every $t \in S$. By Lemma 3.2, we also have

$$\sup_s \inf_t \langle T_{ts}T_{us_0}y - PT_{us_0}y, PT_{us_0}y - Px\rangle \geq 0.$$

So, there exists $v \in S$ such that

$$\langle T_{tv}T_{us_0}y - PT_{us_0}y, PT_{us_0}y - Px\rangle > -\varepsilon$$

for every $t \in S$. Then, from $PT_{us_0}y = Py$, we have

$$\langle T_{tv}T_{us_0}y - Py, Py - Px\rangle > -\varepsilon$$

for every $t \in S$. Therefore, we obtain

$$\begin{aligned} -2\varepsilon &< \langle T_{uvus_0}x - Px, Px - Py\rangle + \langle T_{uvus_0}y - Py, Py - Px\rangle \\ &\leq \|T_{uvus_0}x - T_{uvus_0}y\|\|Px - Py\| - \|Px - Py\|^2 \\ &\leq (1+\varepsilon)\|x - y\|\|Px - Py\| - \|Px - Py\|^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$.

(2) $\Rightarrow$ (1). Let $x \in C$. Then it is obvious that $Px \in F(\mathcal{S})$. Since

$$Px = PT_sx \in \overline{\mathrm{co}}\{T_tT_sx : t \in S\} = \overline{\mathrm{co}}\{T_{ts}x : t \in S\}$$

for every $s \in S$, we have

$$Px \in \bigcap_{s\in S} \overline{\mathrm{co}}\{T_{ts}x : t \in S\} = Q(x). \ \square$$

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