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Controllability of Second Order Delay Integrodifferential Systems in Banach Spaces

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Abstract. Sufficient conditions for controllability of second order delay integrodifferential systems in Banach spaces are established. The results are obtained by using the theory of strongly continuous cosine family of operators and the Schaefer fixed point theorem.

Key words. Controllability, Second order delay differential system, Schaefer's theorem.

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1.Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the concept to infinite dimensional systems in Banach Spaces with bounded operators. Chukwu and Lenhart [3] have studied the controllability of nonlinear systems in abstract spaces. Naito [7,8] has studied the controllability for semilinear systems and nonlinear Volterra integrodifferential systems. Quinn and Carmichael [11] have shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. Balachandran et al [1] established sufficient conditions for controllability of nonlinear integrodifferential systems in Banach spaces.

In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. For example Fitzgibbon [4] used the second order abstract differential equations for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful tool for the study of abstract second order equations is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory [13,14]. Motivation for second order systems can be found in [5,6]. Recently, Park et al [10] have discussed the controllability of second order nonlinear systems in Banach spaces with the help of the Schauder fixed point theorem. The purpose of this paper is to study the controllability of second order integrodifferential systems in Banach spaces by using the Schaefer theorem .

2. Preliminaries.

Consider the second order delay control system of the form

$$\begin{aligned} x''(t) &= Ax(t) + \int_0^t f(s, x_s, x'(s))ds + Bu(t), \qquad t \in J = [0, T], \\ x_0 &= \phi, \qquad x'(0) = y, \end{aligned} \tag{1}$$

where the state $x(\cdot)$ takes values in the Banach space $X, y \in X$, A is the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in X, f is a nonlinear mapping from $J \times C \times X$ to X, B is a bounded linear operator from U to X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U as a Banach space. Here $C = C([-r, 0]) : X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the supremum norm

$$\|\phi\| = \sup\{|\phi(s)| : -r \leq s \leq 0\}.$$

Also for $x \in C([-r, T] : X)$ we have $x_t \in C$ for $t \in [0, T], x_t(s) = x(t + s)$ for $s \in [-r, 0]$.

Definition 1.[13] A one parameter family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s, t \in R$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is continuous in t on R for each fixed $x \in X$.

Define the associated sine family $S(t)$, $t \in R$, by

$$S(t)x = \int_0^t C(s)x ds, \qquad x \in X, \quad t \in R.$$

Assume the following conditions on A.

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from X into itself and the adjoint operator A^* is densely defined i.e. $\overline{D(A^*)} = X^*$ (See[2]).

The infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \qquad \mathrm{x} \in \mathrm{D(A)},$$

where $\quad D(A) = \{ x \in X$: $C(t)x$ is twice continuously differentiable in t $\}$.

Let $\quad E = \{ x \in X$: $C(t)x$ is once continuously differentiable in t $\}$.

To establish our main theorem we need the following lemmas.

Lemma 1.[13] Let (H_1) hold. Then

(i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t) - S(t^*)\| \leq M\left|\int_t^{t^*} e^{\omega|s|}ds\right| \quad \text{for } t, t^* \in R;$$

(ii) $S(t)X \subset E$ and $S(t)E \subset D(A)$ for $t \in R$;

(iii) $\frac{d}{dt}C(t)x = AS(t)x$ for $x \in E$ and $t \in R$;

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x$ for $x \in D(A)$ and $t \in R$.

Lemma 2.[13] Let (H_1) hold, let $v : R \to X$ such that v is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then

q is twice continuously differentiable and for $t \in R$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t-s)v(s)ds, \text{ and } q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

Lemma 3.(Schaefer Theorem [12]) Let S be a convex subset of a normed linear space Y and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let

$$\zeta(F) = \{x \in S : x = \lambda Fx \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Let $M = \sup\{\|C(t)\| : t \in J\}$ and $M^* = \sup\{\|AS(t)\| : t \in J\}$.

Let $\mu(t) = \sup\{|x(s)| : s \in [-r, t]\}$, $t \in J$ and $v(t) = \sup\{|x'(s)| : s \in [0, t]\}$, $t \in J$.

Let $c = K_1 + K_2$, where

$$K_1 = M\|\phi\| + MT\|y\| + MT^2\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| + MT\int_0^T\int_0^s m(\tau)\Omega(\mu(\tau) + \|x'(\tau)\|)d\tau ds],$$

and

$$K_2 = M^*\|\phi\| + M\|y\| + MT\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| + MT\int_0^T\int_0^s m(\tau)\Omega(\mu(\tau) + v(\tau))d\tau ds].$$

We make the following assumptions:

(H_2) $f(t,.,.) : C \times X \to X$ is continuous for each $t \in J$ and the function $f(.,x,z) : J \to X$ is strongly measurable for each $(x,z) \in C \times X$.

(H_3) For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|z\| \leq k} \|f(t,x,z)\| \leq \alpha_k(t) \quad \text{for a.a } t \in J.$$

(H_4) There exists an integrable function $m : J \longrightarrow [0, \infty)$ such that

$$\|f(t, \phi, z)\| \leq m(t)\Omega(\max(\|\phi\|, \|z\|)), \quad t \in J, \quad \phi \in C, \quad z \in X,$$

where $\Omega : [0, \infty) \longrightarrow (0, \infty)$ is a continuous nondecreasing function and

$$M(T+1)\int_0^T \int_0^s m(\tau)d\tau ds < \int_c^\infty \frac{ds}{\Omega(s)} < \infty.$$

(H_5) $Bu(t)$ is continuous.

(H_6) The linear operator $W : L^2(J, U) \longrightarrow X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has a bounded invertible operator $W^{-1} : X \longrightarrow L^2(J, U) \setminus kerW$ (see [1].)

(H_7) $C(t), t > 0$ is compact.

Then the system (1) has a mild solution of the form (see [9])

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)y + \int_0^t S(t-s)\left(\int_0^s f(\tau, x_\tau, x'(\tau))d\tau + Bu(s)\right)ds, \ t \in J \quad (2) \\ x_0 &= \phi \end{aligned}$$

Definition 2. The system (1) is said to be controllable on J if for every $\phi \in C$ with $\phi(0) \in D(A)$, $y \in E$ and $x_1 \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

3.Main Result

Theorem: Suppose (H_1)-(H_7) hold. Then the system (1) is controllable on J.

Proof: Using (H_6), for an arbitrary function $x(\cdot)$ we define the control

$$u(t) = W^{-1}[x_1 - C(T)\phi(0) - S(T)y - \int_0^T S(T-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds](t).$$

Using this control we will show that the operator defined by

$$\begin{aligned} (Fx)(t) &= C(t)\phi(0) + S(t)y + \int_0^t S(t-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds \\ &\quad + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\ &\quad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds, \quad t \in J, \\ &= \phi(t), \quad t \in [-r, 0] \end{aligned}$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(T) = x_1$, which means that the control u steers the system from the initial function ϕ to x_1 in time T, provided we obtain a fixed point of the nonlinear operator F.

Consider the space $Z = C([-r,T],X) \cap C^1(J,X)$ with norm

$$\|x\|^* = \max\{\|x\|_r, \|x\|_0\}$$

where $\|x\|_r = \sup\{|x(t)| : -r \le t \le T\}$, $\|x\|_0 = \sup\{|x'(t)| : 0 \le t \le T\}$.

In order to study the controllability problem for the system (1), we have to apply Lemma 3, as in [9], to the following system

$$x''(t) = \lambda Ax(t) + \lambda \int_0^t f(s, x_s, x'(s))ds + \lambda Bu(t), \quad t \in J, \quad \lambda \in (0,1). \tag{3}$$

Let x be a mild solution of the system (3). Then from

$$\begin{aligned} x(t) &= \lambda(C(t)\phi(0) + S(t)y) + \lambda \int_0^t S(t-s) \int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds \\ &\quad + \lambda \int_0^t S(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\ &\qquad - \int_0^T S(T-\theta) \int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds \end{aligned}$$

we have

$$\begin{aligned} \|x(t)\| &\le M\|\phi\| + MT\|y\| + MT \int_0^t \int_0^s m(\tau)\Omega(\max(\|x_\tau\|, \|x'(\tau)\|))d\tau ds \\ &\quad + MT^2\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| \\ &\qquad + MT \int_0^T \int_0^s m(\tau)\Omega(\max(\|x_\tau\|, \|x'(\tau)\|))d\tau ds] \end{aligned}$$

or

$$\begin{aligned} \mu(t) &\le M\|\phi\| + MT\|y\| + MT \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + \|x'(\tau)\|)d\tau ds \\ &\quad + MT^2\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| \\ &\qquad + MT \int_0^T \int_0^s m(\tau)\Omega(\mu(\tau) + \|x'(\tau)\|)d\tau ds] \\ &= K_1 + MT \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + \|x'(\tau)\|)d\tau ds. \end{aligned}$$

Denoting by $p(t)$ the right-hand side of the above inequality we have

$$p(0) = K_1, \qquad \mu(t) \le p(t) \;\; t \in J$$

and

$$p'(t) = MT \textstyle\int_0^t m(s)\Omega(p(s) + \|x'(s)\|)ds, \quad t \in J.$$

But

$$
\begin{aligned}
x'(t) &= \lambda[AS(t)\phi(0) + C(t)y] + \lambda \int_0^t C(t-s) \int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds \\
&\quad + \lambda \int_0^t C(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad\qquad - \int_0^T S(T-\theta) \int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|x'(t)\| &\leq M^*\|\phi\| + M\|y\| + M\int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\| + \|x'(\tau)\|)d\tau ds \\
&\quad + MT\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| \\
&\qquad\qquad + MT\int_0^T \int_0^s m(\tau)\Omega(\|x_\tau\| + \|x'(\tau)\|)d\tau ds]
\end{aligned}
$$

$$
\begin{aligned}
v(t) &\leq M^*\|\phi\| + M\|y\| + M\int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + v(\tau))d\tau ds \\
&\quad + MT\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi\| + MT\|y\| \\
&\qquad\qquad + MT\int_0^T \int_0^s m(\tau)\Omega(\mu(\tau) + v(\tau))d\tau ds] \\
&= K_2 + M\int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + v(\tau))d\tau ds.
\end{aligned}
$$

Denoting by $q(t)$ the right-hand side of the above inequality we have

$$q(0) = K_2, \qquad v(t) \leq q(t)$$

and

$$q'(t) = M\int_0^t m(s)\Omega(\mu(s) + v(s)), \qquad t \in J.$$

Let

$$w(t) = p(t) + q(t), \qquad t \in J.$$

Then

$$w(0) = p(0) + q(0) = c, \text{ and}$$

$$
\begin{aligned}
w'(t) &= p'(t) + q'(t) \\
&\leq MT\int_0^t m(s)\Omega(w(s)) + M\int_0^t m(s)\Omega(w(s))ds \\
&= M(T+1)\int_0^t m(s)\Omega(w(s))ds, \qquad t \in J.
\end{aligned}
$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s)} \leq M(T+1)\int_0^T \int_0^s m(\tau)d\tau ds < \int_c^\infty \frac{ds}{\Omega(s)}, \qquad t \in J.$$

This inequality implies that there is a constant K such that (see [9])

$$w(t) = p(t) + q(t) \leq K, \qquad t \in J.$$

Then
$$\|x(t)\| \le \mu(t) \le p(t), \qquad t \in J,$$
$$\|x'(t)\| \le v(t) \le q(t), \qquad t \in J,$$
and hence
$$\|x\|^* \le K,$$
where K depends only on T and on the functions m and Ω.

We shall now prove that the operator $F : Z \to Z$ defined by

$$\begin{aligned}
(Fx)(t) &= C(t)\phi(0) + S(t)y + \int_0^t S(t-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds \\
&\quad + \int_0^t S(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\quad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds, \quad t \in J, \\
&= \phi(t), \quad t \in [-r, 0]
\end{aligned}$$

is a completely continuous operator.

Let $B_k = \{x \in Z : \|x\|^* \le k\}$ for $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \le T$,

$$\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\| \\
&\le \|C(t_1) - C(t_2)\|\|\phi(0)\| + \|S(t_1) - S(t_2)\|\|y\| \\
&\quad + \|\int_0^{t_1}[S(t_1 - s) - S(t_2 - s)]\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2 - s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds\| \\
&\quad + \|\int_0^{t_1}[S(t_1 - s) - S(t_2 - s)]BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds\| \\
&\quad + \|\int_{t_1}^{t_2} S(t_2 - s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds\| \\
&\le \|C(t_1) - C(t_2)\|\|\phi(0)\| + \|S(t_1) - S(t_2)\|\|y\| \\
&\quad + \int_0^{t_1}\|S(t_1 - s) - S(t_2 - s)\|\int_0^s \alpha_k(\tau)d\tau ds + \int_{t_1}^{t_2}\|S(t_2 - s)\|\int_0^s \alpha_k(\tau)d\tau ds \\
&\quad + \int_0^{t_1}\|S(t_1 - s) - S(t_2 - s)\|\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi(0)\| + MT\|y\| \\
&\qquad + MT\int_0^T\int_0^\theta \alpha_k(\tau)d\tau d\theta]ds
\end{aligned}$$

$$+\int_{t_1}^{t_2}\|S(t_2-s)\|\|B\|\|W^{-1}\|[\|x_1\|+M\|\phi(0)\|+MT\|y\|$$
$$+MT\int_0^T\int_0^\theta \alpha_k(\tau)d\tau d\theta]ds,$$

and similarly

$$\|(Fx)'(t_1)-(Fx)'(t_2)\|$$

$$\begin{aligned}
\leq\ & \|C'(t_1)-C'(t_2)\|\|\phi(0)\|+\|S'(t_1)-S'(t_2)\|\|y\| \\
&+\|\int_0^{t_1}[C(t_1-s)-C(t_2-s)]\int_0^s f(\tau,x_\tau,x'(\tau))d\tau ds\| \\
&+\|\int_{t_1}^{t_2}C(t_2-s)\int_0^s f(\tau,x_\tau,x'(\tau))d\tau ds\| \\
&+\|\int_0^{t_1}[C(t_1-s)-C(t_2-s)]BW^{-1}[x_1-C(T)\phi(0)-S(T)y \\
&\qquad -\int_0^T S(T-\theta)\int_0^\theta f(\tau,x_\tau,x'(\tau))d\tau d\theta](s)ds\| \\
&+\|\int_{t_1}^{t_2}C(t_2-s)BW^{-1}[x_1-C(T)\phi(0)-S(T)y \\
&\qquad -\int_0^T S(T-\theta)\int_0^\theta f(\tau,x_\tau,x'(\tau))d\tau d\theta](s)ds\|
\end{aligned}$$

$$\begin{aligned}
\leq\ & \|A(S(t_1)-S(t_2))\|\|\phi(0)\|+\|C(t_1)-C(t_2)\|\|y\| \\
&+\int_0^{t_1}\|C(t_1-s)-C(t_2-s)\|\int_0^s \alpha_k(\tau)d\tau ds+\int_{t_1}^{t_2}\|C(t_2-s)\|\int_0^s \alpha_k(\tau)d\tau ds \\
&+\int_0^{t_1}\|C(t_1-s)-C(t_2-s)\|\|B\|\|W^{-1}\|[\|x_1\|+M\|\phi(0)\|+MT\|y\| \\
&\qquad +MT\int_0^T\int_0^\theta \alpha_k(\tau)d\tau d\theta]ds \\
&+\int_{t_1}^{t_2}\|C(t_2-s)\|\|B\|\|W^{-1}\|[\|x_1\|+M\|\phi(0)\|+MT\|y\| \\
&\qquad +MT\int_0^T\int_0^\theta \alpha_k(\tau)d\tau d\theta]ds.
\end{aligned}$$

The right-hand sides are independent of $x\in B_k$ and tends to zero as $t_2-t_1\to 0$, since $C(t),S(t)$ are uniformly continuous for $\in J$ and the compactness of $C(t),S(t)$ for $t>0$ imply the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$. Thus F maps B_k into an equicontinuous family of functions.

The equicontinuity for the cases $t_1<t_2\leq 0$ and $t_1\leq 0\leq t_2$ follows from the uniform continuity of ϕ on $[-r,0]$ and from the relation

$$\|(Fx)(t_1)-(Fx)(t_2)\|\leq\|\phi(t_1)-(Fx)(t_2)\|\leq\|(Fx)(t_2)-(Fx)(0)\|+\|\phi(0)-\phi(t_1)\|$$

respectively. It is easy to see that the family FB_k is uniformly bounded.

Next we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$ we define

$$\begin{aligned}
(F_\epsilon x)(t) &= C(t)\phi(0) + S(t)y + \int_0^{t-\epsilon} S(t-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds \\
&\quad + \int_0^{t-\epsilon} S(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)ds.
\end{aligned}$$

Since $C(t), S(t)$ are compact operators, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $x \in B_k$ we have

$$\begin{aligned}
\|(Fx)(t) - (F_\epsilon x)(t)\| &\leq \int_{t-\epsilon}^t \|S(t-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau\|ds \\
&\quad + \int_{t-\epsilon}^t \|S(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)\|ds. \\
&\leq \int_{t-\epsilon}^t \|S(t-s)\| \int_0^s \alpha_k(\tau)d\tau ds \\
&\quad + \int_{t-\epsilon}^t \|S(t-s)\|\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi(0)\| + MT\|y\| \\
&\qquad + MT\int_0^T\int_0^\theta \alpha_k(\tau))d\tau d\theta]ds,
\end{aligned}$$

and

$$\begin{aligned}
\|(Fx)'(t) - (F_\epsilon x)'(t)\| &\leq \int_{t-\epsilon}^t \|C(t-s)\int_0^s f(\tau, x_\tau, x'(\tau))d\tau\|ds \\
&\quad + \int_{t-\epsilon}^t \|C(t-s)BW^{-1}[x_1 - C(T)\phi(0) - S(T)y \\
&\qquad - \int_0^T S(T-\theta)\int_0^\theta f(\tau, x_\tau, x'(\tau))d\tau d\theta](s)\|ds. \\
&\leq \int_{t-\epsilon}^t \|C(t-s)\| \int_0^s \alpha_k(\tau)d\tau ds \\
&\quad + \int_{t-\epsilon}^t \|C(t-s)\|\|B\|\|W^{-1}\|[\|x_1\| + M\|\phi(0)\| + MT\|y\| \\
&\qquad + MT\int_0^T\int_0^\theta \alpha_k(\tau))d\tau d\theta]ds.
\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $F : Z \to Z$ is continuous. For that consider the space $C_T^0 = \{x \in C([-r, T] : X) : x_0 = \phi = 0\}$. Let $\{x_n\}_0^\infty \subseteq C_T^0$ with $x_n \to x$ in C_T^0.

Then there is an integer ν such that $\|x_n(t)\| \leq \nu, \|x_n'(t)\| \leq \nu$ for all n and $t \in J$, so $\|x(t)\| \leq \nu, \|x'(t)\| \leq \nu$ and $x, x' \in B_\nu$. By (H_2)

$$f(s, x_n(s), x_n{}'(s)) \to f(s, x(s), x'(s))$$

for each $t \in J$ and since

$$\|f(s, x_n(s), x_n{}'(s)) - f(s, x(s), x'(s))\| \leq 2\alpha_\nu(s),$$

we have by dominated convergence theorem

$$\begin{aligned}
\|Fx_n - Fx\| &= \sup_{t\in J} \| \int_0^t S(t-s)[\int_0^s f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^s f(\tau, x(\tau), x'(\tau))d\tau]ds \\
&\quad - \int_0^t S(t-s)BW^{-1} \int_0^T S(T-\theta) \\
&\quad [\int_0^\theta f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^\theta f(\tau, x(\tau), x'(\tau))d\tau]d\theta ds\| \\
&\leq \int_0^T \|S(t-s)[\int_0^s f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^s f(\tau, x(\tau), x'(\tau))d\tau]\|ds \\
&\quad + \int_0^T \|S(t-s)BW^{-1} \int_0^T S(T-\theta) \\
&\quad [\int_0^\theta f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^\theta f(\tau, x(\tau), x'(\tau))d\tau]d\theta\|ds \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\|(Fx_n)' - (Fx)'\| \\
&= \sup_{t\in J} \| \int_0^t C(t-s)[\int_0^s f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^s f(\tau, x(\tau), x'(\tau))d\tau]ds \\
&\quad - \int_0^t C(t-s)BW^{-1} \int_0^T S(T-\theta) \\
&\quad [\int_0^\theta f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^\theta f(\tau, x(\tau), x'(\tau))d\tau]d\theta ds\| \\
&\leq \int_0^T \|C(t-s)[\int_0^s f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^s f(\tau, x(\tau), x'(\tau))d\tau]\|ds \\
&\quad + \int_0^T \|C(t-s)BW^{-1} \int_0^T S(T-\theta) \\
&\quad [\int_0^\theta f(\tau, x_n(\tau), x_n{}'(\tau))d\tau - \int_0^\theta f(\tau, x(\tau), x'(\tau))d\tau]d\theta\|ds \to 0.
\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

We have already proved that the set $\zeta(F) = \{x \in Z : x = \lambda Fx, \lambda \in (0,1)\}$ is bounded. Hence by Schaefer's theorem the operator F has a fixed point in Z. This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus the system (1) is controllable on J.

4.Example

Consider the partial differential equation

$$\begin{aligned} z_{tt}(y,t) &= z_{yy}(y,t)+\mu(y,t)+\int_0^t \sigma(s,z(y,s-r),z_s(y,s))ds, \\ z(0,t) &= z(\pi,t)=0, \quad \text{for} \quad t>0, \\ z(y,t) &= \phi(y,t), \quad \text{for} \quad -r\le t\le 0, \\ z_t(y,0) &= z_1(y), \quad \text{for} \quad 0<y<\pi, \quad t\in J=[0,T]. \end{aligned} \tag{4}$$

Now we have to show that there exists a control μ which steers (4) from any specified initial state to the final state in a Banach space X.

Let $X=L^2[0,\pi]$ and let $A:X\to X$ be defined by

$$Aw=w'', \qquad w\in D(A),$$

where $D(A)=\{w\in X: w,w'$ are absolutely continuous, $w''\in X$, $w(0)=w(\pi)=0\}$.

Then, $$Aw=\sum_{n=1}^{\infty}-n^2(w,w_n)w_n, \qquad w\in D(A),$$

where $w_n(s)=\sqrt{2/\pi}\sin ns$, $n=1,2,3,....$ is the orthogonal set of eigenvalues of A.

It can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in R$, in X and is given by

$$C(t)w=\sum_{n=1}^{\infty}\cos nt(w,w_n)w_n, \qquad w\in X.$$

The associated sine family is given by

$$S(t)w=\sum_{n=1}^{\infty}\frac{1}{n}\sin nt(w,w_n)w_n, \qquad w\in X.$$

Let $f:J\times C\times X\to X$ be defined by

$$f(t,v,w)(y)=\sigma(t,v(y),w(y)), \qquad v\in C,\ w\in X, \qquad y\in[0,\pi],$$

where $\sigma: J\times[0,\pi]\times[0,\pi]\to[0,\pi]$ is continuous and strongly measurable and $\phi: [0,\pi]\times[-r,0]\to[0,\pi]$ is continuous.

Let $u:J\to U\subset X$ be defined by

$$(u(t))(y)=\mu(y,t), \qquad y\in[0,\pi].$$

where $\mu:[0,\pi]\times J\to[0,\pi]$ is continuous.

Assume that there exists a bounded invertible operator W^{-1} (with range $L^2(J,U) \setminus kerW$) such that

$$Wu = \int_0^T S(T-s)u(s)ds$$

Further the function σ satisfies the following condition:

There exists a continuous function $p : J \to [0,\infty)$ such that

$$\|\sigma(t,v,w)\| \leq p(t)\Omega(\max(\|v\|,\|w\|)), \quad t \in J, \quad v \in C, \; w \in X,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function and

$$M(T+1)\int_0^T \int_0^s p(s)ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

where c is a known constant.

With this choice of A, f, and $B = I$ (Identity operator), (1) is an abstract formulation of (4). Furthermore, all the conditions stated in the above theorem are satisfied. Hence, system (4) is controllable on J.

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