The second dual of a tensor product of C*-algebras

By

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1. Introduction

Let f be a positive linear functional on a C*-algebra A. Then there exists a canonically associated representation $\pi_f: A \to L(H_f)$ with a cyclic vector ξ_f such that

$$f(x)=(\pi_f(x)\xi_f, \xi_f), \quad x \in A.$$

We denote by $Q(A)$ the set of all positive linear functional on A. Let π_A denote the representation $\sum_{f\in Q(A)} \oplus \pi_f$ of A and let H_A denote the Hilbert space $\sum_{f\in Q(A)} \oplus H_f$. The weak closure of $\pi_A(A)$ in $L(H_A)$ is $*$-isomorhpic to the second dual A^{**} of A [3: 12. 1. 3.]. Therefore we may regard A^{**} as a W*-algebra on H_A.

Let A and B be C*-algebras, $A \otimes_\alpha B$ their C*-tensor product, and $A^{**} \otimes B^{**}$ the W*-tensor product of A^{**} and B^{**} [5: Theorem 1], [7: Theorem 1]. Then $\pi_A \otimes \pi_B$ is the $*$-isomorphism of $A \otimes_\alpha B$ onto $\pi_A \otimes \pi_B(A \otimes_\alpha B)$. Therefore $A \otimes_\alpha B$ may be regarded as a subalgebra of $A^{**} \otimes B^{**}$.

This paper is concerned with embedding of $A^{**} \otimes B^{**}$ in $(A \otimes_\alpha B)^{**}$.

2. Theorems

THEOREM *Let A and B be C*-algebras. Then there exists the central projection z of $(A \otimes_\alpha B)^{**}$ which has the following properties:*

(a) *$(A \otimes_\alpha B)^{**} z$ is $*$-isomorphic to $A^{**} \otimes B^{**}$.*

(b) *For a positive linear functional f on $A \otimes_\alpha B$ to have the normal extension to $A^{**} \otimes B^{**}$ it is necessary and sufficient that f has the support such that $supp(f) \leqq z$.*

(c) *$A^* \otimes_{\alpha'} B^* = (A \otimes_\alpha B)^* z$, where $A^* \otimes_{\alpha'} B^*$ denotes the norm closure of the algebraic tensor product $A^* \otimes B^*$ of linear spaces A^* and B^* in $(A \otimes_\alpha B)^*$.*

PROOF. Let $Q(A) \times Q(B)$ be the set of all positive linear functionals on $A \otimes_\alpha B$ which can be written as follows:

$$f\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = f^1 \otimes f^2\left(\sum_{i=1}^{n} x_i \otimes y_i\right),$$

for $f^1 \in Q(A)$, $f^2 \in Q(B)$, and $\sum_{i=1}^{n} x_i \otimes y_i \in A \otimes_\alpha B$.

$H = \sum_{f \in Q(A) \times Q(B)} \oplus H_f$ is invariant with respect to $\pi_{A \otimes_\alpha B}$, and we denote by π the restriction of $\pi_{A \otimes_\alpha B}$ to H. Then we have

$$\pi(x) = \pi_A \otimes \pi_B\ (x), \quad x \in A \otimes_\alpha B.$$

Let $\bar{\pi}$ be the representation which is the extension of π to $(A \otimes_\alpha B)^{**}$. Since ker $\bar{\pi}$ is a w*-closed two-sided ideal in $(A \otimes_\alpha B)^{**}$, there exists the central projection z of $(A \otimes_\alpha B)^{**}$ such that

$$(A \otimes_\alpha B)^{**}(I-z) = \ker \bar{\pi}.$$

Then we have

$$\bar{\pi}((A \otimes_\alpha B)^{**}\ z) = A^{**} \otimes B^{**}.$$

Hence, we obtain the $*$-isomorphism of $(A \otimes_\alpha B)^{**}\ z$ onto $A^{**} \otimes B^{**}$ such that

$$\varphi:\ xz \longrightarrow \bar{\pi}\ (xz), \quad xz \in (A \otimes_\alpha B)^{**}\ z.$$

By [6: Theorem 1], $(A^{**} \otimes B^{**})_*$ can be identified with $A^* \otimes_{\alpha'} B^*$. Using the $*$-isomorphism φ of $(A \otimes_\alpha B)^{**}\ z$ onto $A^{**} \otimes B^{**}$, we have

$$(A \otimes_\alpha B)^*\ z = A^* \otimes_{\alpha'} B^*.$$

This completes the proof.

Now, a *-isomorphism φ of $A^{**} \otimes B^{**}$ to $(A \otimes_\alpha B)^{**}$ is said *canonical* if $\varphi^{-1}(x) = \pi_A \otimes \pi_B(x)$, $x \in A \otimes_\alpha B$.

COROLLARY. *$A^{**} \otimes B^{**}$ is canonically $*$-isomorphic to $(A \otimes_\alpha B)^{**}$ if and only if every positive linear fnnctional on $A \otimes_\alpha B$ has the normal extension to $A^{**} \otimes B^{**}$.*

PROOF. Suppose that every positive linear functional on $A \otimes_\alpha B$ has the normal extension to $A^{**} \otimes B^{**}$. From (b) of Theorem, the central projection z is the identity of $(A \otimes_\alpha B)^{**}$, and $A^{**} \otimes B^{**}$ is canonically $*$-isomorphic to $(A \otimes_\alpha B)^{**}$.

Conversely, suppose that there exists a canonically $*$-isomorphism φ from $A^{**} \otimes B^{**}$ onto $(A \otimes_\alpha B)^{**}$. For a positive linear functional f on $A \otimes_\alpha B$, we have

$$f(\sum_{i=1}^{n} x_i \otimes y_i) = \tilde{f}(\varphi(\sum_{i=1}^{n} x_i \otimes y_i)), \quad x_i \in A, \ y_i \in B,$$

where $\tilde{f}$ denotes the normal extension of f to $(A \otimes_\alpha B)^{**}$.

Then the linear functional: $x \longrightarrow \tilde{f}(\varphi(x))$ may be regarded as the normal extension of f to $A^{**} \otimes B^{**}$. This completes the proof.

3. Examples

Wwe consider a case of dual C*-algebras. We begin with the following definition.

Let A be a C*-algebra that does not necessarily contain a unit element. A projection $P \in A^{**}$ is open if there exists a net $\{a_\alpha\} \subset A$ such that $0 \leqq a_\alpha \uparrow P$. If P is open, we say $P' = I - P$ is closed [1: Definition II. 1]. As [1: Proposition II. 2], a projection $P \in A^{**}$ is closed if and only if P supports a weak* closed left invariant subspace in A^*.

In case A is a dual C*-algebra, by [2: Theorem II. 5] A is a two-sided ideal in A^{**}. Hence every projection $P \in A^{**}$ is open and closed.

Lemma. *Let A and B be dual C*-algebras. Then $A \otimes_\alpha B$ is a dual C*-algebra.*

Proof. Let $\hat{C}$ denote the spectrum of any C*-algebra C [3: 2. 9. 7., 3. 1. 5.].

Since A and B are dual C*-algebras, $\hat{A}$ and $\hat{B}$ are discrete, and there exists the homeomorphism $(\pi \times \nu) \longrightarrow \pi \otimes \nu$ of $\hat{A} \times \hat{B}$ onto $(A \otimes_\alpha B)^\wedge$. Hence, $(A \otimes_\alpha B)^\wedge$ is discrete, and every irreducible representation of $A \otimes_\alpha B$ is a compact one.

Let π_t be any element of the equivalence class $t \in (A_\alpha \otimes B)^\wedge$. Then a representation $\sum_{t \in (A \otimes_\alpha B)^\wedge} \pi_t$ of $A \otimes_\alpha B$ is faithful.

Let ε be a positive number, and x an element in $A \otimes_\alpha B$. By [3: 3. 3. 7.], $\{t \in (A \otimes_\alpha B)^\wedge \mid \|\pi_t(x)\| \geqq \varepsilon\}$ is compact, i. e. it consists of finite elements. Consequently, $A \otimes_\alpha B$ is a dual C*-algebra.

Exampla *Let A and B be dual C*-algebras. Then $A^{**} \otimes B^{**}$ is canonically $*$-isomorphic to $(A \otimes_\alpha B)^{**}$.*

Proof. Since $A^* \otimes_{\alpha'} B^*$ is invariant, there exists the central projection z such that

$$A^* \otimes_{\alpha'} B^* = (A \otimes_\alpha B)^* z.$$

Since $A \otimes_\alpha B$ is a dual C*-algebra, z is closed. Hence $(A \otimes_\alpha B)^* z$ is the weakly $*$-closed subspace of $(A \otimes_\alpha B)^*$.

On the other hand, $A^* \otimes_{\alpha'} B^*$ is the weakly *-dense subset of $(A \otimes_\alpha B)^*$. Therefore, we have

$$(A \otimes_\alpha B)^* = (A \otimes_\alpha B)^* z.$$

Now we get

$$(A \otimes_\alpha B)^* = A^* \otimes_{\alpha'} B^*.$$

By Theorem, $A^{**} \otimes B^{**}$ is canonically $*$-isomorphic to $(A \otimes_\alpha B)^{**}$.

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