Some hypersurfaces in a Euclidean space

By

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1. Introduction.

The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

(*) $\qquad R(X, Y)\cdot R=0, \qquad$ for any tangent vectors X and Y,

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. A result of K. Nomizu [2] tells us that the converse is affirmative in the case where M is a certain hypersurface in a Euclidean space. That is

THEOREM A. *Let M be an m-dimensional, connected and complete Rimannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x)\geqq 3$ at least at one point x. If M satisfies the condition* (*), *then it is of the form $M=S^k\times E^{m-k}$, where S^k is a hypersphere in a Euclidean subspace E^{k+1} of E^{m+1} and E^{m-k} is a Euclidean subspace orthogonal to E^{k+1}.*

Now, let R_1 be the Ricci tensor of M and R^1 be the symmetric endomorphism given by $R_1(X, Y)=g(R^1X, Y)$. Then, the condition (*) implies in particular

(**) $\qquad R(X, Y)\cdot R_1=0, \qquad$ for any tangent vectors X and Y.

Recently, S. Tanno [4] gave the following

THEOLEM B. *Let M be an m-dimensional, connected and complete Rimannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x)\geqq 3$ at least at one point x. If M satisfies the condition* (**) *and have the positive scalar curvature, then it is of the form $M=S^k\times E^{m-k}$.*

In the present paper, we shall show that the assumption of having the positive scalar curvature in theorem B can be replaced by some other conditions. That is :

THEOREM C. *Let M be an m-dimensional, connected and complete Riemannian manifold which isometrically immered in a Euclidean space E^{m+1} so that M is not minimal and the type number $k(x)\geqq 3$ at least at one point x. If M satisfies the condition* (**), *then it is of* *the form* $M=S^k\times E^{m-k}$.

THEOREM D. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x)$ is greater than 2 and odd at least at one point x. If M satisfies the condition (**), then it is of the form* $M=S^k\times E^{m-k}$.

2. Reduction of the condition (**).

Let M be a connected hypersurface in a Euclidean space E^{m+1} and let g be the induced metric on M. Let U be a neighborhood of a point x_0 of M on which we can choose a unit vector field ξ normal to M. For local vector fields X and Y on U tangent to M, we have the formulas of Gauss and Weingarten :

(2. 1) $$D_XY=\nabla_XY+H(X, Y)\xi,$$

(2. 2) $$D_X\xi=-AX,$$ where D_X and ∇_X denote

covariant differentiation for the Euclidean connection of E^{m+1} and the Riemannian connection on M, respectively. H is the second fundamental form and A is a symmetric endomorphism given by $H(X, Y)=g(AX, Y)$. Then, the equation of Gauss is

(2. 3) $$R(X, Y)=AX\wedge AY,$$ where, in general, $X\wedge Y$

denotes the endomorphism which maps Z upon $g(Z, Y)X-g(Z, X)Y$. The type number $k(x)$ at a point x is, by definition, the rank of A at x. From (2, 3), the Ricci tensor R_1 of M is given by

(2. 4) $$R_1(X, Y)=(trace\ A)g(AX, Y)-g(A^2X, Y).$$

For a point x of M, take an orthonormal basis $\{e_1,\cdots,e_m\}$ of the tangent space $T_x(M)$ such that $Ae_i=\lambda_ie_i$, $1\leq i\leq m$. Then, the equation (2. 3) implies

(2. 5) $$R(e_i, e_j)=\lambda_i\lambda_je_i\wedge e_j, \qquad 1\leq i, j\leq m,$$

and (2. 4) implies

(2. 6) $$R_1(e_i, e_i)=\lambda_i\sum_{h=1}^m\lambda_h-\lambda_i^2,$$ and otherwise zero.

From (2. 5) and (2. 6), by direct computing, the condition (**) is euivalent to

(2. 7) $$\lambda_i\lambda_j(\lambda_i-\lambda_j)(\sum_{h=1}^m\lambda_h-\lambda_i-\lambda_j)=0, \text{ for } i\neq j.$$

From (2. 7), for each point $x\in M$, we see that the following cases are possible at x :

I. $\lambda_1=\cdots\lambda_k=\lambda,\ \lambda_{k+1}=\cdots=\lambda_m=0,$

II. $\lambda_1=\cdots=\lambda_t=\lambda,\ \lambda_{t+1}=\cdots=\lambda_{t+t'}=\mu,\ \lambda_{t+t'+1}=\cdots=\lambda_m=0,$

where $k=k(x)$, and, for II, $\lambda\neq\mu$, $t=t(x)\geqq 2$, $t'=t'(x)\geqq 2$, $k=t+t'$, $(t-1)\lambda+(t'-1)\mu=0$. If M satisfies the condition (*), then II can not occur. In future, we shall show that II can not occur under some conditions.

3. Lemmas.

Now, we assume that $k(z)\geqq 3$ at some point $z\in M$ and II is valid at z. Then, by the continuity argument for the characteristic polynomial of A, we see that II is valid and furthermore, t and t' are constant near z and hence, let $W=\{x\in M;\ k(x)\geqq 3$ and II is valid at $x\}$, which is an open set of M. For each point $x_o\in W$, let W_o be the connected component of x_o in W. Then, non-zero eigenvalues of A, λ and μ, are differentiable functions on W_o and hence, we can define three differentiable distributions, T_λ, T_μ and T_o corresponding λ, μ and 0, respectively on W_o. Here, if $k=m$ on W_o, then we consider $T_o(x)$ as $\{0\}$, for $x\in W_o$. Let $T_1(x)=T_\lambda(x)+T_\mu(x)$ (direct sum), for each point $x\in W_o$. Then, T_1 is differentiable, and, from (2. 4) and II, we have

$$R^1X=\lambda\mu X,\ \text{for}\ X\in T_1,\ \text{and}\ R^1X=0,\ \text{for}\ X\in T_o. \tag{3. 1}$$

For any $Z\in T_x(M)$, Z_λ, Z_μ and Z_o will denote the components of Z in $T_\lambda(x)$, $T_\mu(x)$ and $T_o(x)$, respectively. Then we have

LEMMA 3. 1. *T_λ and T_μ are involutive.*

PROOF. We recall the Codazzi equation

$$(\nabla_X A)Y=(\nabla_Y A)X.$$

Suppose that X and Y are vector fields belonging to T_λ. Then

$$(\nabla_X A)Y=X\lambda Y+(\lambda-\mu)(\nabla_X Y)_\mu+\lambda(\nabla_X Y)_o,$$

$$(\nabla_Y A)X=Y\lambda X+(\lambda-\mu)(\nabla_Y X)_\mu+\lambda(\nabla_Y X)_o.$$

Thus, we have

$$X\lambda Y-Y\lambda X=0,\ \text{and}\ [X,Y]_\mu=[X,Y]_o=0.$$

The second identity shows that $[X,Y]\in T_\lambda$, proving that T_λ is involutive. Similarly, T_μ is involutive, and furthermore, for any vector fields X and Y belonging to T_μ, we have $X\mu Y-Y\mu X=0$. Q. E. D.

For each point $x\in W_o$, let $M_\lambda(x)$ and $M_\mu(x)$ be the maximal integral manifolds through x of T_λ and T_μ, respectively. Since $t\geqq 2$, $t'\geqq 2$, from the proof of lemma 3. 1, we have

LEMMA 3. 2. *λ and hence μ are constant on W_o.*

Next, suppose that $X\in T_\lambda$, $Y\in T_\mu$, and compute the both sides of Codazzi equation :

$$(3.\ 2)\qquad \begin{aligned}(\nabla_X A)Y&=X\mu Y+(\mu-\lambda)(\nabla_X Y)_\lambda+\mu(\nabla_X Y)_o,\\ (\nabla_Y A)X&=Y\lambda X+(\lambda-\mu)(\nabla_Y X)_\mu+\lambda(\nabla_Y X)_o.\end{aligned}$$

Thus, by virtue of lemma 3. 2, we have

$$(3.\ 3)\qquad (\nabla_Y X)_\mu=0,\ \text{and}\ (\nabla_X Y)_\lambda=0,\ \text{for}\ X\epsilon T_\lambda,\ Y\epsilon T_\mu.$$

Furthermore, suppose that $X\epsilon T_\lambda$, $Y\epsilon T_o$, and compute the both sides of the Codazzi equation :

$$(3.\ 4)\qquad \begin{aligned}(\nabla_X A)Y&=-\lambda(\nabla_X Y)_\lambda-\mu(\nabla_X Y)_\mu,\\ (\nabla_Y A)X&=Y\lambda X+(\lambda-\mu)(\nabla_Y X)_\mu+\lambda(\nabla_Y X)_o.\end{aligned}$$

Thus, we have $(\nabla_Y X)_o=0$, that is, $\nabla_Y X\epsilon T_1$. Similarly, for $X\epsilon T_\mu$, $Y\epsilon T_o$, we have $\nabla_Y X\epsilon T_1$. Thus we have $\nabla_Y T_1\subset T_1$, for $Y\epsilon T_o$, and hence, $\nabla_Y T_o\subset T_o$, for $Y\epsilon T_o$. Therefore, of course, T_o is involutive and furthermore, for each point $x\epsilon W_o$, let $M_o(x)$ be the maximal integral manifold through x of T_o, then

LEMMA 3. 3. *Each $M_o(x)$ is totally geodesic.*

4. Main results.

Since T_λ, T_μ and T_o are differentiable on W_o, for each point $x\epsilon W_o$, we may choose a differentiable field of orthonormal basis $\{X_i\}$ near x so that $\{X_a\}$, $\{X_p\}$ and $\{X_u\}$ are bases for T_λ, T_μ and T_o, respectively. Here $1\leqq a, b, c, \cdots\leqq t$, $t+1\leqq p, q, r, \cdots\leqq t+t'=k$, $k+1\leqq u, v, w,\cdots\leqq m$. From (2. 3) and II, with respect to the above basis $\{X_i\}$, we have

$$(4.\ 1)\qquad \begin{aligned}&R(X_a, X_b)=\lambda^2 X_a\wedge X_b,\\ &R(X_a, X_p)=\lambda\mu X_a\wedge X_p,\\ &R(X_p, X_q)=\mu^2 X_p\wedge X_q,\qquad \text{and otherwise zero.}\end{aligned}$$

On the other hand, in general, for a local differentiable field of orthonormal basis $\{X_i\}$ in a Riemannian manifold (M, g), we may put

$$(4.\ 2)\qquad \nabla_{X^i}X_j=\sum_{h=1}^{m}\gamma_{ijh}X_x,\qquad \text{where } \nabla_X \text{ denotes the}$$

covariant differentiation with respect to the Riemannian connection given by g and $\gamma_{ijh}=-\gamma_{ihj}$, $m=\dim M$.

First, we assume that $k(z)=m$ at some point $z\epsilon M$. Then, the type number is also m near z and hence, let $W=\{x\epsilon M;\ k(x)=m \text{ at } x\}$, which is an open set of M. For each point $x_o\epsilon W$, let W_o be the connected component of X_o in W. Then, I is valid at each point of W_o. Because, if II is valid at some point of W_o, then we see that II is valid everywhere on W_o and furthermore, considering $T_o(x)$ as $\{0\}$ for each point $x\epsilon W_o$, we may think that this case is the special case in the arguments in § 3. Thus, from (3. 2), T_λ and T_μ are parallel on the open subspace W_0. Therefore, in particular, it must follow that $R(X, Y)=0$, for $X\epsilon T_\lambda$, $Y\epsilon T_\mu$. But, this contradicts to (4. 1). Thus we have

PROPOSITION 4. 1. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x)=m$ at least at one point x. If M satisfies the condition (**), then it is a hypersphere.*

Next, we assume that $3\leqq k(z)<m$ at some point $z\epsilon M$. Then, by the arguments in § 3, we can take non-trivial three differentiable distributions, T_λ, T_μ and T_0 on an open set W_0. we shall now study the properties of T_λ, T_μ and T_0.

From (3. 4) and (4. 2), we have the followings

(4. 3) $\gamma_{aub}=0$, for $a\neq b$, similarly, $\gamma_{puq}=0$, for $p\neq q$,

(4. 4) $\gamma_{aua}=-X_u\lambda/\lambda$, similarly, $\gamma_{pup}=-X_u\mu/\mu$.

(4. 5) $(\lambda-\mu)\gamma_{uap}+\mu\gamma_{aup}=0,$

similarly

(4. 6) $(\mu-\lambda)\gamma_{upa}+\lambda\gamma_{pua}=0.$

Ffom (4. 5) and (4. 6), we have

(4. 7) $\lambda\gamma_{pua}-\mu\gamma_{aup}=0.$

From (4. 2) and the fact of lemma 3. 3, we have

(4. 8) $\gamma_{uav}=0$, similarly $\gamma_{upv}=0$.

Since $(t-1)\lambda+(t'-1)\mu=0$, from (4. 4), we have

(4. 9) $\gamma_{aua}=\gamma_{bub}=\gamma_{pup}=\gamma_{quq}=-X_u\lambda/\lambda$, for $a\neq b$, $p\neq q$.

On the other hand, from (4. 1), we have

(4. 10)
$$\begin{aligned} R(X_a, X_u)X_v &= \nabla_{X_a}\nabla_{X_u}X_v-\nabla_{X_u}\nabla_{X_a}X_v-\nabla_{[X_a, X_u]}X_v \\ &= \sum_{i=1}^{m}(X_a\gamma_{uvi}+\sum_{h=1}^{m}\gamma_{uvh}\gamma_{ahi}-X_u\gamma_{avi}-\sum_{h=1}^{m}\gamma_{avh}\gamma_{uhi} \\ &\quad -\sum_{h=1}^{m}(\gamma_{auh}-\gamma_{uah})\gamma_{hvi})X_i=0. \end{aligned}$$

Thus, from (4. 10), by virtue of (4. 3) and (4. 8), we have

(4. 11)
$$X_u\gamma_{ava}+\gamma_{aua}\gamma_{ava}+\sum_{p=t+1}^{k}\gamma_{avp}\gamma_{upa}+\sum_{p=t+1}^{k}\gamma_{aup}\gamma_{pva}-\sum_{p=t+1}^{k}\gamma_{uap}\gamma_{pva}=0,$$

similarly, considering $R(X_p, X_u)X_v=0$,

(4. 12)
$$X_u\gamma_{pvp}+\gamma_{pup}\gamma_{pvp}+\sum_{a=1}^{t}\gamma_{pva}\gamma_{uap}+\sum_{a=1}^{t}\gamma_{pua}\gamma_{avp}-\sum_{a=1}^{t}\gamma_{upa}\gamma_{avp}=0.$$

Taking $u=v$ in (4. 11), (4. 12) and using (4. 5), (4. 6), (4. 7) and (4. 9), we have

$$\lambda\sum_{q=t+1}^{k}(\gamma_{auq})^2+\mu\sum_{b=1}^{t}(\gamma_{aup})^2=0. \tag{4. 13}$$

Thus, from (4. 13), we have

$$(t\mu+t'\lambda)\sum_{p=t+1}^{k}\sum_{a=1}^{t}(\gamma_{aup})^2=0. \tag{4. 14}$$

Since $(t-1)\lambda+(t'-1)\mu=0$, (4. 14) implies

$$(t-t')(t+t'-1)\sum_{p=t+1}^{k}\sum_{a=1}^{t}(\gamma_{aup})^2=0. \tag{4. 15}$$

Thus, from (4. 15), if $t\neq t'$, then we have

$$\gamma_{aup}=0, \text{ and hence, } \gamma_{pua}=0. \tag{4. 16}$$

From (4. 16) (4. 11) implies

$$X_u\gamma_{ava}+\gamma_{aua}\gamma_{ava}=0. \tag{4. 17}$$

Now, let $L(s)$ be a geodesic starting from any point $x\in W_0$ with any initial direction belonging to $T_0(x)$, where s denotes the arc-length parameter. Then, by lemma 3. 3, $L(s)$ is contained in $M_0(x)$ for sufficiently small s. From (4. 9) and (4. 17), we have

$$\frac{d^2}{ds^2}(1/\lambda)=0, \qquad \text{along } L(s). \tag{4. 18}$$

Thus, if M is complete, then, by the same ones as the arguments in [2], we can show that $L(s)$ is infinitely extendible in W_0 and furthermore, λ is constant along $L(s)$, and hence, is constant on each $M_0(x)$. Thus, from (4. 9), we have

$$\gamma_{aua}=\gamma_{pup}=0. \tag{4. 19}$$

Therefore, from (3. 3), (4. 3), (4. 8), (4. 16) and (4. 19), we can show that T_λ, T_μ and T_0 are parallel on W_0. Thus, in particular, it must follow that $R(X, Y)=0$, for $X\in T_\lambda$, $Y\in T_\mu$. But, this contradicts to (4. 1). Therefore, in this case, II can not occur. If $t=t'$, then we see that M is minimal in E^{m+1}. Thus, we have theorem C. If k is odd, of course, it must follow that $t\neq t'$. Thus, we have theorem D. Lastly, we assume that M is a space of constant scalar curvature. Then, from (4. 4), (4. 19) is valid, and hence, from (4. 11), we have (4. 16). Therefore, we can also show that II can not occur in this case. That is

PROPOSITION 4. 2. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x)\geqq 3$ at least at one point x. If M satisfies the condition* (**) *and has the constant scalar curvature, then it is the form $M=S^k\times E^{m-k}$.*

Remark. In our arguments, if the type number $k(x)=3$ at some point x, then we see that II can not occur near x.

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