# On 3-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor

By

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## 1. Introduction

If a Riemannian manifold $M$ is locally symmetric, then its curvature tensor $R$ satisfies

(*) $\quad R(X, Y)\cdot R=0 \quad$ for all tangent vectors $X$ and $Y$,

where the endomorphism $R(X, Y)$ operates on $R$ as a derivation of tensor algebra at each point of $M$.

Conversely, does this algebraic condition (*) on the curvature tensor field $R$ imply that $M$ is locally symmetric (i. e. $\nabla R=0$) ?

One must exclude the 2-dimentional case, as was already observed by E. Cartan, 1. K. Nomizu has conjectured that the answer is affirmative in the case where $M$ is irreducible and complete and dim. $M\geqq 3$. There are some partial or related results in this direction.

The main purpose of the present paper is to deal with the same problem about 3-dimensional Riemannian manifolds.

## 2. Reduction of condition (*) and some results

Let $M$ be a 3-dimensional connected Riemannian manifold, then it is well known that the curvature tensor $R$ of $M$ is written in the form

$$R(X, Y)=AX\wedge Y+X\wedge AY-\frac{1}{2}(\text{trace } A)X\wedge Y \tag{2.1}$$

where $A$ is a field of symmetric endomorphism which corresponds to the Ricci tensor field $S$, that is, $g(AX, Y)=S(X, Y)$, $g$ being the Riemannian metric and $X\wedge Y$ denotes the endomorphism which maps $Z$ upon $g(Z, Y)X-g(Z, X)Y$.

At a point $x\in M$, let $\{e_1, e_2, e_3\}$ be an orthogonal basis of the tangent space $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $i=1, 2, 3$.

Then, the equation (2. 1) implies

(2.2) $$R(e_i, e_j)=(\lambda_i+\lambda_j-\frac{1}{2}\sum_{k=1}^{3}\lambda_k)e_i\wedge e_j.$$

By computing

$$(R(e_i, e_j)\cdot R)(e_k, e_l)=[R(e_i, e_j), R(e_k,e_l)]-R(R(e_i, e_j)e_k, e_l)$$
$$-R(e_k, R(e_i, e_j)e_l),$$

we find that it is zero except possibly in the case where $k=i$ and $l\neq i, j$ $(i\neq j)$. For this case we have

(2.3) $$(R(e_i, e_j)\cdot R)(e_i, e_l)=(\lambda_j-\lambda_i)(\lambda_j+\lambda_i-\frac{1}{2}\sum_{k=1}^{3}\lambda_k)e_j\wedge e_l.$$

Thus, we see that the condition (*) is equivalent to

(2.4) $$(\lambda_j-\lambda_i)(2(\lambda_j+\lambda_i)-\sum_{k=1}^{3}\lambda_k)=0, \qquad \text{for} \quad i\neq j.$$

Then, we have the following

THEOREM. *Let $M$ be a 3-dimensional connected Riemannian manifold whose curvature tensor $R$ satisfies the condition (*). If the rank of the Ricci form is* 3 *at some point of $M$, then $M$ is a space of constant curvature.*

PROOF. We assume that the rank of the Ricci form is 3 at a point $x_0\in M$. Then, if $\lambda_1=\lambda_2$, $\lambda_2\neq\lambda_3$, then from (2.4), we get

$$2(\lambda_1+\lambda_3)-(2\lambda_1+\lambda_3)=0.$$

Thus, we get $\lambda_3=0$. This is a contradiction.

Similarly, if $\lambda_1\neq\lambda_2$, $\lambda_2\neq\lambda_3$, $\lambda_3\neq\lambda_1$, then from (2.4), we get

$$2(\lambda_1+\lambda_2)-\sum_{k=1}^{3}\lambda_k=0$$

and

$$2(\lambda_2+\lambda_3)-\sum_{k=1}^{3}\lambda_k=0.$$

Thus, we get $\lambda_1=\lambda_3$. This is a contradiction. Therefore, we can conclude that $\lambda_1=\lambda_2=\lambda_3$ at $x_0$.

Now, let $W=\{x\in M$ ; the rank of $S$ is 3 at $x\}$, which is an open set. Let $W_0$ be the connected component of $x_0$ in $W$. Then, we can easily see that $\lambda_1=\lambda_2=\lambda_3\equiv\lambda(\neq 0)$ on $W_0$ and hence $\lambda$ is constant on $W_0$. We now show that $W_0$ is actually equal to $M$. Let $x$ be a point of $\overline{W}_0-W_0$. By the continuity argument for the characteristic polynomial of $A$, we see that the the rank of $S$ is equal to 3 at $x$. Thus, $W_0$ is open and closed so that $W_0=M$. Therefore, $M$ is an Einstein space with Ricci tensor $S=\lambda g$, and hence, by virtue of (2.1), we see that $M$ is a space of constant curvature $\frac{\lambda}{2}$ $(\lambda\neq 0)$. q. e. d.

In the next place, we assume that the rank of $A$ (or $S$) is 2 at some pont, say $x_0 \in M$. In this case, if $\lambda_3=0$ at $x_0$, then we see that $\lambda_1=\lambda_2 \neq 0$.

We shall now state a few examples of non-symmetric and irreducible Riemannian manifolds satisfying the condition (*).

Let $M$ be a 2-dimensional Riemannian manifold with metric $g$, $I$ an open interval of a real line $R$ with natural metric $dt^2$ and $\overline{M}=M\times I$. The tangent space $T_{\bar{p}}(\overline{M})$ at a point $\bar{p}\in\overline{M}(\bar{p}=(p,t)$, $p\in M$ and $t\in I)$ is considered as the direct sum $T_p(M)+T_t(I)$, where $T_p(M)$ and $T_t(I)$ are the tangent spaces at $p\in M$ and $t\in I$ respectively. That is, any $X\in T_p(M)$ is uniquely decomposed as

$$\overline{X}=X+X_I, \quad X\in T_p(M), \quad X_I\in T_t(I).$$

Now, we shall define the following Riemannian metric $\bar{g}$ on $\overline{M}$;

$$\bar{g}(\overline{X}, \overline{Y})=e^{-2\lambda}g(X, Y)+e^{-2\mu}dt(X_I)dt(Y_I),$$

where $\lambda$ and $\mu$ are some functions of $t\in I$.

If we denote by $\{X, Y\}$ an orthonormal basis of vector fields on a neighborhood $U\subset M$, then $\left(\overline{X}=e^{\lambda}X,\ \overline{Y}=e^{\lambda}Y,\ \overline{Z}=e^{\mu}\frac{\partial}{\partial t}\right)$ is an orthonormal basis of vector fields on $U\times I\subset M$.

Between the Riemannian connections $\nabla$ and $\overline{\nabla}$ corresponding to $g$ and $\bar{g}$, the following relations are valid;

$$\overline{\nabla}_{\overline{X}}\overline{X}=e^{\lambda}g(Y, \nabla_X X)\overline{Y}+\lambda' e^{\mu}\overline{Z}=e^{\lambda}\nabla_X X+\lambda' e^{2\mu}\frac{\partial}{\partial t}$$

$$\overline{\nabla}_{\overline{Y}}\overline{Y}=e^{\lambda}g(X, \nabla_Y Y)\overline{Y}+\lambda' e^{\mu}\overline{Z}=e^{\lambda}\nabla_Y Y+\lambda' e^{2\mu}\frac{\partial}{\partial t}$$

$$\overline{\nabla}_{\overline{X}}\overline{Y}=e^{\lambda}g(X, \nabla_X Y)\overline{X}=e^{2\lambda}\nabla_X Y$$

$$\overline{\nabla}_{\overline{Y}}\overline{X}=e^{\lambda}g(Y, \nabla_Y X)\overline{Y}=e^{2\lambda}\nabla_Y X$$

$$\overline{\nabla}_{\overline{X}}\overline{Z}=-\lambda' e^{\mu}\overline{X}=-\lambda' e^{\lambda+\mu}X \tag{2.5}$$

$$\overline{\nabla}_{\overline{Y}}\overline{Z}=-\lambda' e^{\mu}\overline{Y}=-\lambda' e^{\lambda+\mu}Y$$

$$\overline{\nabla}_{\overline{Z}}\overline{X}=0$$

$$\overline{\nabla}_{\overline{Z}}\overline{Y}=0$$

$$\overline{\nabla}_{\overline{Z}}\overline{Z}=0$$

and

$$[\overline{Z}, \overline{X}]=\lambda' e^{\mu}\overline{X}=\lambda' e^{\lambda+\mu}X$$

$$[\overline{Z}, \overline{Y}]=\lambda' e^{\mu}\overline{Y}=\lambda' e^{\lambda+\mu}Y$$

$$[\overline{X}, \overline{Y}]=e^{\lambda}\{g(X, \nabla_X Y)\overline{X}-g(Y, \nabla_Y X)\overline{Y}\}=e^{2\lambda}[X, Y].$$

Using these equations, we get the following relations between the curvature tensors $\overline{R}$ and $R$ corresponding to $\overline{\nabla}$ and $\nabla$;

$$\bar{R}(\bar{Y}, \bar{X})\bar{X} = e^{3\lambda}R(Y, X)X - \lambda'^2 e^{\lambda+2\mu}Y$$

$$\bar{R}(\bar{X}, \bar{Z})\bar{X} = -e^{3\mu}(\lambda'' + \lambda'\mu' - \lambda'^2)\frac{\partial}{\partial t}$$

$$\bar{R}(\bar{Y}, \bar{Z})\bar{X} = 0$$

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = 0$$

$$\bar{R}(\bar{X}, \bar{Z})\bar{Z} = e^{\lambda+2\mu}(\lambda'' + \lambda'\mu' - \lambda'^2)X.$$

Now, let us assume the condition

(2. 6) $$\lambda'' + \lambda'\mu' - \lambda^2 = 0,$$

then the rank of the Ricci form $\bar{S}$ of $\bar{M}$ is 2 or 0 at every point of $\bar{M}$. In fact, we can see that

$$\bar{S}(\bar{X}, \bar{X}) = \bar{S}(\bar{Y}, \bar{Y}) = \bar{g}(\bar{R}(\bar{Y}, \bar{X})\bar{X}, \bar{Y})$$

$$= e^{2\lambda}(e^{\lambda}g(R(Y, X)X, Y) - \lambda'^2 e^{2\mu})$$

$$\bar{S}(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Z}) = \bar{S}(\bar{Y}, \bar{Z}) = \bar{S}(\bar{Z}, \bar{Z}) = 0,$$

that is,

(2. 7) $$\bar{S} = \begin{pmatrix} e^{2\lambda}(Ke^{2\lambda} - \lambda'^2 e^{2\mu}) & 0 & 0 \\ 0 & e^{2\lambda}(Ke^{2\lambda} - \lambda'^2 e^{2\mu}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $K$ is the Gaussian curvature of $M$.

To find out the pairs of functions $\lambda$, $\mu$ which satisfy the differential equation (2. 6), we assume that $\mu$ is given. Then by Bernoulli's formula, we get $\frac{1}{\lambda'} = -e^{\mu}\int e^{-\mu}dt.$ Using the last equation, we can choose the pairs

(I) $\begin{cases} \lambda = t \\ \mu = t \end{cases}$ $(t \in I = R)$, (II) $\begin{cases} \lambda = -\log t \\ \mu = 0 \end{cases}$ $(t \in I = R_+)$

and etc. ($R_+$: a positive half line)

Therefore, we see that the Riemannian manifolds $\bar{M}$ with the metric $\bar{g}$ corresponding to the pairs of functions $\lambda$, $\mu$ like (I), (II) are irreducible by (2. 5) and these curvature tensors satisfy the condition (*). And moreover, they are not symmetric, because any 3-dimensional symmetric Riemannian manifold whose Ricci tensor has the rank equal to 2 or 0 is reducible.

But, as is easily seen, they are not complete. Therefore, with respect to Nomizu's conjecture, the assumption of completeness is essential.

REMARK 1. In the case (I), we assumed that $K \not\equiv 1$.

REMARK 2. For 3-dimentional Riemannian manifolds, the condition $R(X, Y)\cdot R = 0$ is equivalent to the condition $R(X, Y)\cdot S=0$.

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