On harmonic tensor and C-analytic tensor in compact K-contact Riemannian manifolds.

By

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S. Tachibana (4) has introduced a C-analytic 1-form in a Sasakian manifold which satisfies the following

$$\varphi_k{}^l \partial_{[l} v_{i]} - \partial_{[k}(\varphi_i{}^l{}_{]} v_l)^{1)} = 0.$$

The vector v_i given by the above relation corresponds to an almost covarient analytic vector in an almost complex manifold.

In this paper, we shall define a C-analytic covariant tensor in a contact manifold, and consider some properties of C-analytic covariant tensor corresponding to almost analytic covariant tensors in an almost complex manifold. In §1 we shall define a C-tensor and a C-analytic tensor in an almost contact manifold. In §2 we shall give some identities in a K-contact Riemannian manifold and a Sasakian manifold. In §3 we shall state some properties of C-analytic tensors in a compact K-contact Riemannian manifold. The last §4 will be devoted to the discussion of a C-analytic tensor and a harmonic tensor in a compact Sasakian manifold.

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1. C-tensor and C-analytic tensor in an almost contact manifold.

Let M be an $n(=2m+1)$-dimensional differentiable manifold with local coordinates $\{x^i\}$. If there exist a tensor field $\varphi_j{}^i$, a contravariant vector field ξ^i and a covariant vector field η_i on M satisfying the following relations:

(1. 1) $$\xi^i \eta_i = 1,$$

(1. 2) $$rank|\varphi_j{}^i| = 2m,$$

(1. 3) $$\varphi_j{}^i \xi^j = 0,$$

(1. 4) $$\varphi_j{}^i \eta_i = 0,$$

1) As to the notations, we follow K. Yano's.

(1. 5) $$\varphi_j{}^i\varphi_h{}^j = -\delta_h^i+\eta_h\xi^i,$$

then the manifold M is called an almost contact manifold [1].

First of all, we shall introduce a C-tensor in an almost contact manifold which corresponds to a pure tensor in an almost complex manifold. If a tensor $T_{j_p\cdots j_1}$ (for brevity $T_{(j)}$) satisfies the following

(1. 6) $$\varphi_{j_t}{}^a T_{j_p\cdots a\cdots j_s\cdots j_1} = \varphi_{j_s}{}^a T_{j_p\cdots j_t\cdots a\cdots j_1} \quad \text{for every } j_s,\ j_t,$$

then we say that the tensor $T_{(j)}$ is a C-tensor.

Transvecting (1. 6) with $\varphi_i{}^{j_t}$ and putting $\widetilde{T}_{(j)} = \varphi_{j_s}{}^a T_{j_p\cdots a\cdots j_1}$, we have

$$\varphi_i{}^{j_t}\widetilde{T}_{j_p\cdots j_t\cdots j_s\cdots j_1} = \varphi_{j_s}{}^a\widetilde{T}_{j_p\cdots i\cdots a\cdots j_1}.$$

Hence, we have the following

LEMMA 1. 1. *In an almost contact manifold, if $T_{(j)}$ is a C-tensor, then $\widetilde{T}_{(j)}$ is also a C-tensor.*

Using (1. 3) and (1. 6), we have

(1. 7) $$\eta^a\widetilde{T}_{j_p\cdots a\cdots j_1}=0.$$

By lemma 1. 1, we have immediately

(1. 8) $$\eta^a T_{j_p\cdots a\cdots j_1}=0.$$

Hence, we have the following

LEMMA 1. 2. *In an almost contact manifold, C-tensors $T_{(j)}$ and $\widetilde{T}_{(j)}$ are both orthogonal to η^i.*

Next, we shall introduce a C-analytic tensor which is corresponding to an almost analytic tensor in an almost complex manifold. In an almost contact manifold, if a skew-symmetric C-tensor $T_{(j)}$ satisfies

$$\partial_{[k}\widetilde{T}_{j_p\cdots j_1]}=\varphi_k{}^l\partial_{[l}T_{j_p\cdots j_1]}$$

then we shall say that $T_{(j)}$ is a C-analytic tensor.

This equation can be written in the tensor form

(1. 9) $$\nabla_{[k}\widetilde{T}_{j_p\cdots j_1]}=\varphi_k{}^l\nabla_{[l}T_{j_p\cdots j_1]}$$

where ∇_k denotes the operator of Rimannian derivative. In this place, we remark that C-analytic tensor is always skew-symmetric.

Now, (1. 9) can be written explicitly as

(1. 10) $$\nabla_k\widetilde{T}_{j_p\cdots j_1}-\sum_{s=1}^{p}\nabla_{j_s}\widetilde{T}_{j_p\cdots k\cdots j_1}=\varphi_k{}^l\nabla_l T_{j_p\cdots j_1}-\varphi_k{}^l\sum_{s=1}^{p}\nabla_{j_s}T_{j_p\cdots l\cdots j_1}.$$

Taking account of (1. 6), (1. 10) turns to

$$\nabla_k \widetilde{T}_{j_p \cdots j_1} - \sum_{s=1}^{p} (\nabla_{j_s} \varphi_k{}^l) T_{j_p \cdots l \cdots j_1} - \varphi_k{}^l \sum_{s=1}^{p} \nabla_{j_s} T_{j_p \cdots l \cdots j_1}$$

$$= \varphi_k{}^l \nabla_l T_{j_p \cdots j_1} - \varphi_k{}^l \sum_{s=1}^{p} \nabla_{j_s} T_{j_p \cdots l \cdots j_1}.$$

Hence, we have

(1. 11) $$\varphi_k{}^s \nabla_s T_{(j)} - \nabla_k \widetilde{T}_{(j)} + \sum_{r=1}^{p} (\nabla_{j_r} \varphi_k{}^l) T_{j_p \cdots l \cdots j_1} = 0$$

which is corresponding to the relation given by an almost analytic tensor in an almost complex manifold[2].

2. Identities in almost Sasakian manifolds.

Let M be an n-dimensional differentiable manifold. If there exists a 1-form η over M such that

$$\eta \wedge (d\eta)^n \neq 0$$

where $d\eta$ means the exterior differential of η and the operator $\wedge$ means the exterior multiplication. Then the manifold M is said to be a contact manifold [1].

It is well known that in a contact manifold we can find four tensors $\varphi_j{}^i$, ξ^i, η_i and g_{ji} so that they define an almost contact Riemannian structure. We call the manifold with an almost contact Riemannian structure briefly an almost Sasakian manifold.

In an almost Sasakian manifold, we know the relations between tensors $\varphi_j{}^i$, ξ^i, η_i and g_{ji} such that

(2. 1) $$g_{ji} \xi^j = \eta_i \text{ [3]}$$

(2. 2) $$g_{ji} \varphi_h{}^j \varphi_k{}^i = g_{hk} - \eta_h \eta_k$$

(2. 3) $$2\varphi_{ji} = \partial_j \eta_i - \partial_i \eta_j$$

where $\partial_j = \partial / \partial x_j$.

In an almost contact manifold, there are four tensors $N_{kj}{}^i$, N_{kj}, $N_j{}^i$ and N_j which correspond to the Nijenhuis tensor in an almost complex manifold. The contact Riemannian manifold with vanishing $N_j{}^i$ and $N_{kj}{}^i$ are called a K-contact Riemannian manifold and a normal contact Riemannian manifold (for brevity a Sasakian manifold) respectively.

We know that in a K-contact Riemannian manifold, η^i is a killing vector field with respect to the Riemannian metric, that is, the following relation is satisfied:

2) For an almost complex manifold, see S. Tachibana [3].
3) In this paper, we use a notation η^j in stead of ξ^j.

(2. 4) $$\nabla_j\eta_i+\nabla_i\eta_j=0.$$

In a Sasakian manifold the following relation is satisfied:

(2. 5) $$\nabla_k\varphi_{ji}=\eta_j g_{ik}-\eta_i g_{jk}.$$

Hence, it will be shown that for a contact Riemannian manifold if η^i is a Killing vector field, then the manifold reduces to K-contact Riemannian manifold and for a K-contact Riemannian manifold, if the relation (2. 5) is satisfied, then the manifold reduces to Sasakian manifold.

Now, we shall prepare some useful identities in a K-contact Riemannian manifold.

From (2. 3) and (2. 4), we have

(2. 6) $$\varphi_{ji}=\nabla_j\eta_i.$$

From (2. 6), we easily find

(2. 7) $$\nabla_k\varphi_{ji}+\nabla_i\varphi_{kj}+\nabla_j\varphi_{ik}=0.$$

Operating ∇_k to (1. 5), by (2. 6) we get

(2. 8) $$\varphi_h{}^j\nabla_k\varphi_j{}^i+\varphi_j{}^i\nabla_k\varphi_h{}^j=\varphi_{kh}\eta^i+\eta_h\varphi_k{}^i.$$

In (2. 8) contracting i and h, by (1. 3) and (1. 4), we have

(2. 9) $$\varphi_h{}^j\nabla_k\varphi_j{}^h=0.$$

Transvecting (2. 7) with φ^{kj}, by (2. 9) we have

(2. 10) $$\varphi^{kj}\nabla_k\varphi_{ji}=0.$$

Operating ∇_h to (1. 3), by (2. 6) and (1. 5), we have

(2. 11) $$\eta^j\nabla_h\varphi_j{}^i=\delta_h^i-\eta_h\eta^i.$$

Transvecting (2. 7) with η^h, we have

(2. 12) $$\eta^h\nabla_h\varphi_j{}^i=0.$$

Transvecting (2. 8) with φ^{hk} and using (1. 5), (2. 10) and (2. 12), we have

(2. 13) $$\nabla^r\varphi_{ir}=(n-1)\eta_i.$$

Finally substituting (2. 13) into the Ricci's identity, we get

(2. 14) $$\nabla^k\nabla_j\varphi_k{}^i=R_{ja}\varphi^{ai}+\frac{1}{2}\varphi^{ab}R_{abj}{}^i+(1-n)\varphi_j{}^i$$

where $R_{abj}{}^i$ and R_{ja} are Riemannian and Ricci tensors respectively.

3. C-analytic tensors in a compact K-contact Riemannian manifold.

In this section we assume that M is a compact K-contact Riemannian manifold and $T_{(j)}$ is a C-analytic tensor.

Operating ∇^k to (1.8), we have

(3. 1) $$\eta^a\nabla_k T_{j_p\cdots a\cdots j_1}+T_{j_p\cdots a\cdots j_1}\nabla_k\eta^a=0.$$

Taking account of (2.6), (3.1) is written as

(3. 2) $$\eta^a\nabla_k T_{j_p\cdots a\cdots j_1}=-\tilde{T}_{j_p\cdots k\cdots j_1}.$$

Transvecting (3.2) with η^k, by (1.7) we have

(3. 3) $$\eta^k\eta^a\nabla_k T_{j_p\cdots a\cdots j_1}=0.$$

Next, transvecting (1.11) with η^k, by (1.3) we have

(3.4) $$\eta^k\nabla_k\tilde{T}_{(j)}=\eta^k\sum_{r=1}^{p}(\nabla_{j_r}\varphi_k{}^l)T_{j_p\cdots l\cdots j_1}.$$

The right hand side of the above equation can be written as

$$\eta^k\sum_{r=1}^{p}(\nabla_{j_r}\varphi_k{}^l)T_{j_p\cdots l\cdots j_1}=-\varphi_k{}^l\sum_{r=1}^{p}(\nabla_{j_r}\eta^k)T_{j_p\cdots l\cdots j_1}$$

$$=-\sum_{r=1}^{p}(\nabla_{j_r}\eta^k)\tilde{T}_{j_p\cdots k\cdots j_1}.$$

Since $\varphi_r{}^k\tilde{T}_{j_p\cdots k\cdots j_1}=-T_{j_p\cdots j_r\cdots j_1}$, from (3.4), we have

(3. 5) $$\eta^k\nabla_k\tilde{T}_{(j)}=pT_{(j)}.$$

Transvecting (3.5) with $\varphi_h{}^{j_r}$, we have

$$\eta^k\varphi_h{}^{j_r}(\nabla_k\varphi_{j_r}{}^a)T_{j_p\cdots a\cdots j_1}+\eta^k\varphi_h{}^{j_r}\varphi_{j_r}{}^a\nabla_k T_{j_p\cdots a\cdots j_1}=p\tilde{T}_{(j)}.$$

In this place, making use of (2.12), (1.5) and (3.3) we have

(3. 6) $$\eta^k\nabla_k T_{(j)}=-p\tilde{T}_{(j)}.$$

Hence, taking account of (3.1), (3.6) can be written as

$$\eta^k\nabla_{[k}T_{j_p\cdots j_1]}=0.$$

On the other hand, transvecting (1.9) with $\varphi_h{}^k$, by (1.5), we have

$$\varphi_h{}^k\nabla_{[k}\tilde{T}_{j_p\cdots j_1]}=-\nabla_{[h}T_{j_p\cdots j_1]}+\eta_h\eta^k\nabla_{[k}T_{j_p\cdots j_1]}.$$

Consequently, since $\tilde{\tilde{T}}_{(j)}=-T_{(j)}$, by (3. 7), we obtain

(3. 8) $$\varphi_h{}^k\nabla_{[k}\tilde{T}_{j_p\cdots j_1]}=\nabla_{[h}\tilde{\tilde{T}}_{j_p\cdots j_1]}.$$

Thus, we have the following

THEOREM 3.1. *In a K-contact Riemannian manifold, if $T_{(j)}$ is a C-analytic tensor, then $\tilde{T}_{(j)}$ is also C-analytic.*

By this theorem, we have immediately the following

THEOREM 3. 2. *In a K-contact Riemannian manifold, if skew-symmetric C-tensors $T_{(j)}$ and $\tilde{T}_{(j)}$ are both closed, then they are both C-analytic.*

Transvecting (1. 11) with $\varphi_h{}^k$, we have

$$\nabla_h T_{(j)} - \eta_h \eta^s \nabla_s T_{(j)} + \varphi_h{}^k \nabla_k \tilde{T}_{(j)} - \sum_{r=1}^{p} \varphi_h{}^k (\nabla_{j_r} \varphi_k{}^l) T_{j_p \cdots l \cdots j_1} = 0 \tag{3. 9}$$

from which it follows

$$\nabla_h T_{(j)} - \eta_h \eta^s \nabla_s T_{(j)} + \varphi_h{}^k \varphi_{j_1}{}^a \nabla_k T_{j_p \cdots j_2 a} - \sum_{r=2}^{p} \varphi_h{}^k (\nabla_{j_r} \varphi_k{}^a) T_{j_p \cdots a \cdots j_1} \tag{3.10}$$
$$= \varphi_h{}^k (\nabla_{j_1} \varphi_k{}^a - \nabla_k \varphi_{j_1}{}^a) T_{j_p \cdots j_2 a}.$$

Operating $(\varphi_t{}^{j_1} g^{j_2 h} + \delta_t^{j_1} \varphi^{j_2 h})$ to (3. 10), we have

$$\varphi_t{}^{j_1} \nabla^{j_2} T_{(j)} + \varphi^{j_2 h} \nabla_h T_{j_p \cdots j_2 t} - \varphi^{j_2 k} \nabla_k T_{j_p \cdots j_2 t} + \eta_t \varphi^{j_2 k} \eta^{j_1} \nabla_k T_{(j)}$$
$$- \varphi_t{}^{j_1} \nabla^{j_2} T_{(j)} - \eta^{j_2} \eta^s \varphi_t{}^{j_1} \nabla_s T_{(j)} + \eta^{j_2} \eta^s \varphi_t{}^{j_1} \nabla_s T_{(j)}$$
$$= \varphi^{j_2 k} \varphi_t{}^a (\nabla^{j_1} \varphi_{ka}) T_{(j)} + \varphi^{j_2 h} \varphi_h{}^k (\nabla^{j_1} \varphi_{kt}) T_{(j)}.$$

In this equation, taking account of (3. 2), (3. 3) and (3. 5), the 4-th, 6-th, and 7-th terms of the left hand side vanish respectively, and therefore it is easy to see that the left hand side vanishes, but, the right hand side, making use of (1. 4), (2. 7), (1. 7) and (1. 8), reduces to $2(\nabla^{j_1} \varphi_t{}^{j_2}) T_{(j)}$.

Hence, we get

$$(\nabla^{j_1} \varphi_t{}^{j_2}) T_{(j)} = 0 \tag{3. 11}$$

from which, by (2. 7), it follows

$$(\nabla_t \varphi^{j_2 j_1}) T_{(j)} = 0. \tag{3. 12}$$

On the other hand, transvecting (3. 10) with g^{hj_1}, we get

$$2 \nabla^h T_{j_p \cdots j_2 h} = \sum_{r=2}^{p} \varphi^{hk} (\nabla_{j_r} \varphi_k{}^a) T_{j_p \cdots a \cdots h}. \tag{3. 14}$$

Accordingly, by (3. 12), we have

$$\nabla^h T_{j_p \cdots j_2 h} = 0. \tag{3. 15}$$

Next, operating ∇^h to (1. 11), we have

$$\nabla^k \nabla_k \tilde{T}_{(j)} = (\nabla^k \varphi_k{}^s) \nabla_s T_{(j)} + \varphi_k{}^s \nabla^k \nabla_s T_{(j)} + \sum_{r=1}^{p} (\nabla^k \nabla_{j_r} \varphi_k{}^l) T_{j_p \cdots l \cdots j_1}. \tag{3. 16}$$

Multiplying (3. 16) by $\tilde{T}^{(j)}$, and making use of the Ricci's identity and (2. 14), we have

$$\tilde{T}^{(j)} \nabla^k \nabla_k \tilde{T}_{(j)} = p(n-1) \tilde{T}^{(j)} \tilde{T}_{(j)} - \frac{1}{2} \varphi^{ks} R_{ksj_1}{}^l T_{j_p \cdots l} \tilde{T}^{(j)} \tag{3. 17}$$

$$+p\tilde{T}^{(j)}\{R_{j_1a}\varphi^{al}T_{j_p\cdots l}+\frac{1}{2}\varphi^{ab}R_{abj_1}{}^{l}T_{j_p\cdots l}+(1-n)\tilde{T}^{(j)}T_{(j)}+(\nabla_{j_1}\varphi_{k}{}^{l})\nabla^{k}T_{j_p\cdots l}\}$$

$$=p\tilde{T}^{(j)}R_{j_1}{}^{a}\tilde{T}_{j_p\cdots a}+pT^{(j)}(\nabla_{j_1}\varphi^{kl})\nabla_{k}T_{j_p\cdots l}.$$

On the other hand, operating ∇^{k} to $\nabla_{j_1}T_{j_p\cdots k}$, and making use of the Ricci's identity and (3.15), we have

$$\nabla^{k}\nabla_{j_1}\tilde{T}_{j_p\cdots k}=R_{j_1}{}^{l}\tilde{T}_{j_p\cdots l}-\sum_{s=2}^{p}R^{k}{}_{j_1j_s}{}^{l}T_{j_p\cdots l\cdots k}. \tag{3.18}$$

Multiplying (3.18) by $p\ \tilde{T}^{(j)}$, we have

$$p\tilde{T}^{(j)}\nabla^{k}\nabla_{j_1}\tilde{T}_{j_p\cdots k}=pR_{j_1}{}^{t}\tilde{T}_{j_p\cdots t}\tilde{T}^{(j)}-\frac{p(p-1)}{2}R^{kt}{}_{j_1j_2}\tilde{T}_{j_p\cdots tk}\tilde{T}^{(j)}. \tag{3.19}$$

Hence, subtracting (3.19) from (3.17), we have

$$\tilde{T}^{(j)}\nabla^{k}\nabla_{[k}\tilde{T}_{j_p\cdots j_1]}=\frac{p(p-1)}{2}R^{kt}{}_{j_1j_2}T_{j_p\cdots tk}\tilde{T}^{(j)}+p\ \tilde{T}^{(j)}(\nabla_{j_1}\varphi^{kl})\nabla_{k}T_{j_p\cdots l}. \tag{3.20}$$

Operating ∇_{k} to (3.12), we have

$$(\nabla_{k}\nabla_{t}\varphi^{j_1}{}_{j_2})T_{(j)}+(\nabla_{t}\varphi^{j_2}{}_{j_1})\nabla_{k}T_{(j)}=0.$$

Transvecting the above equation with $T^{j_p\cdots kt}$ and making use of the Ricci's identity, we have

$$T^{j_p\cdots kt}R_{kts}{}^{j_1}\varphi^{s}{}_{j_2}T_{(j)}=-\tilde{T}^{j_p\cdots kt}(\nabla_{t}\varphi^{j_1}{}_{j_2})\nabla_{k}T_{(j)}. \tag{3.21}$$

Substituting (3.21) into (3.20), we have

$$T^{(j)}\nabla^{k}\nabla_{[k}\tilde{T}_{j_p\cdots j_1]}=\frac{1}{2}\tilde{T}^{(j)}(\nabla_{j_1}\varphi^{kl})\{\nabla_{k}T_{j_p\cdots l}-\nabla_{l}T_{j_p\cdots k} \tag{3.22}$$

$$+(p-1)\nabla_{j_2}T_{j_p\cdots lk}\}$$

$$=\frac{1}{2}T^{(j)}(\nabla_{k_1}\varphi^{ml})\nabla_{[m}T_{j_p\cdots l]}.$$

In this place, taking account of (2.1) and (1.4), the right hand side of (3.22) turns to

$$T^{j_p\cdots j}{}_{1}\nabla_{j_1}\varphi^{ml}\nabla_{[m}T_{j_p\cdots l]}=\frac{1}{2}\tilde{T}^{j}{}_{p\cdots ba}(\delta_{a}{}^{j_1}\delta_{b}{}^{j_2}-\varphi_{a}{}^{j_1}\varphi_{b}{}^{j_2})(\nabla_{j_1}\varphi^{ml})(\nabla_{[m}T_{j_p\cdots j_2l]}$$

$$=\frac{1}{2}T^{j_p\cdots ba}\{(\nabla_{a}\varphi^{ml})\nabla_{[m}T_{j_p\cdots bl]}+\varphi_{a}{}^{j_1}(\nabla^{l}\varphi_{j_1}{}^{m}-\nabla^{m}\varphi_{j_1}{}^{l})\nabla_{[m}T_{j_p\cdots j_2l]}\}$$

$$=\frac{1}{2}T^{j_p\cdots ba}\{(\nabla_{a}\varphi^{ml})\nabla_{[m}T_{j_p\cdots bl]}+2\varphi_{a}{}^{j_1}(\nabla^{l}\varphi_{j_1}{}^{m})\nabla_{[m}T_{j_p\cdots j_2l]}\}$$

$$=\frac{1}{2}\tilde{T}^{j}{}_{p}{}^{\cdots ba}\{(\nabla_{a}\varphi^{ml})\nabla_{[m}T_{j_p\cdots bl]}+(\nabla^{l}\varphi_{a}{}^{j}{}_{1})\nabla_{[j_1}T_{j_p\cdots j_2l]}\}$$

$$=0.$$

Consequently we get

$$\nabla^k(\widetilde{T}^{(j)}\nabla_{[k}\widetilde{T}_{j_p\cdots j_1]})=\frac{1}{p+1}\nabla^{[k}\widetilde{T}^{j_p\cdots j_1]}\nabla_{[k}\widetilde{T}_{j_p\cdots j_1]}.$$

If the manifold is compact, then applying Green's theorem we have

$$\int_M(\nabla^{[k}\widetilde{T}^{j_p\cdots j_1]}\nabla_{[k}\widetilde{T}_{j_p\cdots j_1]})\,d\sigma=0,$$

where $d\sigma$ means the volume element of M.

Hence we can deduce $\nabla_{[k}\widetilde{T}_{j_p\cdots j_1]}=0$.

Thus taking account of (3.15) and theorem 3.1, we have the following

THEOREM 3.3.[4] *In a compact Riemannian manifold, if $T_{(j)}$ is a C-analytic tensor, then $T_{(j)}$ and $\widetilde{T}_{(j)}$ are both harmonic.*

4. Harmonic tensors in a compact Sasakian manifold.

In this section we assume that M is a compact Sasakian manifold. First of all, we shall try to prove the converse of theorem 3.2.

Let $T_{(j)}$ be a harmonic C-tensor. Operating ∇^h to $\nabla_h\widetilde{T}_{(j)}$ and taking account of (1.8) and (2.5), we have

$$\begin{aligned}(4.1)\qquad \nabla^h\nabla_h\widetilde{T}_{(j)}&=\nabla^h(\eta_{j_2}T_{j_p\cdots hj_1}+\varphi_{j_2}{}^l\nabla_hT_{j_p\cdots lj_1})\\&=-\widetilde{T}_{j_p\cdots j_1}+\eta^l\nabla_{j_2}T_{j_p\cdots lj_1}+\varphi_{j_2}{}^l\nabla^h\nabla_hT_{j_p\cdots lj_1}\\&=\varphi_{j_2}{}^l\nabla^h\nabla_hT_{j_p\cdots l_1}.\end{aligned}$$

Operating ∇^h to $\nabla_{j_1}\widetilde{T}_{j_p\cdots j_2h}$, we have

$$\begin{aligned}(4.2)\qquad \nabla^h\nabla_{j_1}\widetilde{T}_{j_p\cdots j_2h}&=\nabla^h(\eta_{j_2}T_{j_p\cdots j_1h}+\varphi_{j_2}{}^l\nabla_{j_1}T_{j_p\cdots lh})\\&=-\widetilde{T}_{j_p\cdots j_1j_2}-\eta^l\nabla_{j_1}T_{j_p\cdots lj_2}+\varphi_2{}^l\nabla^h\nabla_{j_1}T_{j_p\cdots lh}\\&=\varphi_{j_2}{}^l\nabla^h\nabla_{j_1}T_{j_p\cdots lh}.\end{aligned}$$

Subtracting (4.2)$\times p$ from (4.1), since $\widetilde{\widetilde{T}}_{(j)}=-T_{(j)}$ we find

$$(4.3)\qquad \nabla^h\nabla_h\widetilde{T}_{(j)}-p\nabla^h\nabla_{j_1}T_{j_p\cdots j_2h}=\varphi_{j_2}{}^l(\nabla^h\nabla_hT_{j_p\cdots lj_1}-p\nabla^h\nabla_{j_1}T_{j_p\cdots lh}).$$

Multiplying (4.3) by $\widetilde{T}^{(j)}$, we have

$$\begin{aligned}(4.4)\qquad &\widetilde{T}^{(j)}\nabla^h\nabla_h\widetilde{T}_{(j)}-p\widetilde{T}^{(j)}\nabla^h\nabla_{j_1}\widetilde{T}_{j_p\cdots j_2h}\\&=T^{j_p\cdots lj_1}(\nabla^h\nabla_hT_{j_p\cdots lj_1}-p\nabla^h\nabla_{j_1}T_{j_p\cdots lh})\\&=T^{j_p\cdots lj_1}(\nabla^h\nabla_hT_{j_p\cdots lj_1}-pR_{j_1}{}^tT_{j_p\cdots lt}-\frac{p(p-1)}{2}R^{ht}{}_{j_1l}T_{j_p\cdots th})\end{aligned}$$

4) For an almost Kahler and a Tachibaba space, see S. Sawaki and S. Kotô [2].

$$=T^{j_p\cdots lj_1}(\Delta T_{j_p\cdots lj_1}),$$

where $\Delta T_{(j)}$ is Laplacian of $T_{(j)}$.

Consequently, if $T_{(j)}$ is harmonic, then we have

$$\nabla^h(\widetilde{T}^{(j)}\nabla_{[h}\widetilde{T}_{j_p\cdots j_1]})=\frac{1}{p+1}\nabla^{[h}\widetilde{T}^{j_p\cdots j_1]}\nabla_{[h}\widetilde{T}_{j_p\cdots j_1]}. \tag{4. 5}$$

Accordingly, applying Green's theorem to (4. 5), we have

$$\int_M(\nabla^{[h}\widetilde{T}^{j_p\cdots j_1]}\nabla_{[h}\widetilde{T}_{j_p\cdots j_1]})\,d\sigma=0,$$

from which we can deduce $\nabla_{[h}\widetilde{T}_{j_p\cdots j_1]}=0$.

Since it is easy to see that $\nabla^h\widetilde{T}_{j_p\cdots j_2 h}=0$, we have the following

THEOREM 4. 1. *In a compact Sasakian manifold, if a skew-symmetric C-tensor* $T_{(j)}$ *is harmonic, then* $\widetilde{T}_{(j)}$ *is also harmonic.*

By this theorem and theorem 3. 2, we can see that in a compact Sasakian manifold a harmonic C-tensor is C-analytic.

Thus, taking account of theorem 3. 3, we have the following

THEOREM 4. 2. *In a compact Sasakian manifold, a necessary and sufficient condition that a skew-symmetric C-tensor be harmonic is that it is C-analytic.*

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