On the non-embeddability of Dold's manifold

By

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1. Introduction

Let $P(m,n)$ be Dold's manifold of type (m,n). $P(m,n)$ is defined as follows. Let S^m be the unit sphere in R^{m+1}: $S^m=\{x=(x_0, x_1, \cdots, x_m)\in R^{m+1};\ \sum x_i^2=1\}$ and CP_n the complex n-dimensional projective space: $CP_n=\{z=[z_0, z_1 \cdots, z_n]\ z_i$ complex number$\}$. Now $P(m,n)$ is the manifold obtained from $S^m\times CP_n$ by identifying (x,z) with $(-x,\bar{z})$ where $-x$ denotes the antipodal of x and $\bar{z}$ the conjugate of z. The dimension of $P(m,n)$ is $m+2n$.

Y. Ando showed that when $m=2^k$ and $n=2^l+1$ $(0<k<1)$ $P(m,n)\not\subset$ (cannot be embedded) $R^{2(m+2n)-5}$ [1]. In this paper we will show some general results in analogous cases by the similar methods.

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2. Stiefel-Whitney class

By A. Dold [2] we know that

(1) $H^*(P(m,n))$, the cohomology ring of the coefficient Z_2 of $P(m,n)$, generated by $c\in H^1(P(m,n))$ and $d\in H^2(P(m,n))$ such that $c^{m+1}=0$, $d^{n+1}=0$ and $Sq^1d=cd$;

(2) $w(P(m,n))=(1+c)^m(1+c+d)^{n+1}$ where w denotes the total Stiefel-Whitney class.

Now, we consider the case of $m=2^k$ and $n=2^l+r$ $(1\leqq k<1,\ 1\leqq r<2^l)$. There exists a p such that $2^{p-1}<r+1\leqq 2^p$ (for $r\geqq 2$), $2^{p-1}\leqq r+1<2^p$ (for $r=1$), and it is unique. Clearly $p\leqq 1$. Using this p, we have

$$(3)\qquad r+1=2^p-\sum_{i=2}^{p-2}a_i\,2^i-\begin{cases}2+1\\2\\1\\0\end{cases}\quad \text{for } r\equiv\begin{cases}0\\1\\2\\3\end{cases}\pmod 4$$

where $a_i=0$ or 1. Then, from (2) and (1)

$w(P(m,n))=(1+c)^{2^k}(1+c+d)^{2^l+r+1}$

(4) $$=(1+c^{2^k})(1+d^{2^l})(1+c+d)^{2^p}\Pi_{i=0}^{p-2}(1+c+d)^{-a_i 2^i}.$$

Hence, by $w\bar{w}=1$ and (1), we obtain the dual total Stiefel-Whitney class of $P(m, n)$ ($m=2^k$, $n=2^l+r$):

$$\bar{w}(P(m,n))=(1+c^{2^k})(1+d^{2^l})(1+c+d)^{-2^p}\Pi_{i=0}^{p-2}(1+c+d)^{a_i 2^i}$$

(5) $$=(1+c^{2^k})(1+d^{2^l})\sum\binom{s+t}{s}c^{2^p s}d^{2^p t}\begin{Bmatrix}(1+c+d)(1+c^2+d^2)\\(1+c^2+d^2)\\(1+c+d)\\1\end{Bmatrix}$$

$$\cdot\Pi_{i=2}^{p-2}(1+c+d)^{a_i 2^i} \qquad \text{for } r\equiv\begin{cases}0\\1\\2\\3\end{cases} \pmod 4$$

where the sum $\sum$ is taken for the integers s, t such that $0\leqq s\leqq 2^{k-p}$ (if $k\geqq p$), $s=0$ (if $k<p$) and $0\leqq t\leqq 2^{l-p}$.

In the expansion of (5), the number of the terms of the form $c^i d^{2^l+j}$ $(0\leqq i\leqq 2^k$, $0\leqq j\leqq r)$ is even. For, $\binom{s+0}{s}=\binom{s+2^{l-p}}{s}$ (mod 2) for any s. On the other hand, the number of the term $c^{2^k}d^{2^l-(r+1)}$ equals 1: $s=0$, $t=2^{l-p}-1$. (Cf. [6] §2) Hence,

(6) $$\bar{w}(P(m,n))=1+\cdots\cdots+c^m d^{n-2r-1}.$$

Therefore, we have

$$\bar{w}_{m+2n-4r-2}(P(m,n))\neq 0$$

and

(7) $$P(m,n)\not\subset R^{2(m+2n)-4r-2}$$

where $m=2^k$ and $n=2^l+r$ $(1\leqq k<1,\ 1\leqq r<2^l)$.

Next, we consider the case of $r=0$: $m=2^k$ and $n=2^l$ $(1\leqq k<1)$. Then, similarly to above methods, we have

$$w(P(m,n))=(1+c^{2^k})(1+d^{2^l})(1+c+d)$$

and

$$\bar{w}(P(m,n))=(1+c^{2^k})(1+d^{2^l})\sum\binom{s+t}{s}c^s d^t=1+\cdots\cdots+c^{2^k}d^{2^l-1}.$$

Hence,

(8) $$P(m,n)\not\subset R^{2(m+2n)-2}$$

where $m=2^k$ and $n=2^l$ $(1\leqq k<1)$.

3. Steenrod square

The Steenrod square operation is a homomorphism Sq^i: $H^j(X)\rightarrow H^{i+j}(X)$ such that

(9) $$Sq^0x=x,$$

(10) $$Sq^ix=\begin{cases} x & (\text{if } \dim x=i) \\ 0 & (\text{if } \dim x<i), \end{cases}$$

(11) $$Sq^i(xy)=\sum_{j=0}^{i} Sq^jx\cdot Sq^{i-j}y \text{ (Cartan formula)},$$

(12) $$Sq^aSq^b=\sum_{j=0}^{[\frac{a}{2}]}\binom{b-1-j}{a-2j}Sq^{a+b-j}Sq^j \quad (0<a<2b) \text{ (Adem relation)}$$

where the binomial coefficient is taken mod 2. (Cf. [5] Chapter 1, §1) From (12), in particular,

(13) $$Sq^1Sq^i=\begin{cases} 0 & (\text{for odd } i) \\ Sq^{1+i} & (\text{for even } i) \end{cases} \quad (i\geqq 0),$$

(14) $$Sq^2Sq^i=\begin{cases} Sq^{1+i}Sq^1 & (\text{for } i=1, 2 \pmod 4) \\ Sq^{2+i}+Sq^{1+i}Sq^1 & (\text{for } i=0, 3 \pmod 4) \end{cases} \quad (i\geqq 2).$$

For c, d (the generators of $H^*(P(m, n))$), we have

(15) $$Sq^ic^j=\binom{i}{j}c^{i+j},$$

(16) $$Sq^id^j=\sum\binom{j-t}{s}\binom{j}{t}c^sd^{j+t},$$

where the sum $\sum$ is taken for the non-negative integers s, t such that $s+2t=i$. The equalities (15), (16) are proved by the induction on i, j. (Cf. [5] §2.4, p5) From (16), in particular,

$$Sq^1d^j=\begin{cases} cd^j & (\text{for odd } j) \\ 0 & (\text{for even } j) \end{cases}$$

(17) $$Sq^2d^j=\begin{cases} \binom{j}{2}c^2d^j+d^{j+1} & (\text{for odd } j) \\ \binom{j}{2}c^2d^j & (\text{for even } j) \end{cases}$$

$$Sq^3d^j=\begin{cases} \binom{j}{3}c^3d^j & (\text{for odd } j) \\ 0 & (\text{for even } j). \end{cases}$$

4. Non-embedding theorem

In this section, we give the main theorem which includes the result by Y. Ando [1], and prove it by the similar methods to Massy [4], Y. Ando [1].

THEOREM. *Let* $m=2^k$, $n=2^l+r$ *where* $1\leqq k<1$, $1\leqq r<2^l$. *Then* $P(m, n)\not\subset R^{2(m+2n)-4r-1}$.

PROOF. We assume $P(m, n)\subset R^{2(m+2n)-4r-1}$. Let $(E, p, P(m, n), S^{m+2n-4r-2})$ be the normal sphere bundle of this embeding. Then, by Massey [4], we obtain the following results.

There exists a subring $A^*=\sum A^q$ of the cohomology ring $H^*(E)$ and element a of $A^{m+2n-4r-2}$ such that

(18) A^* is closed under any cohomology operation (e.g. Sq-operation),

(19) $H^q(E)=p^*(H^q(P(m, n)))+A^q$ (direct sum, $0<q<2(m+2n)-4r-2$),

(20) $$A^{2(m+2n)-4r-2}=0,$$

(21) any element $y\in H^*(E)$ can be expressed uniquely in the form $y=p^*(y_1)+ap^*(y_2)$ where $y_i\in H^*(P(m, n))$,

(22) if $y=Sq^i(a)$ in (21), then $y_2=\bar{w}_i$.

From (22), we have

(23) $$a^2=ap^*(c^md^{n-2r-1})+\begin{cases}0 & (\text{if } m+2n>8r+4)\\ p^*(\sum_{i,j}h_{i,j}c^id^j) & (\text{if } m+2n\leqq 8r+4)\end{cases}$$

where $h_{i,j}=0$ or $i+2j=2m+4n-8r-4$, i, $j\geqq 0$.

Now, we consider an element x of $A^{m+2n-2r-2}$ such that

(24) $$x=p^*(v_1c^md^{n-r-1}+v_2c^{m-2}d^{n-r}+\cdots\cdots+v_{r+2}c^{m-2r-2}d^n)+ap^*(d^r)$$

where $v_i=0$ or 1; if the power number of c is negative, then the coefficient v_i of its term equals 0.

(A) The case of $r\equiv 0 \pmod 4$.

From (5),

$$\bar{w}(P(m, n))=1+c+\begin{cases}d+0+\cdots & (k=1)\\ (c^2+d)+c^3+\cdots & (k\geqq 2).\end{cases}$$

Hence, by (18), (21) and (22)

$$Sq^1(a)=p^*(s_1c^{m-1}d^{n-2r}+s_2c^{m-3}d^{n-2r+1}+\cdots\cdots+s_{2r+1}c^{m-4r-1}d^n)+ap^*(c),$$

$$Sq^2(a)=p^*(t_1c^md^{n-2r}+t_2c^{m-2}d^{n-2r+1}+\cdots\cdots t_{2r+1}c^{m-4r}d^n)$$
$$+\begin{cases}ap^*(d) & (k=1)\\ ap^*(c^2+d) & (k\geqq 2),\end{cases}$$

$$Sq^3(a)=\begin{cases}0 & (k=1)\\ p^*(u_1c^{m-1}\,d^{n-2r+1}+u_2c^{m-3}\,d^{n-2r+2}+\cdots\cdots u_{2r}c^{m-4r+1}\,d^n)+ap^*(c^3) & (k\geqq 2)\end{cases}$$

where s_i, t_i, $u_i=0$ or 1; if the power number of c is negative, then the coefficient of its terms equals 0. Then, by the calculation of Sq-operation we have

$$\begin{aligned}Sq^2(x)&=p^*(v_1Sq^2(c^md^{n-r-1})+v_2Sq^2(c^{m-2}d^{n-r})+\cdots)+Sq^2(a)p^*(d^r)\\&\quad+Sq^1(a)p^*(Sq^1d^r)+ap^*(Sq^2d^r)\\&=\begin{Bmatrix}p^*(v_1c^2d^{n-r}+\cdots\cdots)\\ p^*(v_1\,c^m\,d^{n-r}+v_2\,c^m\,d^{n-r}+\cdots\cdots)\end{Bmatrix}+p^*(t_1c^md^{n-r}+\cdots\cdots)\\&\quad+\begin{cases}ap^*(d^{r+1}) & (k=1)\\ ap^*(c^2d^r+d^{r+1}) & (k\geqq 2)\end{cases}\\&=\begin{cases}p^*((v_1+t_1)c^2d^{n-r}+\cdots\cdots)+ap^*(d^{r+1}) & (k=1)\\ p^*((v_1+v_2+t_1)c^md^{n-r}+\cdots\cdots)+ap^*(c^2d^r+d^{r+1}) & (k\geqq 2).\end{cases}\end{aligned}$$

Hence, using (23),

$$\begin{aligned}xSq^2(x)&=t_1ap^*(c^md^n)+\begin{cases}a^2p^*(d^{2r+1}) & (k=1)\\ a^2p^*(c^2d^{2r}+d^{2r+1}) & (k\geqq 2)\end{cases}\\&=(1+t_1)ap^*(c^md^n).\end{aligned}$$

Here, note that $n<(n-r)+(n-r+1)$ and $n<2r+\left(\frac{m}{2}+2n-4r-2\right)$.

On the other hand, using (13), (14) we obtain the following results.
From

$$0=Sq^1Sq^1(a)=p^*(s_2c^{m-2}d^{n-2r+1}+\cdots\cdots),$$

$s_2=0$ $(k\geqq 2)$. From

$$Sq^3(a)=Sq^1Sq^2(a)=p^*((t_2+s_1)c^{m-1}d^{n-2r+1}+\cdots\cdots)+\begin{cases}0 & (k=1)\\ ap^*(c^3) & (k\geqq 2),\end{cases}$$

$$t_2+s_1=\begin{cases}0 & (k=1)\\ u_1 & (k\geqq 2).\end{cases}$$ From

$$Sq^2Sq^2(a)=p^*((t_1+s_1)c^md^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3Sq^1(a)=\begin{Bmatrix}p^*(t_2c^2d^{n-2r+1}+\cdots\cdots) & (k=1)\\ p^*((t_2+u_1)c^md^{n-2r+1}+\cdots) & (k\geqq 2)\end{Bmatrix}+ap^*(c^2d)$$

and $Sq^2Sq^2=Sq^3Sq^1$, $t_1+s_1=\begin{cases} t_2 & (k=1) \\ t_2+u_1 & (k\geqq 2). \end{cases}$

Hence, we have $t_1=0$ and by (20)

$$0=A^{2(m+2n)-4r-2}\ni xSq^2(x)=ap^*(c^m d^n). \tag{25}$$

But this is a contradiction. Thus $P(m,n)\not\subset R^{2(m+2n)-4r-1}$.

(B) The case of $r\equiv 1 \pmod 4$.

From (5),

$$\bar{w}(P(m,n))=1+0+\begin{Bmatrix} 0 \\ c^2 \end{Bmatrix}+0+\cdots\cdots \qquad \begin{matrix} (k=1) \\ (k\geqq 2). \end{matrix}$$

Hence,

$$Sq^1(a)=Sq^3(a)=0$$

$$Sq^2(a)=\begin{cases} 0 & (k=1) \\ p^*(t_1c^md^{n-2r}+t_2c^{m-2}d^{n-2r+1}+\cdots\cdots+t_{2r+1}c^{m-4r}d^n)+ap^*(c^2) & (k\geqq 2) \end{cases}$$

where $t_i=0$ or 1; if the power number of c is negative, then the coefficient of its serm equals 0. Then, we have

$$Sq^2(x)=\begin{cases} p^*(v_1c^2d^{n-r}+\cdots\cdots)+ap^*(d^{r+1}) & (k=1) \\ p^*((v_1+t_1+v_2)c^md^{n-r}+\cdots\cdots)+ap^*(c^2d^r+d^{r+1}) & (k\geqq 2). \end{cases}$$

Hence,

$$xSq^2(x)=\begin{cases} ap^*(c^2d^n) & (k=1) \\ (1+t_1)ap^*(c^md^n) & (k\geqq 2). \end{cases}$$

On the other hand, when $k\geqq 2$, from

$$0=Sq^3Sq^1(a)=Sq^2Sq^2(a)=p^*(t_1c^md^{n-2r+1}+\cdots\cdots)$$

$t_1=0$. Thus, we have a contradiction (25).

(C) The case of $r\equiv 2 \pmod 4$.

From (5),

$$\bar{w}(P(m,n))=1+c+\begin{Bmatrix} (c^2+d) \\ d \end{Bmatrix}+0+\cdots\cdots \qquad \begin{matrix} (k=1) \\ (k\geqq 2). \end{matrix}$$

Hence,

$$Sq^1(a)=p^*(s_1c^{m-1}d^{n-2r}+s_2c^{m-3}d^{n-2r+1}+\cdots\cdots+s_{2r+1}c^{m-4r-1}d^n)+ap^*(c),$$

$$Sq^2(a)=p^*(t_1c^md^{n-2r}+t_2c^{m-2}d^{n-2r+1}+\cdots\cdots+t_{2r+1}c^{m-4r}d^n)$$

$$+\begin{cases} ap^*(c^2+d) & (k=1) \\ ap^*(d) & (k\geqq 2), \end{cases}$$

$$Sq^3(a)=0$$

where the remarks for the coefficients s_i, t_i are same with before. Then, we have

$$Sq^2(x)=\begin{cases} p^*((v_1+t_1)c^2d^{n-r}+\cdots\cdots)+ap^*(d^{r+1}) & (k=1) \\ p^*((v_1+v_2+t_1)c^md^{n-r}+\cdots\cdots)+ap^*(c^2d^r+d^{r+1}) & (k\geqq 2). \end{cases}$$

Hence,

$$xSq^2(x)=(1+t_1)ap^*(c^md^n).$$

On the other hand, from

$$0=Sq^3(a)=Sq^1Sq^2(a)=p^*((t_2+s_1)c^{m-1}d^{n-2r+1}+\cdots\cdots)$$

$t_2+s_1=0$, and from

$$Sq^2Sq^2(a)=p^*((t_1+s_1)c^md^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$$Sq^3Sq^1(a)=p^*(t_2c^md^{n-2r+1}+\cdots\cdots)+ap^*(c^2d)$$

$t_1+s_1=t_2$. Hence, we have $t_1=0$ and a contradiction (25).

(D) The case of $r\equiv 3 \pmod 4$.

From (5),

$$\bar{w}(P(m,n))=1+0+\begin{Bmatrix} c^2 \\ 0 \end{Bmatrix}+0+\cdots\cdots \quad \begin{matrix} (k=1) \\ (k\geqq 2). \end{matrix}$$

Hence,

$$Sq^1(a)=0=Sq^3(a)$$

$$Sq^2(a)=\begin{cases} p^*(t_1c^2d^{n-2r}+t_2d^{n-2r+1})+ap^*(c^2) & (k=1) \\ 0 & (k\geqq 2) \end{cases}$$

where $t_i=0$ or 1. Then, we have

$$Sq^2(x)=\begin{cases} p^*(v_1+t_1)c^2d^{n-r}+\cdots\cdots)+ap^*(d^{r+1}) & (k=1) \\ p^*((v_1+v_2)c^md^{n-r}+\cdots\cdots)+ap^*(c^2d^r+d^{r+1}) & (k\geqq 2). \end{cases}$$

Hence,

$$xSq^2(x)=\begin{cases} (1+t_1)ap^*(c^2d^n) & (k=1) \\ ap^*(c^md^n) & (k\geqq 2). \end{cases}$$

On the other hand, when $k=1$, from

$$0 = Sq^3 Sq^1(a) = Sq^2 Sq^2(a) = p^*(t_1 c^2 d^{n-2r+1})$$

$t_1 = 0$. Thus, we have a contradiction (25).

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