Fixed Points of Expanding Maps

By

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1. Introduction

Let $\{f_i\}_{i=1}^{\infty}$ be a convergent sequence of maps from a space X into itself and let f_0 be a limit map. When does there exist a sequence of fixed points a_i of f_i such that $\{a_i\}_{i=1}^{\infty}$ converges to a_0 for each fixed point a_0 of f_0. In [3] Rosen proved that it holds when X is a compact connected ANR and f_i is an open ϵ-locally expansive map for $i=0$, 1, 2, ..., and $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. In [2] Hu and Rosen recently showed that for a compact connected locally connected metric space, the ANR requirement can be dropped.

In this paper we show that in the hypothesis of the Theorem 4.8 in [2], if $\{f_i\}_{i=1}^{\infty}$ is a sequence of expanding maps with common ε and λ, the uniform convergence may be replaced by pointwise convergence and f_0 may be any map with a fixed point.

2. Definition and lemmas

Let (X, d) be a compact metric space. A continuous map $f: X \to X$ is called an ε-local expansion if there are $\varepsilon > 0$ and skewness $\lambda > 1$ such that $0 < d(x, y) < \varepsilon$ implies $d(f(x), f(y)) > \lambda d(x, y)$.

We call a continuous map f to be expanding if f is open and ε-local expansion for some $\varepsilon > 0$ and $\lambda > 1$.

Rosenholtz showed in [4] that if X is a compact connected metric space, such map f has a fixed point.

LEMMA 1. *If X is a compact connected locally connected space with metric d and if $\{f_i\}_{i=1}^{\infty}$ is a sequence of expanding maps of X onto itself with common ε and λ, then there is $\delta_0 > 0$ $(\delta_0 < \varepsilon)$ such that $x, y \in X$ with $d(f_i(x), y) < \delta_0$ implies $B_{\delta_0/\lambda}(x) \cap f_i^{-1}(y) \neq \phi$ for $i=1$, 2, 3, ..., where $B_\alpha(x) = \{y \in X : d(x, y) < \alpha\}$.*

PROOF. According to Lemma 2 in [3], there is a finite open cover $\{W_\beta\}$ of X such that for each β and for $i=1, 2, 3, \dots$, W_β is connected and diam $W_\beta < \varepsilon$ and f_i maps every component of $f_i^{-1}(W_\beta)$ homeomorphically onto W_β and furthermore evey component C of $f_i^{-1}(W_\beta)$ has diameter $< \varepsilon$. Let $\delta_0 > 0$ $(\delta_0 < \varepsilon)$ be a Lebesgue number for $\{W_\beta\}$. If x, $y \in X$ and $d(f_i(x), y) < \delta_0$, then there is some W_β containing $f_i(x)$ and y. Let C be the component of $f_i^{-1}(W_\beta)$ containing x. Then there exists a point z in C with the property that $f_i(z)=y$ and $d(x, z)<\delta_0/\lambda$. Hence $B_{\delta_0/\lambda}(x)\cap f_i^{-1}(y)\neq\phi$ for $i=1, 2, 3, \dots$, and the proof is completed.

Let f be a continuous map of (X, d) into itself. Given $\delta>0$, a sequence $\{x_i\}_{i=0}^n$ $(0\leqq n\leqq\infty)$ is called δ-pseudo-orbit for f if $d(f(x_i), x_{i+1})<\delta$ for $0\leqq i<n$. Given $\epsilon>0$, $\{x_i\}_{i=0}^n$ is called to be ε-traced by a point $y\in X$ if $d(f^i(y), x_i)<\varepsilon$ for $0\leqq i\leqq n$. We call f to have pseudo-orbit tracing property (abbrev. P.O.T.P.) if for any $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit for f can be ε-traced by some point in X. It is well known result that if f is expanding then f has the P.O.T.P. (see, [5]).

LEMMA 2. *Let X be a compact connected locally connected space with metric d and let $\{f_i\}_{i=1}^\infty$ be a sequence of expanding maps of X onto itself with common ε and λ. Then for any $\eta>0$, there is $\delta>0$ such that every δ-pseudo-orbit for f_i can be η-traced for $i=1, 2, 3, \dots$.*

PROOF. For any $\eta>0$, choose $\delta>0$ with $\delta<\min\{(\lambda-1)\eta/2, (\lambda-1)\delta_0/\lambda\}$ where $\delta_0>0$ is given in Lemma 1. Let $\{x_j\}_{j=0}^\infty$ be a δ-pseudo-orbit for f_i. Define $\{x_j^n\}_{j=0}^n$ by $x_j^n=x_j$ ($j=0, 1, 2, \dots$, n), then we have

$$d(f_i(x_{n-1}^n), x_n^n)<\delta<\delta_0.$$

Hence by Lemma 1 there is $y_{n-1}^n\in B_{\delta_0/\lambda}(x_{n-1}^n)$ such that $f_i(y_{n-1}^n)=x_n^n$. Here $d(x_{n-1}^n, y_{n-1}^n)<\delta/\lambda<\varepsilon$ and f_i is ε-local expansion, so we have

$$d(x_{n-1}^n, y_{n-1}^n)\leqq d(f_i(x_{n-1}^n), x_n^n)/\lambda<\delta_0/\lambda<\eta/2.$$

Accordingly

$$d(f_i(x_{n-2}^n), y_{n-1}^n)\leqq d(f_i(x_{n-2}^n), x_{n-1}^n)+d(x_{n-1}^n, y_{n-1}^n)<(1+1/\lambda)\delta<\delta_0.$$

There is by Lemma 1, $y_{n-2}^n\in B_{\delta_0/\lambda}(x_{n-2}^n)$ such that $f_i(y_{n-2}^n)=y_{n-1}^n$.
And it is easily seen that

$$d(x_{n-2}^n, y_{n-2}^n)<(1+1/\lambda)\delta/\lambda<\eta/2.$$

By an iterative procedure we get $\{y_{n-k}^n\}_{k=1}^n$ such that $f_i(y_{n-k}^n)=y_{n-k+1}^n$ ($k=2, 3, \dots, n$), and

$$d(x_{n-k}^n, y_{n-k}^n)<(1+1/\lambda+\dots+1/\lambda^{k-1})\delta/\lambda<\eta/2\ (k=1, 2, \dots, n).$$

When we define $\{y_{n-k}^n\}_{k=1}^n$ as above for every positive integer n, we have for each j a sequence $\{y_j^n\}_{n=j+1}^\infty$ such that $d(x_j, y_j^n)<\eta/2$ ($n=j+1, j+2, \dots$).

Since X is compact, this sequence has a convergent subsequence for each j. Using diagonal method we can get a subsequence $\{n'\}$ of $\{n\}$ such that $y_j^{n'}\longrightarrow y_j$ as $n'\to\infty$ and $f_i(y_j)=y_{j+1}$ and $d(x_j, y_j)\leqq\eta/2<\eta$ for $i=1, 2, \dots$, and $j=0, 1, 2, \dots$.

These relations mean that $\{x_j\}_{j=0}^\infty$ is η-traced by y_0, i.e., $d(f_i^j(y_0), x_j)<\eta$ ($j=0, 1, 2, \dots$).

This completes the proof.

Remark 1. In Lemmas 1 and 2, the connectedness is not essential.

3. The result

THEOREM. *Let X be a compact connected locally connected space with metric d and let $\{f_i\}_{i=1}^{\infty}$ be a sequence of expanding maps of X onto itself with common ε and λ. Assume that $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f_0. Then for each fixed point a_0 of f_0 there exist fixed points a_i of f_i such that $\{a\}_{i=1}^{\infty}$ converges to a_0.*

PROOF. For any $\eta \in (0, \varepsilon/2)$, there is $\delta > 0$ which is given in Lemma 2. There is $N > 0$ such that

$$d(f_i(a_0), f_0(a_0)) < \delta \qquad \text{for } i \geqq N.$$

We have

$$d(f_i(f_0^{n-1}(a_0)), f_0^n(a_0)) = d(f_i(a_0), f_0(a_0)) < \delta,$$

thus $\{f_0^n(a_0)\}_{n=0}^{\infty} = \{a_0, a_0, \dots\}$ is a δ-pseudo-orbit for f_i. Hence by Lemma 2 there is $a_i \in X$ such that

$$d(f_i^n(a_i), a_0) < \eta \text{ for } i \geqq N, n=0, 1, 2, \dots. \qquad (*)$$

Hence

$$d(f_i^n(f_i(a_i)), f_i^n(a_i)) \leqq d(f_i^n(f_i(a_i)), a_0) + d(a_0, f_i^n(a_i))$$
$$< \eta + \eta < \varepsilon \qquad \text{for } i \geqq N, n=0, 1, 2, \dots.$$

Then, since f_i is an ε-local expansion, $f_i(a_i) = a_i$ for $i \geqq N$, thus a_i is a fixed point of f_i. Now from (*),

$$d(a_i, a_0) < \eta \qquad \text{for } i \geqq N.$$

This means $a_i \to a_0$ as $i \to \infty$ and the proof is completed.

Remark 2. In our theorem we assumed that the maps f_i ($i=1, 2, 3, \dots$) have a common skewness λ. This assumption cannot be omitted as the examples given by Hu and Rosen [2] shows.

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