On the imbeddability and immersibility of the total spaces of sphere bundles over real projective space

By

Tsuyoshi WATABE and Tomoaki YOSHIOKA

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1. Introduction

Let ξ be a k-vector bundle over real projective space P_n, and $B(\xi)$ the total space of the associated sphere bundle of ξ. $B(\xi)$ may be considered as a differentiable manifold. In this paper we will consider the imbedding and immersion problem for such a manifold.

Notations and terminologies used in this paper are made clear in Section 2. Section 3 contains some general results and particular cases are examined in Section 4. As an application, we obtain an imbedding of Dold's manifold, which is contained in Section 5.

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2. Notaions and Terminologies

Let ξ be a k-vector bundle over real projective space P_n, (ξ) the associated $(k-1)$-sphere bundle of ξ, and $E(\xi)$, $B(\xi)$ the total space of ξ, (ξ), resp. Let x denote the canonical line bundle over P_n and ε^k the triviak k-vector bundle over P_n.

We recall the definition of $\tilde{KO}$ (P_n) : this is the group of stable classes of real vector bundles over P_n. We denote the stable class of a vector bundle ξ by ξ_o. It is known that $\tilde{KO}$ $(P\ )$ is a cyclic group of order $2^{\varphi(n)}$ generated by x_o. where φ (n) is the number of integer s such that $o < s \leqq n$ and $s \equiv 0, 1, 2, 4 \bmod 8$.

We define the geometric dimension of the element $\tilde{KO}$ (P_n) as follows. Let E (P_n) be the semi group of equivalence cllases of real vector bundles over P_n. Define $\breve{KO}$ $(P_n)=\tilde{KO}(P_n)+Z$. Let θ be the canonical homomorphism; E $(P_n)\rightarrow\breve{KO}(P_n)$ defined by $\theta(\xi)=\xi_o.+k$. Then the geometric dimension of α (denoted by $g(\alpha)$) is the least integer k such that $\alpha+k$ is in the inage of θ. The following properties of geometric dimension are clear by the definition.

(i) *if ξ_o is the stable class of a vector bundle ξ, then $g(-\xi_0)\leq l$ if and only if there exists an l-vector bundle η such that $\xi\oplus\eta=\varepsilon^m$.*

(ii) $g\ (\alpha+\beta)\ \leqq g\ (\alpha)+g\ (\beta)$

In what follows all imbeddings and immersions are differentiable. We will write $M\subset R$,

$M \subseteq R$ if there exists an imbedding of M in R, an immersion of M in R, resp.

3. General results

From now on, let ξ denote a k-vector bundle over P.

3. 1. The following theorem is due to B. J. Sanderson [2].

(3.1.1) *$E(\xi) \subseteq R^{n+k+r}$ if and only if $g(\nu_0-\xi_0) \leqq r$, where ν_0 is the stable class of normal vector bundle of P. Moreover if $k+r>n$, $g(\nu_0-\xi_0) \leqq r$ implies $E(\xi) \subset R^{n+k+r}$.*

From (3.1.1), we have

(3.1.2) *$B(\xi) \subset R^{2n+k}$ and $\subseteq R^{2n+k-1}$.*

Proof. Let P be imbedded in R^{n+m} with normal bundle ν, where m is large enough. Then ξ can be imbedded in ν as a sub-bundle, in ohter words, there exists an (m-1) vector bundle η such that

$$\xi \oplus \eta = \tau$$

Since $\pi_i(V_{m-km-k-n})=0$ for $i<n$, η admits $(m-k-n)$ linearly independent cross sections: $\eta=\varepsilon^{m-k-n}+\xi$.

Thus we have

$$\xi \oplus \xi^n \oplus \varepsilon^{m-k-n} = \nu$$

Hence $g(\nu_0-\xi_0) \leqq n$. According to (3.1.1). $E(\eta) \subseteq R^{2n+k}$ and since $k>0$, $E(\xi) \subset R^{2n+k}$. Now (3.1.2) follows from

(3.1.3) (*i*) $E(\xi) \subset R \Rightarrow B(\xi) \subset R^m$

(*ii*) $E(\xi) \subset R \Rightarrow B(\xi) \subset R^{m-1}$.

Proof. (i) is clear and for the proof of (ii), see [4].

3. 2. The following two lemmas are useful.

(3.2.1) *Let M, N be m-manifold, n-manifold, resp.. Suppose $M \subseteq R^k$, $N \subset R^l$. Then if 2dim. $M < k+l$, $M \times N \subset R^{k+l}$.*

This is due to Sanderson-Schwarzenberger.

(3.2.2) *Let M, N be as in (3.2.1), and N s-parallelizable. Then $M \times N \subseteq R^{m+n+r}$ if and only if $M \subseteq R^{m+r}$.*

One can prove this easily, using the fundamental theorem on immersion; $M \subseteq R^{m+r}$ if and only if $g(\nu_0) \leqq k$, where ν_0 is the stable normal bundle.

3. 3. Let η be the bundle defined in (3.1.2). The obstruction to the existence of $(m-k-n+1)$ linearly independent cross sections of η is an element o_n of $H^n(P\ \{\pi_{n-1}(V_{m-k-1,m-k-1-n+1})\})$, where $\{F\}$ denotes the bundle of coefficients of fibre F. If n is even, $o_n=w_n(\eta)$, and if η is orientable and n is odd, $o_n=\delta\, w_{n-1}(\eta)$, where δ is a homomorphism $H^{n-1}(P_n;Z_2) \rightarrow H^n(P_n;Z)$.

The $(n-1)$-th and n-th Stiefel-Whitney classes of η can be calculated by the formula

$$w(\xi).\ w(\eta)=w(\nu)$$

For example, if $\xi_o = lx_0$, then

$$w(\eta) = (1+a)^{-(n+l+1)}$$

and hence

$$w_i(\eta) = \binom{n+l+i}{i} a^i \bmod 2$$

where a is a generator of $H^*(P\ ;Z_2)$.

If $o_n=0$, then $g(\nu_0-\xi_0) \leqq n-1$, which implies

$$B(\xi) \subset R^{2n+k-1} \quad (k>1)$$

3. 4. We calculate the Stiefel-Whitney classes of $B(\xi)$. As is well known,

$$\tau(B(\xi)) \oplus \varepsilon^1 = \pi^*\{\tau(P_n) \oplus \xi\}$$

Hence we have

$$w(B(\xi)) = \pi^*(w(P_n)w(\xi))$$

From this formula, we can calculate $w\ (B(\xi))$ and hence normal Stiefel-Whitney classes $w_{\cdot}\ (B(\xi))$.

3. 5. Now we will discuss the relations between the stable class of ξ and imbeddability of $B(\xi)$. Let the stable class of ξ be lx_0 $(0 \leqq l < 2^{\varphi(n)})$.

Then we have

$$g(\nu_0-\xi\ ) = g(-(n+l+1)x_0)$$

If $n+l+1 < 2^{\varphi(n)}$, we have

$$g(\nu_0-\xi_0) \leqq 2^{\varphi(n)}-(n+l+1)$$

By (3.1.1.), we have

$$E(\xi) \subseteq R^{2\varphi(n)-l+k+1}$$

and if $2n+1 < 2^{\varphi(n)}+k-l$,

$$B(\xi) \subset R^{2\varphi(n)-l+k-1} \text{ and } \subseteq R^{2\varphi(n)+k-l-2}$$

If $n+l+1 > 2^{\varphi(n)}$, then

$$g(\nu-\xi_0) = g((2^{\varphi(n)}-(n+1))x_0 + (2^{\varphi(n)}-l)x_0) \leqq 2^{\varphi(n)+1}-(n+l+1).$$

Hence we have

$$E(\xi) \subseteq R^{2\varphi(n)+1+k-l-1}$$

and if $2n+1 < 2^{\varphi(n)+1}+k-l$,

$$B(\xi) \subset R^{2\varphi(n)+1+k-l-1} \text{ and } \subseteq R^{2\varphi(n)+1+k-l-2}$$

As spacial cases, we have

(i) if $\nu_0=0$, then $B(\xi) \subset R^{n+k+1+2\varphi(n)-l}$ $(2^{\varphi(n)}+k+1+l>n)$

(ii) if $\xi_0=0$, then $B(\xi) \subset R^{2\varphi(n)-1+k}$ $(2^{\varphi(n)}-1+k>2n)$

3. 6. We conclude this section with a non-imbedding thoorem for $B(\xi)$. Consider Grothendieck's operators. [1]

$$\gamma^i;\ KO(X)\rightarrow KO(X)$$

$$\gamma_t;\ KO(X)\rightarrow A(X)$$

where $A(X)$ denotes the multiplicative group of formal power series in t with coefficients in $KO(X)$ and constant term 1. Explicitly, the operators γ^i and γ_t are defined by the following equations

$$\gamma_t(a)=\sum_{i=0}^{\infty}\gamma^i(a)t^i=\sum_{i=0}^{\infty}\lambda^i(a)t^i(1-t)^{-i} \quad a\varepsilon\ KO(X)$$

where $\lambda^i;\ KO(X)\rightarrow KO(X)$ denotes the exterior operator.

M. F. Atiyah proved in [1] the following properties of operators γ^i and γ_t.

(3.6.1) (i) *if* $a\varepsilon KO(X)$, then $\gamma^0(a)=1$ and $\gamma^1(a)=a$.

(ii) γ *is a natural ring homomordhism*

(iii) *if* $a_0\varepsilon \tilde{KO}(X)$, *then* $\gamma\ (a_0)=0$ *for* $i>g(a_0)$

(iv) *let* M *be a compact differentiable manifold of dimension* m. *If* $M \subseteq R^{m+k}$ *(resp.* $M \subset R^{m+k}$*), then* $\gamma\ (-\nu_0(M))=0$ *for* $i> k$*(resp.* $i \geq k$*)*

(v) $\gamma_t(x_0)=1+x_0t$, *where* x_0 *is the generator of* $\tilde{KO}(P_n)$.

Suppose $\xi_o=lx_0$. Since

$$\tau_0(B(\xi))=\pi^!\ \{\tau_0(P_n)+\xi_0\}=\pi^!\ \{(n+l+1)x_0\}$$

we have

$$\nu_0(B(\xi))=\pi^!\ (-(n+l+1)x_0)$$

By (3.6.1) (v) and the naturarity of γ^i, we have

$$\gamma_t(\nu_0(B(\xi))=\pi^!\ (1+x_0t)^{-(n+l+1)}$$

Since

$$\gamma^i\ (-(n+l+1)x_0)=\pm2^{i-1}\binom{n+l+i}{i}x_0,\ \gamma^i(\nu_0(B))=\pi^!\ (\pm2^{i-1}\binom{n+l+i}{i}x_o)$$

We define σ by

$$\sigma=\max\{\,i\,|\,2^{i-1}\binom{n+l+i}{i}\not\equiv0 \mod 2^{\varphi(n)}\}$$

Then

$$\gamma^\sigma\ (-(n+l+1)x_0)\neq0$$

Thus if $\pi^!$ is an isomorphism, then

$$\gamma^{\sigma}(\nu_0(B(\xi)))\neq 0.$$

this implies

$$B(\xi) \not\subset R^{n+k-1+\sigma} \text{ and } \not\subseteq R^{n+k-1+\sigma-1}$$

Therefore we have obtained the following

(3.6.2) THEOREM. *Suppose* $\pi^! ; KO(P) \to K\tilde{O}(B(\xi))$ *be an isomorphism.*
Then

$$B(\xi) \not\subset R^{n+k-1+\sigma} \textit{ and } \not\subseteq R^{n+k-1+\sigma-1}$$

4. Special cases

In this section, we will show some results on the imbeddability and immer ibility of $B(\xi)$ ($n=2, 3, 4$; $k\geq 1$). We quote from [3] the results of the classification of vector bundles over P_n.

4. 1. *The k-sphere bundle over* P_2

We have

(i) $B(\varepsilon^{k+1}) \subset R^{4+k}$ and $\subseteq R^{3+k}$ ($k\geq 1$)

These results are best possible.

These follows from (3.2.1) and (3.5), and the fact that $w_1(B(\xi)=0$ implies $B \not\subset R^{3+k}$.

(ii) $B(x\oplus\varepsilon^k) \subset R^{4+k}$ *and* $\subseteq R^{3+k}$ ($k\geq 1$)

We have $o_2(\eta)=0$, where η is the bundle in (3.3). In fact, since $w(\eta)=(1+a)^{-4}$, $w_2(\eta)=o_2(\eta)=0$.

(iii) $B(2x\oplus\varepsilon^{k+1}) \subset R^{5+k}$ *and* $\subseteq R^{4+k}$

For $k\geq 2$, these results are best possible.

Since $2x\oplus\varepsilon^{k-1}$ has a cross section for $k\geq 2$, $\pi^!$; $K\tilde{O}(P_2)\to K\tilde{O}(B)$ *is isomorphic.*

Combining this and the fact that σ in (3.6) is 2, we have $B \not\subset R^{4+k}$ and $\not\subseteq R^{3+k}$

(iv) $B(3x\oplus\varepsilon^{k-2}) \subset R^{5+k}$ and $\subseteq R^{4+k}$.

For any k, these are best possible.

These follows from (3.5).

4. 2. *The k-sphere bundle over* P_3

(i) $B(\varepsilon^{k+1}) \subset R^{5+k}$ *and* $\subseteq R^{4+k}$ ($k\geq 2$)

For $k=1$, $B\subset R^7$ and $\subseteq R^5$.

These follows from (3.2.1) and (3.2.2).

(ii) $B(x\oplus\varepsilon^k) < R^{7+k}$ *and* $\leq R^{6+k}$ ($k\geq 1$)

These are best possible. In fact, we have $\bar{w}_3(B)=\pi^* a_3$. Since $w_{1+k}(x\oplus\varepsilon^k)=0$, π^* is an isomorphism by the exactness of Gysin sequence. Thus we have $\bar{w}_3(B)\neq 0$

(iii) $B(2x\oplus\varepsilon^{k-1}) \subset B^{\sigma+k}$ *and* $\subseteq R^{5+k}$ ($k\geq 2$)

(3. 6) implies these are best possible for $k\geq 2$.

(iv) $B(3x\oplus\varepsilon^{k-2}) \subset R^{5+k}$ *and* $\subseteq R^{4+k}$ ($k\geq 2$)

These are best possible for $k \geq 3$. These follows from (3.5) and (3.6)

4. 3. *The k-sphere bunble over* P_4.

(i) $B(\varepsilon^{k+1}) \subset R^{8+k}$ *and* $\subseteq R^{7+k}$

These results are derived from (3.5), and (3.6) implies these are best possible.

(ii) $B(x \oplus \varepsilon^k) \subset R^{8+k}$ *and* $\subseteq R^{7+k}$.

(iii) $B(2x \oplus \varepsilon^{k-1}) \subset R^{8+k}$ *Nand* $\subseteq R^{7+k}$ $(k \geq 1)$.

(iv) $B(3x \oplus \varepsilon^{k-2}) \subset R^{8+k}$ *and* $\subseteq R^{7+k}$ $(k \geq 2)$.

(v) $B(4x \oplus \varepsilon^{k-3}) \subset R^{9+k}$ *and* $\subseteq R^{8+k}$ $(k \geq 3)$.

(vi) $B(5x \oplus \varepsilon^{k-\varphi}) \subset R^{9+k}$ *and* $\subseteq R^{8+k}$ $(k \geq 4)$.

(vii) $B(6x \oplus \varepsilon^{k-5}) \subset R^{9+k}$ *and* $\subseteq R^{8+k} (k \geq 5)$.

(viii) $B(7x \oplus \varepsilon^{k-6}) \subset R^{9+k}$ *and* $\subseteq R^{8+k}$ $(k \geq 6)$.

It is easy to see that B in (vi)～(viii) is not imbeddable in R^{6+k} and not immersible in R^{5+k}.

5. Imbedding of Dold' s manifold

In this section we will discuss the imbeddability of Dold's manifold of type $(n,1)$. We denote it by $P(n,1)$.

$P(n,1)$ is defined as follows. Let S^n be the unit sphere in R^{n+1}; $S = \{(x_0, x_1, \cdots\cdots x_n)$ ε R^{n+1}, $\Sigma x^2 = 1\}$ and CP_1 the complex 1-dimensional projective space; $CP = \{z = (z_0, z_1)$ z; complex number$\}$. Now $P(n,1)$ is the manifold obtained from $S^n \times CP_1$ by identifying (x,z) with $(-x, \bar{z})$ where $-x$ denotes the antipodal of x and $\bar{z}$ the congugate of z.

It is obvious that ρ; $P(n,1) \to P_{n}$ defined by $p(x,z) = x$ is a fibre map. We denote this bundle by δ;

$$\delta = \{P(n,1), p, P_n, \ CP_1 \ O(1)\}$$

5. 1. We will first prove that $P(n,1)$ is the total space of the associated sphere bundle of a vector bundle with cross section.

We identify CP_1 with S^2 by the map

$$\varphi : [z_0, z_1] \to \left(\frac{2R(z_0 \bar{z}_1)}{|z_0|^2 + |z_1|^2} \quad \frac{2I(z_0 \bar{z}_1)}{|z_0|^1 + |z_1|^2} \quad \frac{z_0^2 - z_1^2}{|z_0|^2 + |z_1|^2} \right)$$

Let A be the non-trivial element of O(1). By the definition

$$A[z_0, z_1] = [\bar{z}_0, \bar{z}_1]$$

Hence A operates on S^2 by the formula

$$A(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

Thus the operation of A on S^2 is the suspension of the operation of A on S^0. This implies δ is the associated sphere bundle of a 3-vector bundle ξ. It is easy to see that ξ has a cross section.

5. 2. According to a result of J.Levine [3], it is known that the stable class of ξ is x_0 if $n>2$.

From the above considerations, we have the following

(5.2.1) THEOREM. $P(n,1) \subset R^{2n+3}$ *and* $\subseteq R^{2n+2}$.

(5.2.2) THEOREM. $P(n,1) \subset R^{2n+2}$ *and* $\subseteq R^{2n+1}$ *for n even.*

(5.2.3) THEOREM. $P(n,1) \subset R^{2\varphi(n)+1}$ *and* $\subseteq R^{2\varphi(n)}$ *if* $2n-1 < 2^{\varphi(n)}$.

(5.2.4) THEOREM. $P(n,1) \subset R^{n+\sigma+2}$ *and* $\subseteq R^{n+\sigma+1}$

where

$$\sigma = max\{i \mid 2^{i-1}\binom{n+1+i}{i} \neq o \ mod \ 2^{\varphi(n)}\}$$

(5.2.1) follows from (3.1.2), and (5.2.2) from (3.3). (5.2.3) follows from (3.5) and (5.2.4) from (3.6.2).

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References

1. M. F. ATIYAH, ***Immersions and imbeddings of manifolds***, Topology 1 (1962), 125–132.
2. B. J. SANDERSON, ***Immersions and Embeddings of Projective spaces***, Proc. London Math. Soc. (3) 14 (1964) 137–153.
3. J. LEVINE, ***Spaces mith involution and bundles over P_n***, Amer. Jour. Math. vol. 85 (3) (1963) 516–540.
4. T. WATABE, ***A note on the imbeddability and immersibility of the total spaces of sphere bundles over sphere***, to appear in this Journal.