A note on the imbeddability and immersibility of the total spaces of sphere bundle over sphere

By

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1. Introduction

In this note we will consider the total space of an oriental sphere bundle over sphere. The total space of such a bunble may be considered as a differentiable manifold. Throughout we will consider the total space as a differentiable manifold.

We will be concerned with the following two questions about the total space of a sphere bundle over sphere.

1. *Can it be imbedded or immersed in R^m?*
2. *Cannot it be imbedded or immersed in R^m?*

After preparations in Sections 2, 3 and 4, we give answers to Question 1 and 2 in Section 5, and then examine particular cases in Section 6.

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2. Notations and Terminologies

2. 1. In what follows, the word " differentiable " will mean " of class C^∞". A differentiable map of a differentiable manifold M^n in Euclidean space R^m is called an immersion if its diffrential has the maximal rank n ($n<m$) at each point of M, and an immersion which is one-one an imbedding.

We will write $M \subseteq R^m$, $M \subset R^m$ when M is immersed in R^m, imbedded in R^m, respectively.

2. 2. Let $\xi=\{E(\xi), \pi_\xi, S^n, R^{k+1}, SO(k+1)\}$ be a $(k+1)$-dimensional vector bundle over S^n, and $(\xi)=\{B(\xi), p_\xi, S^n, S^k, SO(k+1)\}$ the associated k-sphere bundle. Let $\in^k$ denote trivial k-vector bundle.

By the bnndle classification theorem, the equivalence classes of k-sphere bundle over n-sphere are in one one correspondence with elements of $\pi_{n-1}(SO(k+1))$.

Now we define bundle $\xi_m^{(n,k)}$ as follows;

$$\xi_m^{(n,k)}=\{B_m^{(n,k)},\ p_m^{(n,k)},\ S^n,\ S^k,\ SO(k+1)\}$$

where $\xi_m^{(n,k)}$ corresponds to element m $\in \pi_{n-1}$ (SO(k+1)). If no confusion arises, we write B(ξ), p_ξ for $B_m^{(m,k)}$, $p_m^{(n,k)}$, respectively.

3. Preliminary lemmas

3. 1. First we will prove a lemma on the imbeddadility of a vector bundle over S^n in a vector bundle over S^n.

Let ξ, η be vector bundles over S^n with dimension k, l (we suppose $k>l$). Then we have

(3.1.1) LEMMA. *If l-k is greater than n-1, then ξ can be imbedded in η as a subbundle. In other words, there exists an (l-k) vector bundle ζ over S^n such that*

$$\xi \oplus \zeta = \eta$$

PROOF. Let *Hom* (ξ, η) be the bundle defined by *Hom* $(\xi, \eta)_x = Hom\ (\xi_x, \eta_x)$$\cdots$group of linear transformations of ξ_x into η_x, and L (ξ, η) the sub-bundle of Hom (ξ, η) with fibre $L(\xi_x, \eta_x)$$\cdots$group of linear transformations of maxmal rank k ($k<l$). Then ξ can be imbedded in η as a sub-bundle if and only if L (ξ, η) has a cross section. Since π $(L\ (\xi_x, \eta_x)) = \pi\ \ (V_{l,k}) = 0$, for $i<l$-k, lemma follows from the standard obstruction theory.

3. 2. Following lemmas are basic for the proof of our results.

(3.2.1) LEMMA. *Let ξ be a k-vector bundle over a differentisble manifold M. Suppose M be imbedded in R^m with normal vector bundle ν. If ξ can be imbedded in ν as a sub-dundle of ν, then*

$$E(\xi) \subset R^m$$

This lemma follows from the fact that the assumption implies that E (ξ) is imbedded in $E(\nu)$ and E (ν) is imbedded in R^m as a tubular neighbourhood of M in R^m.

(3.2.2) LEMMA. *Let ξ be a k-vector bundle over a differentiable manifold M. Then we have*

$$\text{(i)}\quad E(\xi) \subseteq R^{n+k+r} \to B(\xi) \subseteq R^{n+k+r-1}$$

$$\text{(ii)}\quad E(\xi) \subset R^{n+k+r} \to B(\xi) \subset R^{n+k+r}$$

The proof of (i). Let ν be the normal bundle of immersion; E $(\xi) \subseteq R^{n+k+r}$
We have

$$\tau(E(\xi)) \oplus \nu = \varepsilon^{n+k+r}$$

and

$$\tau(E(\xi)) = \pi_\xi{}^!\ (\tau(M) \oplus \xi)$$

Hence

$$i^!\tau(E(\xi)) = p_\xi{}^!\ (\tau(M) \oplus \xi) = p_\xi{}^!\ (\tau(M) \oplus \tilde{\xi} \oplus \varepsilon^1 = \tau(B(\xi)) \oplus \varepsilon'$$

where $i^!$ is induced by the inclusion i; $B \subset E$ and $\tilde{\xi}$ the bundle along the fibres.

Since $$i^! \tau(E(\xi)) \oplus i^! \nu = \varepsilon^{n+k+r}$$

we have $$\tau(B(\xi)) \oplus \varepsilon^1 \oplus i^! \nu = \varepsilon^{n+k+r}$$

Therefore $B(\xi)$ can be immersed in R^{n+k+r} with normal bundle $\varepsilon^1 \oplus i^! \nu$. This implies (i).

(ii) is clear.

(3.2.3) LEMMA. *Let ξ be as in (3.2.2). Then we have*

$$\tau(B(\xi)) \oplus \varepsilon^1 = p_\xi{}^! (\tau(M) \oplus \xi)$$

where $\tau(\quad)$ denotes the tangent bundle.

PROOF. See [9].

3. 3. Now let us calculate the Stiefel-Whitney classes of a k-sphere bundle (ξ) over the n-sphere S^n.

Let $\tilde{\xi}$ be the associated principal $SO(k+1)$-bundle of ξ. We may suppose $k+1 \geq n$. The restriction of $\tilde{\xi}$ on the $(n-1)$ skelton of S^n has a cross section f. Let $0(\tilde{\xi}, f)$ be the obstruction to extending f over S^n; $0(\tilde{\xi}, f) \in H^n(S^n : \pi_{n-1}(SO(k+1))$.

Let p denote the natural projection $SO(k+l) \rightarrow V_{k+1,k+1-n+1}$, p_* the homomorphism $\pi_{n-1}(so(k+1)) \rightarrow \pi_{n-1}(V_{k+1,k+1-n+1})$ and p_{**} the induced homomrphisn $H^n(S^n : \pi_{n-1}(SO(k+1)) \rightarrow H^n(S^n : \pi_{n-1}(V_{k+1,k+1-h+1}))$. Then we have

$$w_n(\xi) = p_{**}0(\xi, f)$$

The following result has been proved in [4].

(3.3.1) *If $k \geq n$ and $n \neq 2, 4$ or 8, then $w_n(\xi) = 0$.*

3. 4. We will calculate $w_n(\xi)$ when n=2, 4 *or* 8.

3.4.1. The case n=2.

In this case we can choose the associated sphere bundle of $\theta \oplus \varepsilon^{k-1}$ as (ξ), where θ is the canonical 2- vector bundle over S^2 ; the canonical complex line bundle over $CP_1 = S^2$ regarded as a real vector bundle. Since θ has the total Chern class $c(\theta) = 1 + a$, where a is a generator of $H^2(S^2 : Z)$, $w_2(\xi) = a \bmod 2$.

3.4.2. The case n =4.

Let $\xi_n^{(4,k)}$ $(k \geq 4)$ be the bundle with characteristic map $i(n\sigma)$, where i; $SO(4) \rightarrow SO(r)$ $(r \geq 5)$, $\sigma : S^3 \rightarrow SO(4)$ given by

$$\sigma(u)v = uv$$

where u and v denote quatenions with norm 1. By a result of [10], we have

$$0(\tilde{\xi}_n^{(4,k)}) = \pm n a_4$$

where a_4 is a generator of $H^4(S^4)$. Hence we have

(3.4.2.1) $w_4\ (\xi^{(4,k)}_n)=0$ *if and only if* n *is even.*

Let $\xi^{(4,3)}_{m,n}$ be the bundle with characteristic map $m\rho+n\sigma$, where $\rho; S^3\to SO(3)\subset SO(4)$, given by $\rho(u)v=uvu^{-1}$. Then we have

(3.4.2.2) $w_4\ (\xi^{(4,3)}_{m,n})=\pm n\ a_4 \mod 2$.

3.4.3. The case n=8.

In this case we have the similar results.

$$w_8(\xi^{(8,7)}_{m,n})=\pm n\ a_8 \mod 2$$

$$w_8(\xi^{(8,k)}_{m})=\pm m\ a_8 \mod 2\ (k\geq 8)$$

where a_8 is a generator of $H^8\ (S^8)$.

3. 5. Let us calculate the Stiefel-Whitney classes of B, using (3.2.3).

(3.2.3) shows

$$w(\tau(B(\xi)))=p^*(\mathrm{w}(\xi))$$

where $p^*;\ H^*(S^n: Z_2)\to H^*(\mathrm{B}(\xi): Z_2)$ is the homomorphism induced by p. If $k\geq n$, p^* is an isomorphism. Hence we have

(3.5.1) LEMMA. *Let* ξ *be a* $(k+1)$*-vector bundle over the n-sphere* S^n $(k\geq n)$.
Then we have

(i) $w(B)=1$ *if* $n\neq 2,\ 4,\ 8$

(ii) $w\ (B^{(2,k)})=1+a$

(iii) $w\ (B^{(4,k)}_m)=1+a$ *for m odd*

$=1$ *for m even*

(iv) $w(B^{(8,k)}_m)=1+a$ *for m odd*

$=1$ *for m even*

(3.5.2) LEMMA. $w(B^{(k+1,k)}_{m,n})=1$ *for* $k=3,7$

This follows from $H^r\ (B^{(k,k-1)}_{m,n})=Z_n$ *for* $k=4,8$.

Combining these and a result of J. Milnor, we have

$w\ (B^{(n,k)}_m)\neq 0$ *if and only if* $(n,\ k,\ m)=(2,2+l,\ d),\ (4,4+l,\ d)$ *and* $(8,8+l,\ d)$, *where* l *is nonnegative integer,* d *is an odd integer.*

4. Grothendieck operations on $\widetilde{KO}\ (S^n)$

4. 1. We recall the definitian of the group $\widetilde{KO}(S^n)$; $\widetilde{KO}(S^n)$ is isomorphic to the group of stable classes of real veetor bundles over S^n. It is well known that $\widetilde{KO}(S^n)$

is isomorphic to π_n (BSO), where BSO is the classifying space for stable rotation group SO. The following table is also well known.

(4.1.1)

$n\,(mod 8)$	0	1	2	3	4	5	6	7
$\tilde{KO}\,(S^n)$	Z	Z_2	Z_2	0	Z	0	0	0

4. 2. Let λ^i and ψ_i be operations defined by R. Bott in [2]. There exists the following relation between these operations

(4.2.1) $$\psi_i-\psi_{i-1}\lambda^1+\cdots(-1)^i i\,\lambda^i=0.$$

Moreover we have

(4.2.2) $$\psi_i(x)=i^r x \text{ for } x\in\tilde{KO}(S^n),\ n=2r$$

If x denotes a generator of KU (S^n), then $x^2=0$. for $n\equiv 0$ mod 4. Hence

$$\lambda^i(x)=(-1)^i\, i^{(r-1)}x$$

4. 3. Next we will consider the r operation on $\tilde{KO}$ (S^n) defined by F. Atiyah in [1]. γ^i is defined by the following formula

$$\Sigma\gamma^i t^i=\Sigma\lambda^i t^i(1-t)^{-i}$$

in other words,

$$\gamma^i=\lambda^1+\binom{i-1}{1}\lambda^2+\binom{i-1}{2}\lambda^3+\cdots\cdots+\binom{i-1}{i-1}\lambda^i$$

Therefore, for $n=4r$

$$\gamma^i\,(x)=\lambda^1(x)+\binom{i-1}{1}\,(-1)^1\,2^{2r-1}\lambda^1\,(x)+\cdots\cdots+\binom{i-1}{i-1}(-1)^{i-1}i^{2r-1}\lambda^1(x)$$

Since λ^1 $(x)=x$, we have

$$\gamma^i(x)=x\{1-\binom{i-1}{1}\,2^{2r-1}+\binom{i-1}{2}3^{2r-1}-\cdots\cdots+(-1)^{i-1}\binom{i-1}{i-1}i^{2r-1}\}$$

4.4. If $n\equiv 1$, 2 mod 8, γ^i operations are determined by r^i-operationson $\tilde{KO}$ $(P\ )$, and if $n\not\equiv 0$, 1, 2, 4, mod 8 all γ^i operations are zero.

5. Immersions and imbeddings

5. 1. First we prove the following

(5.1.1) THEOREM. *Let ξ be $(k+1)$ vector bundle over the n-sphere S^n. Then we have*

$$B(\xi)\subset R^{2n+k+1} \text{ and } B(\xi)\subseteq R^{2n+k}.$$

PROOF. Let S^n be imbedded in R^{n+m} with normal vector bundle ν. If m is large enough, ξ can be imbedded in ν as a sub-bundle, there exists an $(m$-k-$1)$ vector bundle η such that

$$\xi \oplus \eta = \nu$$

Since π_i $(V_{m-k-1,\ m-k-1-n})=0$, for $i>n$,

$$\eta = \zeta \oplus \varepsilon^{m-k-1-n}$$

where ξ is an n-vector bundle over S^n, Thus we have

$$\xi \oplus \zeta \oplus \varepsilon^{m-k-1-n} = \nu.$$

We can find an immersion of S^n in R^{2n+k+1} with the normal vector bundle $\xi \oplus \zeta$, and suppose that this immersion is an imbedding. Hence ξ can be imbedded in the normal vector bundle of an imbedding of S^n in R^{2n+k+1}. By (3.2.1), (3.2.2), we have

$$B(\xi) \subset R^{2n+k+1} \text{ and } B(\xi) \subseteq R^{2n+k}$$

In remainder of this section, we will show the general result can be improved in some special cases.

5. 2. We prove, for even n,

(5.2.1) LEMMA. *If* $w_n(\xi)=0$. *then* $B(\xi) \subset R^{2n+k}$ *and* $B(\xi) \subseteq R^{2n+k-1}$.

PROOF. $w_n(\xi)=w_n(\eta)$ is the only one obstruction to the existence of (m-k-1-n+1) linearly independent cross sections of η. If $w_n(\eta)=0$, then ξ can be imbedded in the normal vector bundle of an imbedding of S^n in R^{2n+k}.

Now we consider some spacial cases.

5. 3. The case n=3, 5, 6, 7 mod 8.

In this case we can prove the following

(5.3.1) THEOREM. *If* $k \geq n$, *then* $B(\xi) \subset R^{n+k+1}$.

(5.3.2) THEOREM. *If* $k \leq n-1$, *then* $B(\xi) \subset R^{2n+1}$ *and* $B(\xi) \subseteq R^{2n}$.

PROOF. (5.3.1) follows from the fact that if $k \geq n$, then $(\xi)=S^n \times S^k$.

The proof of (5.3.2) is as follows; since ξ is stably trivial, $\xi \oplus \varepsilon^{n-k} = \varepsilon^{n+1}$ Hence ξ can be imbedded in the normal vector bundle ε^{n+1} of an imbeddng of S^n in R^{2n+1}. The result follows from (3.2.1) and (3.2.2).

Moreover we have

(5.3.3) THEOREM. *For any* $k, B(\xi) \subseteq R^{n+k+1}$.

This follows from that for any k, $B(\xi)$ is stably parallelizable.

5. 4. The case n≡1, 2 mod 8.

(5.4.1) THEOREM. (*i*) *If* $n \equiv 1$ *mod* 4, $k \geq 3$ *and* $n \geq 3$ *and* 4, *then* $B(\xi) \subset R^{2n+k-2}$ *and* $B(\xi) \subseteq R^{2n+k-3}$ (*ii*) *If* $n \equiv 2$ *mod* 4,$k \geq 7$ *and* $n \geq 7$, *then* $B(\xi) \subset R^{2n+k-5}$ *and* $B(\xi) \subseteq R^{2n+k-6}$

PROOF. Let S^n be imbeded in R^{n+m} with normal bundle ν. As in the proof of (5.1.1), we have

$$\xi \oplus \eta = \nu$$

for some $(m-k-1)$ vector bundle η. The only obstruction to the existence of $(m-k-1+3-n)$ linearly independent cross sections of η is an element of H^n $(S^s: \pi_{n-1}(V_{m-k-1,}$

$_{m-k-1-n+3}$)). By the result of [6] for $n=1$ mod 4,

$$\pi_{n-1}(V_{m-k-1,m-k-1-n+3})=0.$$

Hence we have

$$\eta=\varepsilon^{m-k-1-n+3}\oplus\zeta$$

where ζ is an $(n-3)$ vector bundle over S^n.

Then we have

$$\xi\oplus\zeta\oplus\varepsilon^{m-k-1-n+3}=\nu$$

If $k\geq 3$, we can find an imbedding of S^n in R^{2n+k-2} with normal vector bundle $\xi\oplus\zeta$. This implies ξ can be imbedded in R^{2n+k-2}. Lemma (3.2.1) and (3.2.2) complete the proof of (i). The proof of (ii) is similar.

5. 5. The case n≡0 mod 4.

(5.2.1) and (3.3.1) give the following

(5.5.1) THEOREM. If $n\equiv 0$ mod 4, and $n\neq 4, 8$, then $B(\xi)\subset R^{2n+k}$ and $\subseteq R^{2n+k-1}$.

Theorems in Sections 5. 3, 5. 4 and 5. 5 give a partil answer to puestion 1.

Next we will consider question 2.

5. 6. Let $p^!$ be the homomorphism $K\tilde{O}(S^n)\to K\tilde{O}(B(\xi))$ induced by the projection p; $B(\xi)\to S$. The following lemma is due to M. F. Atiyah.

(5.6.1) LEMMA. *Let ξ be non-stably trivial k-bundle over S, and $p^!$ an isomorphism. Then if $r^i(-\xi_0)\neq 0, B(\xi)\not\subset R^{n+k+i}$, and $\not\subseteq R^{n+k+i}$, where ξ_0 denotes the stable class of ξ.*

6. Some special cases

In this section we shall study k-sphere bundle over the n-sphere for $n\leqq 4$.

6. 1. The case n=2.

6.1.1. 1-sphere bundle over S^2.

Since $B(\xi)$ is an orientable manifold of dimension 3 for any $m\in\pi_1(SO(1))$, $B(\xi)$ can be imbedded in R^5 with a trivial normal bundle (3), and hence $B(\xi)$ can be immersed in R^4.

6.1.2. k-sphere bundle over S^2.

In this case it follows from (5.1.1) that $B_m^{(2,k)}$ can de imbedded in R^{5+k} and immersed in R^{4+k}, Since $w_2(B_m^{(2,k)})\neq 0$, there results are best possile.

6. 2. The case n=4.

6.2.1. 2-sphere bundle over S^4.

First we recall some results on group $\pi_3(SO(r))$ $(r\geq 3)$

As is well known, we have

$$\pi_3(SO(3))=Z,\qquad \pi_3(SO(4))=Z+Z,\qquad \pi_3(SO(r))=Z(r\geq 5)$$

Let i ; $SO(r)\to SO(r+1)$ be natural inclusion. Then the generators

$$\{\alpha_3\},\quad \{\alpha_4,\beta_4\},\quad \{\beta_r\}\quad (r\geq 5)$$

of $\pi_3(SO(3))$, $\pi_3(SO(4))$, $\pi_3(SO(r))$ respectively are given as follows;

$$\alpha_3(u)v=uvu^{-1},\quad \alpha_4=(i_3)_*(\alpha_3),\quad \beta_4(u)v=uv.$$

where u and v are quoternions with norm 1. And

$$\beta=(i_{r-1})_*\cdots\cdots(i_4)_*(\beta_4)\,(r\geq 5)$$

$$(i_4)_*(\alpha_4)=-2\beta_5$$

It follows from these that $\xi_m^{(4,3)}$ is not stably trivial for $m\neq 0$. Since its stable class is $2mx$, where x is a generator of $\tilde{KO}(S^n)$, $p^!$ is an isomorphism, and

$$\gamma^2(2mx)=2mx(1-2)\neq 0$$

$$\gamma^3(2mx)=2mx\left(1-\binom{2}{1}2+\binom{2}{2}3\right)=0$$

we have $B(\xi)\not\subset R^8$ and $\not\subseteq R^7$.

That $w_4(\xi_m^{(4,3)})=0$ implies $B(\xi)\subset R^{10}$ and $\subseteq R^9$

6.2.2 3-sphere bundle over S^4.

We have

(i) $B(\xi_{m,o})\subset R^{11}$ *and* $\subseteq R^{10}$ $m\neq 0$

(ii) $B(\xi)\not\subset R^9$ *and* $\not\subseteq R^8$

PROOF. (i) follows from that $\xi_m^{(4,4)}$, has a cross section,. $E(\xi_{m,o}))\subset R^{10}$.

(ii) follows from that $\xi_{m,o}$ is not stably trivial and $\gamma^2(B(\xi_{m,o}))\neq 0$.

Moreover we can prove the following

(iii) $B_{m,n}\subseteq R^8 \Leftrightarrow n=2$ *or* $2n\equiv 0$ *mod* n.

(iv) $B_{m,n}\subset R^{12}$ *and* $\subseteq R^{11}$ *fof a ny* m *and* n.

(v) $B_{m,n}\subset R^{11}$ *and* $\subseteq R^{10}$ *for* n *even.*

(vi) $B_{m,n}\subset R^9$ *and* $\subseteq R^8$ *for any* m

PROOF. of (iii). Let $\tilde{KO}(B_{m,n})$ be the group of stable classes of real vector bundles over $B_{m,n}$. Then the projection $p;B_{m,n}\to S^4$ induces the homomorphism

$$p^!\ ;\ \tilde{KO}(S^4)\to\tilde{KO}(B_{m,n})$$

and

$$\{\tau(B_{m,n})\}=p^!\{\xi_{m,n}\}$$

where $\{\eta\}$ denotes the stable class of η. It is known that the kernel of $p^!$ consists of all integlal multiples of the order of $H_3(B_{m,n}:Z)$ [9]. The stable class of $\xi_{m,n}$ is regarded as the image of $(m\alpha_4+n\beta_4)\in\pi_3(SO(4))$ under the homomorphism $(i_{r-1})_*\cdots(i_4)_*$; $\pi_3(SO(4))\to\pi_3(SO(r))\,(r\geq 5)$. Since

$$(i_{r-1})_* \cdots (i_4)_* (m\alpha_4 + n\beta_4) = (-2m+n)\beta_r$$

and the order of $H_3(B_{m,n})$ is n, $p^! \{\xi_{m,n}\}=0$ if and only if $n=2$ or $2m \equiv 0 \mod n$. $p^! \{\xi_{m,n}\}=0$ implies that $B_{m,n}$ is stably parallelizable and hence $B(\xi) \subseteq R^3$.

6.2.3. k-sphere bundle over $S^4 (k \leq 4)$

In this case we have

(i) $B_m \subset R^{8+k}$ *and* $\subseteq R^{7+k}$ *for* m *even*

(ii) $B_m \subset R^{9+k}$ *and* $\subseteq R^{8+k}$ *for* m *odd.*

The result of (ii) are best possible.

(i) is due to the fact $w_4(\xi_m)=0$. Since $w_4(B_m)=p^* \ w_4(\xi_m)$, p^* is an isomorphism and $w_4(\xi_m) \neq 0$, $w_4(B_m) \neq 0$, which implies $w_4(B_m) \neq 0$. Hence $B \not\subset R^{8+k}$ and $\not\subseteq R^{7+k}$. It is easy to see $B \not\subset R^{6+k}$ and $B \not\subseteq R^{5+k}$ for even $m \neq 0$.

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