On sequential estimators for jumps and reliability

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1. Introduction and Summary

Let $F(x)$ be a probability distribution function on the real line R. Assuming the singular part to be identically zero, it is well known that $F(x)$ is uniquely decomposed into $F(x)=F_1(x)+F_2(x)$ where $F_1(x)$ is an absolutely continuous function and $F_2(x)$ is a pure step function with steps of magnitude, say, S_i at the points $x=x_i$, $i=\pm 1, \pm 2$, and that finally both $F_1(x)$ and $F_2(x)$ are non-decreasing. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with the same distribution function $F(x)$. As in MURTHY [3], we call $R(x)=1-F(x)$ the reliability function. If x is any point of continuity of the distribution $F(x)$ and if the density at x is denoted by $f(x)$, $Z(x)=f(x)/(1-F(x))$ will be also refered to as the hazard rate.

We consider the problem of estimating the jump S_i corresponding to the saltus $x=x_i$ based on random samples $X_1, X_2, \cdots$. Also, considered are the problems of estimating of the reliability function $R(x)$ and the hazard rate $Z(x)$. This problem was considered by MURTHY [3]. He gave consistent classes of estimators in [3], while in this paper we shall give strong consistent classes of sequential estimators where that $\{Y_n\}$ is a strong consistent class of estimators of Y means that with probability one $Y_n \to Y$ as $n\to\infty$.

This paper consists of five sections. In section 2, auxiliary results will be given for proving results in section 3 and 4. In section 3 we shall give a strong consistent class of sequential estimators of the jump S_i. In section 4 strong consistent classes of sequential estimators of the reliability function will be given. Section 5 will give strong consistent classes of estimators for the hazard rate. In section 3 to 5, we assume that the singular part of the distribution $F(x)$ is identically zero.

2. Auxiliary Results

Lemma 2.1, 2.2 and 2.3 are due to WATANABE [4] and [5], while Lemma 2.4 is due to BRAVERMAN and PYATNITSKII [1].

LEMMA 2.1. ([4]). *Let $\{A_n\}$ be a sequence of non-negative numbers. Suppose that there exist three sequences of non-negative numbers $\{a_n\}$, $\{b_n\}$ and $\{L_n\}$, a positive constant*

L and a positive integer n_0 *such that*

$$(2.1)\qquad A_{n+1}\leqslant(1-a_{n+1})A_n+La_{n+1}b_{n+1}+L_{n+1}$$

for all $n\geqslant n_0$,

$$(2.2)\qquad \sum_{n=1}^{\infty}a_n=\infty \quad \text{and} \quad \lim_{n\to\infty}a_n=0,$$

$$(2.3)\qquad \lim_{n\to\infty}b_n=0,$$

$$(2.4)\qquad \sum_{n=1}^{\infty}L_n<\infty.$$

Then, it holds that $\lim_{n\to\infty}A_n=0$.

Further, if $L_n\equiv 0$ *for all* $n\geqslant 1$ *in* (2.1) *and* $\{b_n\}$ *is a sequence of positive numbers such that*

$$(2.5)\qquad (1-a_{n+1})b_n/b_{n+1}\leqslant 1-\alpha a_{n+1} \qquad \text{for all } n\geqslant n_0$$

where α *is some positive constant and in this case* $\{b_n\}$ *need not satisfy the condition* (2.3), *then there exists a constant* $C>0$ *such that*

$$(2.6)\qquad A_n\leqslant C\cdot b_n \qquad \text{for all } n\geqslant 1.$$

LEMMA 2.2. ([5]). *Let* $\{U_n\}$ *and* $\{V_n\}$ *be two sequences of random variables on a probability space* $(\Omega, \mathfrak{A}, P)$. *Let* $\{\mathfrak{A}_n\}$ *be a sequence of sub-σ-fields of* $\mathfrak{A}$, $\mathfrak{A}_n\subset\mathfrak{A}_{n+1}\subset\mathfrak{A}$, *where* U_n *and* V_n *are measurable with respect to* $\mathfrak{A}_n$ *for each* $n\geqslant 1$. *Furthermore, let* $\{a_n\}$ *be a sequence of non-negative numbers satisfying* (2.2). *Suppose that the following conditions are satisfied:*

$$(2.7)\qquad U_n\geqslant 0 \quad a.s. \ \text{for all } n\geqslant 1,$$

$$(2.8)\qquad E[U_1]<\infty,$$

$$(2.9)\qquad E[U_{n+1}|\mathfrak{A}_n]\leqslant(1-a_{n+1})U_n+V_n \qquad a.s.$$

for all $n\geqslant 1$,

$$(2.10)\qquad \sum_{n=1}^{\infty}E[|V_n|]<\infty.$$

Then, it holds that $\lim_{n\to\infty}U_n=0$ *a.s. and* $\lim_{n\to\infty}E[U_n]=0$.

LEMMA 2.3. ([5]). *Let* $\{U_n\}$ *be a sequence of random variables on a probability space* $(\Omega, \mathfrak{A}, P)$. *Let* $\{\mathfrak{A}_n\}$ *be a sequence of sub-σ-fields of* $\mathfrak{A}$, $\mathfrak{A}_n\subset\mathfrak{A}_{n+1}\subset\mathfrak{A}$, *where* U_n *is measurable with respect to* $\mathfrak{A}_n$ *for each* $n\geqslant 1$. *And let* $\{a_n\}$ *and* $\{v_n\}$ *be two sequences of positive numbers. Suppose that there exist a positive integer* n_0 *and two positive numbers* $0<M<\infty$ *and* $0<\lambda<1$ *satisfying*

(2.11) $U_n \geqslant 0$ *a.s. for all* $n \geqslant n_0$,

(2.12) $M \geqslant U_{n_0}$ *a.s.*,

(2.13) $E[U_{n+1} | \mathfrak{A}_n] \leqslant (1-a_{n+1}) U_n + v_{n+1}$ *a.s.*

for all $n \geqslant n_0$,

(2.14) $(1-a_{n+1})(v_n/v_{n+1})^\lambda \leqslant 1$ *for all* $n \geqslant n_0$,

(2.15) $\sum_{n=1}^{\infty} v_n^{1-\lambda} < \infty.$

Then for any $\delta > 0$ *there exists a positive constant* $C(\delta)$ *such that*

$$P\{U_n \leqslant C(\delta) \cdot v_n^\lambda \quad \text{for all } n \geqslant n_0\} > 1-\delta.$$

LEMMA 2.4. ([1]). *Let* $\{S_n\}$ *be a sequence of non-negative numbers satisfying*

$$S_{n+1} \leqslant (1-ra_{n+1}) S_n + La_{n+1}^2 \quad \text{for all } n \geqslant 1$$

where $r>0$, $L>0$ *and* $a_n>0$.

Suppose that there exist a positive number λ $(0<\lambda<2)$ *and a positive integer* n_0 *such that*

$$(1-ra_{n+1})(a_n/a_{n+1})^\lambda \leqslant 1 \quad \text{for all } n \geqslant n_0,$$

and

$$\sum_{n=1}^{\infty} a_n^{2-\lambda} < \infty.$$

Then, there exists a positive constant C such that

$$S_n \leqslant Ca_n^\lambda \quad \text{for all } n \geqslant 1.$$

3. Estimation of the jump S_i at the saltus x_i of the distribution $F(x)$

In this section we shall give a strong consistent class of sequential estimators of the jump S_i. Let $K(y)$ be a real-valued Borel measurable function on R and satisfying

(K 1) $K(y) \geqslant 0$ for all $y \in R$,

(K 2) $\int_{-\infty}^{\infty} K(y)\,dy = 1$,

(K 3) $\int_0^{\infty} K(y)\,dy \neq 0.$

Let $\{h_n\}$ be a sequence of real numbers satisfying

(H) $h_n > 0$ for all $n=1, 2, \cdots$ and $\lim_{n\to\infty} h_n = 0.$

Let $\{a_n\}$ be a sequence of real numbers satisfying

(A 1) $$1 \geqslant a_n > 0 \quad \text{for all } n=1, 2, \cdots,$$

(A 2) $$\sum_{n=1}^{\infty} a_n = \infty,$$

(A 3) $$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

We set

(3. 1) $$G(x) = \int_x^{\infty} K(y) dy,$$

where $K(y)$ satisfies (K1), (K2) and (K3).

Also $U(x)$ is defined as follows:

$$U(x) = 1 \quad \text{for } x > 0$$
$$= 0 \quad \text{for } x \leqslant 0.$$

Since $G(0) \neq 0$ from (K3), $D_n(x, y)$ can be defined as follows:

$$D_n(x, y) = [G(h_n^{-1}(x-y)) - U(y-x)]/G(0)$$

for all $n \geqslant 1$ and all $x, y \in R$.

Now, we shall give sequential estimators of the jump S_i as follows:

(A) $$H_0(x) \equiv 0 \quad \text{for all } x \in R$$
$$H_{n+1}(x) = H_n(x) + a_{n+1}\{D_{n+1}(x, X_{n+1}) - H_n(x)\}$$
for all $n \geqslant 0$ and all $x \in R$.

The following lemma presents an asympotic unbiasedness of S_i.

LEMMA 3. 1. *Let X be a random variable with the distribution $F(x)$. Suppose that $K(y)$ satisfies* (K1), (K2) *and* (K3) *and* $\{h_n\}$ *satisfies* (H). *Then if S_i is any jump corresponding to the saltus $x = x_i$ of the distribution $F(x)$, it holds that*

$$\lim_{n \to \infty} E[D_n(x_i, X)] = S_i.$$

PROOF. The proof is proceeded in the same way as in [3]. It is easily seen that

(3. 2) $$R(x_i) = F_1(\infty) - F_1(x_i) + \sum_{x_j > x_i} S_j,$$

(3. 3) $$E[U(X - x_i)] = R(x_i),$$

(3. 4) $$E[G(h_n^{-1}(x_i - X))]$$
$$= \int_{-\infty}^{\infty} G(h_n^{-1}(x_i - y)) dF_1(y) + \sum_j G(h_n^{-1}(x_i - x_j)) S_j$$
$$\equiv I_n^1 + I_n^2 .$$

Integrating by parts, we have

$$I_n^1 = F_1(\infty) - \int_{-\infty}^{\infty} K(y) F_1(x_i - h_n y)\, dy.$$

Since $F_1(y)$ is a continuous function, it follows, by the dominated convergence theorem and (K2), that

$$\lim_{n\to\infty} I_n^1 = F_1(\infty) - F_1(x_i). \tag{3. 5}$$

Now

$$I_n^2 = J_n^1 + J_n^2 + G(0) S_i$$

where

$$J_n^1 = \sum_{x_j < x_i} G(h_n^{-1}(x_i - x_j)) S_j$$

and

$$J_n^2 = \sum_{x_j > x_i} G(h_n^{-1}(x_i - x_j)) S_j .$$

In the same way as in [3] we have

$$\lim_{n\to\infty} J_n^1 = 0 \quad \text{and} \quad \lim_{n\to\infty} J_n^2 = \sum_{x_j > x_i} S_j .$$

Thus,

$$\lim_{n\to\infty} I_n^2 = \sum_{x_j > x_i} S_j + G(0) S_i . \tag{3. 6}$$

By the relations (3. 2) to (3. 6) we obtain

$$\lim_{n\to\infty} E[D_n(x_i, X)] = S_i ,$$

which completes the proof.

THEOREM 3. 1. *Suppose the conditions of Lemma* 3. 1. *Let* $\{a_n\}$ *satisfy* (A1), (A2) *and* (A3). *Let* $\{H_n(x)\}$ *be given by* (A). *Then for any jump* S_i *corresponding to the saltus* $x = x_i$ *of the distribution* $F(x)$,

$$\lim_{n\to\infty} H_n(x_i) = S_i \qquad a.\ s.$$

and

$$\lim_{n\to\infty} E[(H_n(x_i) - S_i)^2] = 0 .$$

PROOF. Since

$$|H_n(x_i) - S_i| \leqslant |H_n(x_i) - E[H_n(x_i)]| + |E[H_n(x_i)] - S_i| ,$$

in order to prove the first assertion, it suffices to show that

$$\lim_{n\to\infty} |H_n(x_i) - E[H_n(x_i)]| = 0 \qquad \text{a. s.} \tag{3. 7}$$

and

$$\lim_{n\to\infty}|E[H_n(x_i)]-S_i|=0. \tag{3.8}$$

Firstly, let us show (3. 8). From (A), we obtain

$$\begin{aligned}(3.9)\qquad &|E[H_{n+1}(x_i)]-S_i|\\ &\leqslant(1-a_{n+1})|E[H_n(x_i)]-S_i|\\ &\quad+a_{n+1}|E[D_{n+1}(x_i, X_{n+1})]-S_i|.\end{aligned}$$

Taking into account (3. 9), Lemma 3. 1, it follows that $\lim_{n\to\infty}|E[H_n(x_i)]-S_i|=0$, which is (3. 8).

Secondly, we shall show (3. 7). From (A), we get

$$\begin{aligned}(3.10)\qquad &(H_{n+1}(x_i)-E[H_{n+1}(x_i)])^2\\ &=(1-a_{n+1})^2(H_n(x_i)-E[H_n(x_i)])^2\\ &\quad+a_{n+1}^2(D_{n+1}(x_i, X_{n+1})-E[D_{n+1}(x_i, X_{n+1})])^2\\ &\quad+2(1-a_{n+1})a_{n+1}(H_n(x_i)-E[H_n(x_i)])\\ &\qquad\times(D_{n+1}(x_i, X_{n+1})-E[D_{n+1}(x_i, X_{n+1})]).\end{aligned}$$

It follows, from the independence of $\{X_n\}$, that

$$\begin{aligned}(3.11)\qquad &E[(H_n(x_i)-E[H_n(x_i)])(D_{n+1}(x_i, X_{n+1})\\ &\quad-E[D_{n+1}(x_i, X_{n+1})])|X_1, \cdots, X_n]=0 \qquad \text{a. s.}\end{aligned}$$

Taking conditional expectations on both sides of (3. 10) and using (A1) and (3. 11), we obtain

$$\begin{aligned}(3.12)\qquad &E[(H_{n+1}(x_i)-E[H_{n+1}(x_i)])^2|X_1, \cdots, X_n]\\ &\leqslant(1-a_{n+1})(H_n(x_i)-E[H_n(x_i)])^2\\ &\quad+a_{n+1}^2\mathrm{Var}[D_{n+1}(x_i, X_{n+1})].\end{aligned}$$

Since $E[D_{n+1}^2(x_i, X_{n+1})]\leqslant(G(0))^{-2}$, it follows from (3. 12) that

$$\begin{aligned}(3.13)\qquad &E[(H_{n+1}(x_i)-E[H_{n+1}(x_i)])^2|X_1, \cdots, X_n]\\ &\leqslant(1-a_{n+1})(H_n(x_i)-E[H_n(x_i)])^2+(G(0))^{-2}a_{n+1}^2.\end{aligned}$$

According to Lemma 2. 2, we have

$$\lim_{n\to\infty}(H_n(x_i)-E[H_n(x_i)])^2=0 \qquad \text{a. s.}$$

which is equivalent to (3. 7), and

$$\lim_{n\to\infty}E[(H_n(x_i)-E[H_n(x_i)])^2]=0. \tag{3.14}$$

Since

$$E[(H_n(x_i)-S_i)^2]$$
$$=E[(H_n(x_i)-E[H_n(x_i)])^2]+(E[H_n(x_i)]-S_i)^2,$$

the relations (3. 8) and (3. 14) yield

$$\lim_{n\to\infty} E[(H_n(x_i)-S_i)^2]=0$$

which is the second assertion.

Thus the proof is completed.

We shall consider the rate of the variance of $H_n(x_i)$.

THEOREM 3. 2. *Let* $\{a_n\}$ *and* $\{h_n\}$ *satisfy* (A1), (A2), (A3) *and* (H) *and satisfy the following:*

(3. 15) $$(1-a_{n+1})(a_n/a_{n+1})^{2\lambda}\leqslant 1 \qquad \text{for some } 0<\lambda<1 \text{ and all } n\geqslant \text{some } n_0,$$

(3. 16) $$\sum_{n=1}^{\infty} a_n^{2(1-\lambda)}<\infty .$$

Then, under the conditions of Theorem 3. 1, *there exists a positive constant* $L(x_i)$ *depending on* x_i *such that*

$$\text{Var}\ [H_n(x_i)]\leqslant L(x_i)a_n^{2\lambda} \qquad \text{for all } n\geqslant 1.$$

PROOF. Taking expectations on both sides of (3. 13), we get

(3. 17) $$\text{Var}\ [H_{n+1}(x_i)]$$
$$\leqslant (1-a_{n+1})\text{Var}\ [H_n(x_i)]+(G(0))^{-2}\, a_{n+1}^2 .$$

In view of (3. 15), (3. 16), (3. 17) and Lemma 2. 4, the assertion is established. Thus the theorem is proved.

4. Asymptotic equivalence of two classes of estimators for the reliability at a point of continuity of $F(x)$

We define two classes of sequential estimators for the reliability function. The first algorithm is defined as follows:

(R) $$R_0(x)\equiv 1 \qquad \text{for all } x\in R$$
$$R_{n+1}(x)=R_n(x)+a_{n+1}\{U(X_{n+1}-x)-R_n(x)\}$$

for all $n\geqslant 0$ and all $x\in R$, where $\{a_n\}$ satisfies (A1), (A2) and (A3).

The second algorithm is defined as follows:

(R*) $$R_0^*(x)\equiv 1 \qquad \text{for all } x\in R$$
$$R_{n+1}^*(x)=R_n^*(x)+a_{n+1}\{G(h_{n+1}^{-1}(x-X_{n+1}))-R_n^*(x)\}$$

for all $n \geqslant 0$ and all $x \in R$, where $G(y)$ is given by (3. 1) and $\{a_n\}$ and $\{h_n\}$ satisfy (A1), (A2), (A3) and (H), respectively.

It is easily seen that $0 \leqslant R_n(x) \leqslant 1$ and $0 \leqslant R_n^*(x) \leqslant 1$ for all $x \in R$ and all $n \geqslant 1$ because of (K1) and (K2).

LEMMA 4. 1. *We consider the class of sequential estimators $\{R_n(x)\}$ given by (R). Then for any point x, it holds that*

$$\lim_{n\to\infty} R_n(x) = R(x) \quad a.s. \tag{4. 1}$$

and

$$\lim_{n\to\infty} E[(R_n(x)-R(x))^2] = 0. \tag{4. 2}$$

PROOF. From (R) we have

$$E[R_{n+1}(x)] - R(x) = (1-a_{n+1})(E[R_n(x)] - R(x)).$$

Repeating this relation, we obtain

$$E[R_{n+1}(x)] - R(x) = \prod_{k=1}^{n+1} (1-a_k)(1-R(x)).$$

Thus,

$$\begin{aligned} |E[R_{n+1}(x)] - R(x)| &= \prod_{k=1}^{n+1} (1-a_k)(1-R(x)) \\ &\leqslant (1-R(x)) \exp\{-\sum_{k=1}^{n+1} a_k\}. \end{aligned} \tag{4. 3}$$

From (4. 3) and (A2) we get

$$\lim_{n\to\infty} |E[R_n(x)] - R(x)| = 0. \tag{4. 4}$$

From (R) we get

$$\begin{aligned} &(R_{n+1}(x) - E[R_{n+1}(x)])^2 \\ &= (1-a_{n+1})^2 (R_n(x) - E[R_n(x)])^2 \\ &\quad + a_{n+1}^2 \{U(X_{n+1}-x) - E[U(X_{n+1}-x)]\}^2 \\ &\quad + 2(1-a_{n+1}) a_n (R_n(x) - E[R_n(x)]) \\ &\quad \times \{U(X_{n+1}-x) - E[U(X_{n+1}-x)]\}. \end{aligned} \tag{4. 5}$$

Taking conditional expectations on both sides of (4. 5) and using (A1), the indpendence of $\{X_n\}$ and Var $[U(X_{n+1}-x)] \leqslant 1$, we have

$$\begin{aligned} &E[(R_{n+1}(x) - E[R_{n+1}(x)])^2 | X_1, \cdots, X_n] \\ &\leqslant (1-a_{n+1})(R_n(x) - E[R_n(x)])^2 + a_{n+1}^2 \quad \text{a. s. for all } n. \end{aligned} \tag{4. 6}$$

By (4. 6) and Lemma 2. 2, we obtain

$$\lim_{n\to\infty} (R_n(x)-E[R_n(x)])^2=0 \qquad \text{a. s.}$$

which is equivalent to

$$\text{(4. 7)} \qquad \lim_{n\to\infty} |R_n(x)-E[R_n(x)]|=0 \qquad \text{a. s.,}$$

and

$$\text{(4. 8)} \qquad \lim_{n\to\infty} E[(R_n(x)-E[R_n(x)])^2]=0.$$

The triangle inequality, (4. 4) and (4. 7) yield (4. 1). Since

$$E[(R_n(x)-R(x))^2]$$
$$=E[(R_n(x)-E[R_n(x)])^2]+(E[R_n(x)]-R(x))^2,$$

making use of (4. 4) and (4. 8), we get (4. 2).

Thus the proof is completed.

LEMMA 4. 2. *We consider the class of sequential estimators* $\{R_n(x)\}$ *given by* (R). *Let* $\{a_n\}$ *satisfy also the following conditions in addition to* (A1), (A2) *and* (A3):

$$\text{(4. 9)} \qquad (1-a_{n+1})(a_n/a_{n+1})^{2\lambda}\leqslant 1 \qquad \textit{for some } 0<\lambda<1$$

and all $n\geqslant$ *some* n_0,

$$\text{(4. 10)} \qquad \sum_{n=1}^{\infty} a_n^{2(1-\lambda)}<\infty.$$

Then for any positive number δ *and any* $x\in R$, *there exist a positive constant* $M(\delta, x)$ *depending on* δ *and* x *such that*

$$P\{|R_n(x)-R(x)|\leqslant M(\delta, x)d_n \qquad \textit{for all } n\geqslant n_0\}>1-\delta$$

where $d_n=\max\{a_n{}^\lambda, \exp(-\sum_{k=1}^{n} a_k)\}$.

Further, there exists a positive constant $L(x)$ *such that*

$$\text{Var}\,[R_n(x)]\leqq L(x)a_n^{2\lambda} \qquad \textit{for all } n\geqslant 1.$$

PROOF. From (4. 6), (4. 9), (4. 10) and Lemma 2. 3, for any $\delta>0$ and any x, there exists a positive constant $M_1(\delta, x)$ such that

$$\text{(4. 11)} \qquad P\{(R_n(x)-E[R_n(x)])^2\leqslant M_1(\delta, x)a_n^{2\lambda} \qquad \textit{for all } n\geqslant n_0\}$$
$$>1-\delta.$$

If $|R_n(x)-E[R_n(x)]|\leqslant (M_1(\delta, x))^{\frac{1}{2}} a_n{}^\lambda$, (4. 3) implies

$$\text{(4. 12)} \qquad |R_n(x)-R(x)|\leqslant M(\delta, x)d_n \qquad \text{for all } n\geqslant n_0$$

where $M(\delta, x)=(M_1(\delta, x))^{\frac{1}{2}}+1-R(x)$.

From (4. 11) and (4. 12), we have

$$P\{|R_n(x)-R(x)|\leqslant M(\delta, x)d_n \quad \textit{for all } n\geqslant n_0\}$$
$$\geqslant P\{(R_n(x)-E[R_n(x)])^2\leqslant M_1(\delta, x)a_n^{2\lambda} \quad \textit{for all } n\geqslant n_0\}$$
$$>1-\delta.$$

Thus the first assertion is proved.
Taking expectations on both sides of (4. 6) and using (4. 9), (4. 10) and Lemma 2. 4, we have the second assertion.
Therefore the proof is completed.

Now, we shall consider the class of sequential estimators $\{R_n^*(x)\}$ given by (R*) in Lemma 4. 3 and Lemma 4. 4.

LEMMA 4. 3. *Let x be an arbitrary point of continuity of $F(x)$. Then it holds that*

$$\lim_{n\to\infty} R_n^*(x)=R(x) \quad a.\ s. \tag{4.13}$$

and

$$\lim_{n\to\infty} E[(R_n^*(x)-R(x))^2]=0. \tag{4.14}$$

PROOF. From (R*) we have

$$\begin{aligned}&|E[R_{n+1}^*(x)]-R(x)|\\&\leqslant(1-a_{n+1})|E[R_n^*(x)]-R(x)|\\&\quad+a_{n+1}|E[G(h_{n+1}^{-1}(x-X_{n+1}))-R(x)|.\end{aligned} \tag{4.15}$$

Integrating by parts, we get

$$\begin{aligned}&|E[G(h_{n+1}^{-1}(x-X_{n+1}))]-R(x)|\\&=\left|\int_{-\infty}^{\infty}K(y)F(x-h_{n+1}y)dy-F(x)\right|.\end{aligned} \tag{4.16}$$

By (K1), (K2) and the dominated convergence theorem, we have

$$\lim_{n\to\infty}|E[G(h_{n+1}^{-1}(x-X_{n+1}))]-R(x)|=0. \tag{4.17}$$

Combining (4. 15), (4. 17) and Lemma 2. 1, we obtain

$$\lim_{n\to\infty}|E[R_n^*(x)]-R(x)|=0. \tag{4.18}$$

Since $G^2(h_{n+1}^{-1}(x-y))\leqslant 1$ for all n and all y, it follows

$$\mathrm{Var}\,[G(h_{n+1}^{-1}(x-X_{n+1}))]\leqslant 1 \qquad \text{for all } n\geqslant 0. \tag{4.19}$$

Thus (R*), (4. 19) and the independence of $\{X_n\}$ imply

$$E[(R_{n+1}^*(x)-E[R_{n+1}^*(x)])^2|X_1, \cdots, X_n] \tag{4.20}$$

$$\leqslant (1-a_{n+1})(R_n^*(x)-E[R_n^*(x)])^2+a_{n+1}^2 \qquad \text{a. s.} \quad \text{for all } n\geqslant 0.$$

From (4. 20) and Lemma 2. 2, we obtain

$$\lim_{n\to\infty} (R_n^*(x)-E[R_n^*(x)])^2=0 \qquad \text{a. s.}$$

which is equivalent to

$$\text{(4. 21)} \qquad \lim_{n\to\infty} |R_n^*(x)-E[R_n^*(x)]|=0 \qquad \text{a. s,}$$

and

$$\text{(4. 22)} \qquad \lim_{n\to\infty} E[(R_n^*(x)-E[R_n^*(x)])^2]=0.$$

The relations (4. 18) and (4. 21) yield (4. 13). Also, the relations (4. 18) and (4. 22) yield (4. 14).

Therefore the lemma is completed.

LEMMA 4. 4. *Let* $K(y)$ *satisfy the following condition in addition to* (K1) *and* (K2):

$$\text{(K 4)} \qquad \int_{-\infty}^{\infty} |y|K(y)dy<\infty.$$

Let $\{a_n\}$ *and* $\{h_n\}$ *satisfy the following conditions in addition to* (A1), (A2), (A3) *and* (H):

$$\text{(4. 23)} \qquad (1-a_{n+1})h_n/h_{n+1}\leqslant 1-\alpha a_{n+1}$$

for some $\alpha>0$ *and all* $n\geqslant$ *some* n_1,

$$\text{(4. 24)} \qquad (1-a_{n+1})(a_n/a_{n+1})^{2\lambda}\leqslant 1$$

for all $n\geqslant n_1$ *and some* $0<\lambda<1$,

$$\text{(4. 25)} \qquad \sum_{n=1}^{\infty} a_n^{2(1-\lambda)}<\infty .$$

Let x *be an arbitrary point of continuity of the distribution* $F(x)$. *Suppose that there exist positive constants* $\eta(x)$, $C(x)$ *depending on* x *such that*

$$\text{(4. 26)} \qquad |F(x+y)-F(x)|\leqslant C(x)|y| \qquad \textit{for all } |y|\leqslant\eta(x).$$

Then for any $\delta>0$ *there exists a positive constant* $M(\delta, x)$ *depending on* δ *and* x *such that*

$$P\{|R_n^*(x)-R(x)|\leqslant M(\delta, x)d_n \qquad \textit{for all } n\geqslant n_1\}>1-\delta,$$

where $d_n=\max\{a_n^\lambda, h_n\}$.
Furthermore, there exists a positive constant $L(x)$ *such that*

$$\text{Var}\,[R_n^*(x)]\leqslant L(x)a_n^{2\lambda} \qquad \textit{for all } n\geqslant 1.$$

PROOF. Taking into account (4. 20), (4. 24), (4. 25) and Lemma 2. 3, for any $\delta>0$ there exists a positive constant $M_1(\delta, x)$ such that

$$P\{(R_n^*(x)-E[R_n^*(x)])^2 \leqslant M_1(\delta, x)a_n^{2\lambda} \quad \textit{for all } n \geqslant n_1\} > 1-\delta$$

which is equivalent to

$$(4.27) \qquad P\{|R_n^*(x)-E[R_n^*(x)]| \leqslant (M_1(\delta, x))^{\frac{1}{2}} a_n^{\lambda} \quad \textit{for all } n \geqslant n_1\} > 1-\delta.$$

From (4. 16) and (K2) we have

$$(4.28) \qquad |E[G(h_{n+1}^{-1}(x-X_{n+1}))]-R(x)|$$

$$\leqslant \int_{|y| \leqslant \eta(x)h_{n+1}^{-1}} K(y)|F(x-h_{n+1}y)-F(x)|dy$$

$$+\int_{|y|>\eta(x)h_{n+1}^{-1}} K(y)|F(x-h_{n+1}y)-F(x)|dy$$

$$\leqslant C(x)h_{n+1}\int_{-\infty}^{\infty}|y|K(y)dy+2\int_{|y|>\eta(x)h_{n+1}^{-1}} K(y)dy.$$

The condition (K4) implies

$$(4.29) \qquad \int_{|y|>x}|y|K(y)dy \leqslant C_1 \quad \text{for all } x>0$$

where $C_1=\int_{-\infty}^{\infty}|y|K(y)dy$.

Since

$$x\int_{|y|>x}K(y)dy \leqslant \int_{|y|>x}|y|K(y)dy \quad \text{for all } x>0,$$

from (4. 29) we get

$$\int_{|y|>x}K(y)dy \leqslant C_1x^{-1} \quad \text{for all } x>0.$$

Thus we have

$$(4.30) \qquad \int_{|y|>\eta(x)h_{n+1}^{-1}} K(y)dy \leqslant C_1 h_{n+1}\eta(x)^{-1} \quad \text{for all } n.$$

Combining (4. 28) and (4. 30), we obtain

$$(4.31) \qquad |E[G(h_{n+1}^{-1}(x-X_{n+1}))]-R(x)| \leqslant C_2(x)h_{n+1}$$

where $C_2(x)=\{C(x)+2(\eta(x))^{-1}\}\int_{-\infty}^{\infty}|y|K(y)dy$.

The relations (4. 15) and (4. 31) imply

$$(4.32) \qquad |E[R_{n+1}^*(x)]-R(x)|$$

$$\leqslant (1-a_{n+1})|E[R_n^*(x)]-R(x)|+C_2(x)a_{n+1}h_{n+1}.$$

By (4. 23), (4. 32) and Lemma 2. 1, there exists a positive constant $C_3(x)$ such that

$$(4.33) \qquad |E[R_n^*(x)]-R(x)| \leqslant C_3(x)h_n \qquad \text{for all } n.$$

In the same way as proof of Lemma 4. 2, it follows, from (4. 27) and (4. 33), that

$$P\{|R_n^*(x)-R(x)| \leqslant M(\delta, x)d_n \qquad \textit{for all } n \geqslant n_1\} > 1-\delta$$

where $M(\delta, x)=(M_1(\delta, x))^{\frac{1}{2}}+C_3(x)$.
Thus the first statement is established.

Taking expectations on both sides of (4. 20) and using (4. 24), (4. 25) and Lemma 2. 4, we obtain

$$\text{Var}\,[R_n^*(x)] \leqslant L(x)a_n^{2\lambda} \qquad \text{for all } n \geqslant 1$$

which shows the second statement.

Therefore, the lemma is proved.

Definition 4. 1. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \alpha_n=0$. Also, let $\{X_n\}$ be a sequence of random variables and K be a constant. If for any $\delta>0$ there exists a positive constant $C(\delta)$ depending on δ such that

$$P\{|X_n-K| \leqslant C(\delta)\alpha_n \qquad \textit{for all } n \geqslant 1\} > 1-\delta,$$

then we call $\{X_n\}$ an asymptotic optimal sequence of type I with order $\{\alpha_n\}$ at K(AO–I$\{\alpha_n\}$).

Definition 4. 2. Let $\{\alpha_n\}$, $\{X_n\}$ and K be the same as in Definition 4. 1. If $\lim_{n\to\infty}|X_n-K|=0$ a. s. and there exists a positive constant C such that Var $[X_n] \leqslant C\alpha_n$ for all $n \geqslant 1$, then we call $\{X_n\}$ an asymptotic optimal sequence of type II with order $\{\alpha_n\}$ at K(AO–II$\{\alpha_n\}$).

From the previous lemmas, we obtain the following theorem.

Theorem 4. 1. *Let* $\{R_n(x)\}$ *and* $\{R_n^*(x)\}$ *be defined by* (R) *and* (R*) *respectively. Let* $\{a_n\}$ *and* $\{h_n\}$ *satisfy* (A1), (A2), (A3) *and* (H), *and in addition satisfy the following conditions:*

$$(4.34) \qquad (1-a_{n+1})(a_n/a_{n+1})^{2\lambda} \leqslant 1 \qquad \textit{for some } 0<\lambda<1 \textit{ and all } n \geqslant \textit{some } n_0,$$

$$(4.35) \qquad \sum_{n=1}^{\infty} a_n^{2(1-\lambda)} < \infty,$$

$$(4.36) \qquad (1-a_{n+1})h_n/h_{n+1} \leqslant 1-\alpha a_{n+1} \qquad \textit{for some } \alpha>0 \textit{ and all } n \geqslant n_0.$$

Suppose that $K(y)$ *is a real-valued* Borel *measurable function and satisfies* (K1), (K2) *and* (K4). *Let* x *be an arbitrary point of continuity of the distribution* $F(x)$ *with* (4. 26). *We put* $\alpha_n=\max\{a_n^{\lambda}, h_n\}$ and $d_n=\max\{a_n^{\lambda}, \exp(-\sum_{k=1}^{n} a_k)\}$ *for all* $n \geqslant 1$. *Then,* $\{R_n(x)\}$ *and* $\{R_n^*(x)\}$ *are* AO–I$\{d_n\}$ *and* AO–I$\{\alpha_n\}$ *at* $R(x)$, *respectively. Furthermore,* $\{R_n(x)\}$ *and* $\{R_n^*(x)\}$ *are equivalent in the sense of* AO–II $\{a_n^{2\lambda}\}$ *at* $R(x)$.

EXAMPLES OF $\{a_n\}$ AND $\{h_n\}$.

The following examples satisfy the conditions (A1), (A2), (A3), (H), (4. 34), (4. 35) and (4. 36):

$$a_n=n^{-t} \quad 2^{-1}<t\leq 1,\ n=1, 2, \cdots \tag{4.37}$$

$$h_n=n^{-s} \quad 0<s<t,\ n=1, 2, \cdots. \tag{4.38}$$

We shall give the asymptotic normality of $\{R_n^*(x)\}$.

THEOREM 4. 2. *Let* $\{R_n^*(x)\}$ *be defined by* (R*). *Assume the conditions of Lemma* 4. 4. *Let* $\{a_n\}$ *and* $\{h_n\}$ *with* $t=1$ *and* $2^{-1}<s<1$ *be given by* (4. 37) *and* (4. 38), *respectively. Then* $n^{\frac{1}{2}}(R_n^*(x)-R(x))$ *converges in law to the normal distribution with mean zero and variance* $\sigma^2(x)$ *where* $\sigma^2(x)=1-2F(x)\int_{-\infty}^{\infty}G(y)K(y)dy-R^2(x)$.

PROOF. From (R*) we get

$$(n+1)^{\frac{1}{2}}(R_{n+1}^*-R) = (n+1)^{-\frac{1}{2}}\sum_{m=1}^{n+1}v_m+(n+1)^{-\frac{1}{2}}\sum_{m=1}^{n+1}W_m \tag{4.39}$$

where $R_{n+1}^*\equiv R_{n+1}^*(x)$, $R\equiv R(x)$,

$$v_m=G(h_m^{-1}(x-X_m))-E[G(h_m^{-1}(x-X_m))]$$

and

$$W_m=E[G(h_m^{-1}(x-X_m))]-R.$$

The relation (4. 31) implies

$$n^{-\frac{1}{2}}\sum_{m=1}^{n}|W_m|\leq C_2(x)n^{-\frac{1}{2}}\sum_{m=1}^{n}m^{-s}$$

$$\sim C_2(x)(1-s)^{-1}n^{\frac{1}{2}-s}\longrightarrow 0 \quad \text{as } n\to\infty,$$

so that

$$n^{-\frac{1}{2}}\sum_{m=1}^{n}W_m=o(1) \quad \text{as } n\to\infty. \tag{4.40}$$

We note that v_m, $m\geq 1$, are mutually independent and $E[v_m]=0$ for all m.

Putting $S_n=\sum_{m=1}^{n}v_m$ and $s_n^2=E[S_n^2]$, we get $s_n^2=\sum_{m=1}^{n}E[v_m^2]$. Making use of (4. 17) and the dominated convergence theorem after integration by parts, we obtain

$$\lim_{n\to\infty}E[v_n^2]=\sigma^2(x). \tag{4.41}$$

Since $n^{-1}\sum_{m=1}^{n}E[v_m^2]\longrightarrow\sigma^2(x)$ as $n\to\infty$ from (4. 41), we have

(4. 42) $\qquad s_n^2 \sim \sigma^2(x)n \quad \text{as } n \to \infty \quad \text{where } \sigma^2(x) \neq 0.$

In the case of $\sigma^2(x)=0$, it holds that

(4. 43) $\qquad n^{-\frac{1}{2}} S_n \longrightarrow 0 \quad \text{in prob. as } n \to \infty,$

so the relations (4. 39), (4. 40) and (4. 43) yield the conclusion of the theorem. We assume $\sigma^2(x) \neq 0$. Let us verify the Lindeberg condition, i. e.

$$s_n^{-2} \sum_{m=1}^{n} E[v_m^2 I(|v_m| \geqslant \varepsilon s_n)] \longrightarrow 0 \qquad \text{as } n \to \infty$$

for all $\varepsilon > 0$ where $I(A)$ denotes the indicator function of a set A.
If

(4. 44) $\qquad \sum_{n=1}^{\infty} s_n^{-2} E[v_n^2 I(|v_n| \geqslant \varepsilon s_n)] < \infty,$

then by Kronecker's lemma

(4. 45) $\qquad s_n^{-2} \sum_{m=1}^{n} E[v_m^2 I(|v_m| \geqslant \varepsilon s_m)] \longrightarrow 0 \qquad \text{as } n \to \infty,$

so

$$s_n^{-2} \sum_{m=1}^{n} E[v_m^2 I(|v_m| \geqslant \varepsilon s_n)] \longrightarrow 0 \qquad \text{as } n \to \infty$$

because of $I(|v_m| \geqslant \varepsilon s_n) \leqslant I(|v_m| \geqslant \varepsilon s_m)$ for all $m \leqslant n$.
Thus (4. 44) yields the Lindeberg condition.
Now, we shall show that (4. 44) is verified.
Since from (4. 42)

$$s_n^{-2} E[v_n^2 I(|v_n| \geqslant \varepsilon s_n)]$$

$$\sim \sigma^{-2}(x) n^{-1} E[v_n^2 I(|v_n| \geqslant \varepsilon' n^{\frac{1}{2}})] \qquad \text{as } n \to \infty$$

where $\varepsilon' = \varepsilon\sigma(x)$, it suffices to show that

$$\sum_{n=1}^{\infty} n^{-1} E[v_n^2 I(|v_n| \geqslant \varepsilon n^{\frac{1}{2}})] < \infty \qquad \text{for all } \varepsilon > 0.$$

According to $E[|v_n|^3] \leqslant 1$ for all n, it follows that

$$\sum_{n=1}^{\infty} n^{-1} E[v_n^2 I(|v_n| \geqslant \varepsilon n^{\frac{1}{2}})]$$

$$\leqslant \varepsilon^{-1} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty.$$

Thus, it holds that

(4. 46) $\qquad S_n/s_n \longrightarrow N(0, 1) \quad$ in law as $n\to\infty$.

The relations (4. 39), (4. 40), (4. 42) and (4. 46) yield the conclusion of the theorem. Therefore the proof is completed.

5. Estimation of hazard rate

In this section, we shall present strong consistent classes of the hazard rate $Z(x)=f(x)/[1-F(x)]$. The following estimator was given by the author [2]:

(5. 1)
$$f_0(x)\equiv K(x) \qquad \text{for all } x\in R$$
$$f_{n+1}(x)=f_n(x)+a_{n+1}\{K_{n+1}(x, X_{n+1})-f_n(x)\}$$

for all $n\geqslant 0$ and all $x\in R$, where

$$K_n(x, y)=h_n^{-1}K(h_n^{-1}(x-y)) \qquad \text{for } n=1, 2, \cdots,$$

$K(y)$ satisfies (K1), (K2),

(K 5)
$$\sup_{-\infty<x<\infty} K(y)<\infty$$

and

(K 6)
$$\lim_{|y|\to\infty} |y|K(y)=0.$$

$\{a_n\}$ and $\{h_n\}$ satisfy (A1), (A2), (A3) and (H).

Let us now propose $Z_n(x)$ as an estimate of the hazard rate $Z(x)$ where

(5. 2)
$$Z_n(x)=f_n(x)/R_n(x) \qquad \text{if } R_n(x)\neq 0$$
$$=Z_{n-1}(x) \qquad \text{if } R_n(x)=0$$

for $n=1, 2, \cdots$, $f_n(x)$ and $R_n(x)$ being respectively given by (5. 1) and (R).

THEOREM 5. 1. *Let x be an arbitrary point of continuity of $F(x)$ and also $f(x)$ and satisfy the following condition:*

$$\sum_i S_i/|x_i-x|<\infty.$$

If $\sum_{n=1}^{\infty} a_n^2h_n^{-1}<\infty$, *then*

$$\lim_{n\to\infty} Z_n(x)=Z(x) \qquad \textit{with probability one.}$$

PROOF. It was earlier shown by the author (ISOGAI [2]) that $f_n(x)$ is a strong consistent estimator of $f(x)$ at every point of continuity of $F(x)$ and $f(x)$, i.e.

(5. 3) $$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{w. p. 1.}$$

It follows from (4. 1) that

(5. 4) $$\lim_{n\to\infty} R_n(x) = R(x) \qquad \text{w. p. 1.}$$

Combining (5. 3) and (5. 4), we at once have

$$\lim_{n\to\infty} Z_n(x) = Z(x) \qquad \text{w. p. 1,}$$

which concludes the theorem.

The second class of estimators is given by the following:

(5. 5) $$Z_n{}^*(x) = f_n(x)/R_n{}^*(x) \qquad \text{if } R_n{}^*(x) \neq 0$$
$$= Z_{n-1}^*(x) \qquad \text{if } R_n{}^*(x) = 0$$

where $R_n{}^*(x)$ is defined by (R^*) and f_n is the same as (5. 1) with $K(y)$ satisfying (K3) in addition.

The following theorem is observed immediately in the same way as Theorem 5. 1.

THEOREM 5. 2. *Under the conditions of Theorem* 5. 1,

$$\lim_{n\to\infty} Z_n{}^*(x) = Z(x) \qquad \textit{with probability one.}$$

References

[1] BRAVERMAN, E.M. and PYATNITSKII, E.S. (1966). *Estimation of the rate of convergence of algorithms based on the potential functions method.* Automation and Remote control. 27, No. 1, 80–100.

[2] ISOGAI, E. (1978). *On strong consistency of a sequential estimator of probability density.* Sci. Rep. Niigata Univ., Ser. A, 15, 25–33.

[3] MURTHY, V. K. (1965). *Estimation of jumps, reliability and hazard rate.* Ann. Math. Statist. 36, 1032–1040.

[4] WATANABE, M. (1973). *On asymptotically optimal algorithms for pattern classification problems.* Bull. Math. Stat. 15, 3–4, 31–48.

[5] WATANABE, M. (1974). *On convergence of asymtotically optimal discriminant functions for pattern classification problems.* Bull. Math. Stat. 16, 1–2, 23–34.