The second dual of a tensor product of C*-algebras III

By

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1. Introduction

Let D be a C*-algebra, and let D^* denote its dual and D^{**} its second dual. Let π_D be the universal representation of D on the Hilbert space H_D, then D^{**} can be identified with the weak closure of $\pi_D(D)$.

Let A and B be C^*-algebras, and let $A\otimes_\alpha B$ denote the C*-tensor product of A and B and $A^{**}\otimes B^{**}$ the W^*-tensor product of A^{**} and B^{**}. If $\pi_A\otimes\pi_B$ has a normal extension to $(A\otimes_\alpha B)^{**}$ which is a *-isomorphism onto $A^{**}\otimes B^{**}$, we shall say that $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes B^{**}$. It is known that $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes B^{**}$ if and only if $(A\otimes_\alpha B)^*=A^*\otimes_{\alpha'} B^*$, where $A^*\otimes_{\alpha'} B^*$ denotes the uniform closure of the algebraic tensor product of A^* and B^* in $(A\otimes_\alpha B)^*$ ([2], [4]).

We are interested in C*-algebras A having the property:

(*) $(A\otimes_\alpha B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes B^{**}$ for an arbitrary C*-algebra B.

We shall present a characterization of commutative C*-algebras having the property (*).

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2. Theorem

We first consider a commutative C*-algebra A such that $(A\otimes_\alpha A)^{**}$ is not canonically *-isomorphic to $A^{**}\otimes A^{**}$.

Let X be a locally compact Hausdorff space, and let $C_0(X)$ be the C*-algebra of all complex-valued continuous functions on X, which vanish at infinity. Let $M(X)$ be the set of all complex regular Borel measures on X and $M(X)^+$ the set of all positive measures of $M(X)$. From the Riesz-Markov representation theorem we can identify $M(X)$ with $C_0(X)^*$.

Throughout this paper, χ_E denotes the characteristic function of a set E, also δ_t

denotes the evaluation functional for a point t.

LEMMA. *Suppose that there exists a non-atomic measure μ in $M(X)$. Then $(C_0(X)\underset{\alpha}{\otimes} C_0(X))^{**}$ is not canonically *-isomorphic to $C_0(X)^{**}\otimes C_0(X)^{**}$.*

PROOF. $C_0(X)\underset{\alpha}{\otimes} C_0(X)$ can be identified with $C_0(X\times X)$. Hence we define the positive linear functional on $C_0(X\times X)$ as follows:

$$u(a)=\int_X a(t\times t)d\mu(t).$$

Suppose that u is an element of $C_0(X)^*\underset{\alpha'}{\otimes} C_0(X)^*$, then there exists a sequence (u_n) with the uniform limit u of the form

$$u_n=\sum_{i=1}^{m_n}\overset{(n)}{f_i}\otimes\overset{(n)}{g_i},\qquad \overset{(n)}{f_i},\ \overset{(n)}{g_i}\in C_0(X)^*.$$

Let Δ be the diagonal set $(t\times t)_{t\in X}$. For each n we define the functional on $C_0(X\times X)$ by the following

$$v_n(a)=u_n(\chi_\Delta a).$$

Then we have

$$\begin{aligned}|v_n(a)-u(a)|&=|u_n(\chi_\Delta a)-u(\chi_\Delta a)|\\&\leqq\|u_n(\chi_\Delta(\,,\,))-u(\chi_\Delta(\,,\,))\|\,\|a\|\\&\leqq\|u_n-u\|\,\|a\|.\end{aligned}$$

Hence the sequence (v_n) converges to u uniformly on $C_0(X\times X)$.

Now, from the Fubini theorem we have

$$\begin{aligned}v_n(a)&=\sum_{i=1}^{m_n}\overset{(n)}{f_i}\otimes\overset{(n)}{g_i}(\chi_\Delta a)\\&=\sum_{i=1}^{m_n}\overset{(n)}{f_i}(\overset{(n)}{g_i}(\chi_\Delta(s\times t)a(s\times t))\\&=\sum_{i=1}^{m_n}\overset{(n)}{f_i}(a(s\times s)\overset{(n)}{g_i}(\{s\})\end{aligned}$$

for all a in $C_0(X\times X)$. Since for each $\overset{(n)}{g_i}$ the set $(t\in X:\ g_i(\{t\})\neq 0)$ is at most countable, v_n are of the form

$$v_n=\sum_{i=1}^{\infty}\overset{(n)}{\alpha_i}\,\delta_{\overset{(n)}{t_i}\times\overset{(n)}{t_i}}.$$

Moreover the set $(\overset{(n)}{t_i}\times\overset{(n)}{t_i})$ is at most countable, v_n can be written as follows:

$$v_n=\sum_{i=1}^{\infty}\overset{(n)}{\alpha_i}\,\delta_{t_i\times t_i}\,.$$

On the other hand, we have $\|v_n-v_m\|=\sum_{i=1}^{\infty}|\overset{(n)}{\alpha_i}-\overset{(m)}{\alpha_i}|$, and there exists the functional v such that

$$v=\sum_{i=1}^{\infty}\alpha_i\,\delta_{t_i\times t_i}$$

and

$$\lim\|v_n-v\|=0\,.$$

It follows that $u=v$.

Since v is positive, we obtain $\alpha_i\geqq 0$. Hence there exists $\alpha_i>0$.

Now there exists a neighborhood of t_i such that $\mu(U(t_i))<\alpha_i$, and there exists a function a in $C_0(X\times X)$ such that its support lies in $U(t_i)\times U(t_i)$, $a(t_i\times t_i)=1$, and $0\leqq a\leqq 1$.

Then we have

$$v(a)\geqq\alpha_i>u(a).$$

This is a contradiction. Therefore $(C_0(X)\underset{\alpha}{\otimes}C_0(X))^{**}$ is not canonically *-isomorphic to $C_0(X)^{**}\otimes C_0(X)^{**}$.

THEOREM. *A commutative C*-algebra $C_0(X)$ has the property (*) if and only if each measure μ in $M(X)^+$ is of the form*

$$(**)\qquad \mu=\sum_{i=1}^{\infty}\alpha_i\,\delta_{t_i}$$

where each α_i is a non-negative real number.

PROOF. For each μ in $M(X)^+$, considering the set of elements t_i such that $\mu(\{t_i\})\neq 0$, there exists the countable set (t_i) of X such that $\mu-\sum_{i=1}^{\infty}\mu(\{t_i\})\delta_{t_i}$ belongs to $M(X)^+$, and $(\mu-\sum_{i=1}^{\infty}\mu(\{t_i\})\delta_{t_i})(s)=0$ for all s in X.

Suppose $\mu-\sum_{i=1}^{\infty}\mu(\{t_i\})\delta_{t_i}\neq 0$, from Lemma, $C_0(X)$ has not the property (*). Hence if $C_0(X)$ has the property (*), μ is of the form (**).

Conversely, let each measure μ in $M(X)^+$ be of the form (**).

Let B be an arbitrary C*-algebra. For each u in $(C_0(X)\underset{\alpha}{\otimes}B)^*$, from [1. Proposition 32], there exist a measure μ in $M(X)^+$ and a weakly measurable function on X into B^* such that

$$u=\int_X\delta_t\otimes f(t)\,d\mu(t).$$

Since $f(t)$ is μ-separably-valued, weakly measurable and bounded, it is Bochner μ-intagrable. Hence there exists a sequence of finite-valued functions $f_n(t)$ strongly convergent to $f(t)$ μ-a.e. on X. Then $f_n(t)$ is of the form

$$f_n(t)=\sum_{i=1}^{l_n}\chi_{E_i^{(n)}}(t)\overset{(n)}{g_i}, \qquad \overset{(n)}{g_i}\in B^*.$$

Then we have

$$\lim \|\sum_{i=1}^{l_n}\chi_{E_i^{(n)}}\mu\otimes\overset{(n)}{g_i}-u\|=0,$$

where $\chi_{E_i^{(n)}}\mu$ denotes the positive functional on $C_0(X)$ such that

$$h\longrightarrow\int_X h(t)\chi_{E_i^{(n)}}(t)\,d\mu(t).$$

Hence $C_0(X)$ has the property $(*)$.

REMARK. *A C*-algebra A has the property* $(*)$ *if* A^{**} *is atomic.*

PROOF. Let B be an arbitrary C*-algebra and π be a non-degenerate representation of $A\underset{\alpha}{\otimes}B$.

Then there exist representations π_1 and π_2 of A and B such that

$$\pi(a\otimes b)=\pi_1(a)\,\pi_2(b)=\pi_2(b)\,\pi_1(a)$$

for a in A and b in B.

Since the weak closure of $\pi_1(A)$ is atomic, π is unitary equivalent to a representation of the form

$$\sum_\beta \pi_{1\beta}\otimes\pi_{2\beta}$$

where $\pi_{1\beta}$ and $\pi_{2\beta}$ are representations of A and B respectively. Hence π has a normal extension to $A^{**}\otimes B^{**}$. It follows that every positive functional on $A\underset{\alpha}{\otimes}B$ has a normal extension to $A^{**}\otimes B^{**}$, so $(A\underset{\alpha}{\otimes}B)^*=A^*\underset{\alpha'}{\otimes}B^*$. Thus $(A\underset{\alpha}{\otimes}B)^{**}$ is canonically *-isomorphic to $A^{**}\otimes B^{**}$.

3. Examples

EXAMPLE 1. *Let X be a discrete topological space. Then* $C_0(X)$ *has the property* $(*)$.

PROOF. For each μ in $M(X)^+$, the set $I=(t\in X\colon \mu(\{t\})\neq 0)$ is countable. Then, $\nu=\mu-\Sigma_{t\in I}\,\mu(\{t\})\delta_t$ is non-atomic and positive.

For each f in $C_0(X)$ and every $\varepsilon>0$, the set $K=(t\in X\colon |f(t)|\geqq\varepsilon)$ is finite, so that $\nu(K)=0$. Then we obtain $|\nu(f)|\leq\varepsilon\|\nu\|$. Hence $\nu(f)=0$, and, so $\nu=0$. Therefore, we have $\mu=\Sigma_{t\in I}\,\mu(\{t\})\delta_t$. From Theorem $C_0(X)$ has the property $(*)$.

EXAMPLE 2. *Let X be a locally compact Hausdorff space, which is a countable set. Then*

$C_0(X)$ *has the property* (*).

PROOF. Since X is countable, every positive measure is of the form (**). From Theorem $C_0(X)$ has the property (*).

Let [0 1] be the unit interval of real numbers. Since the Lebesgue measure on [0 1] is non-atomic, we have the following example.

EXAMPLE 3. $C([0\ 1])$ *has not the property* (*).

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