On a 6-dimensional K-space

By

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1. Introduction

Let M be an n-dimensional almost Hermitian manifold with almost Hermitian structure (F_i^h, g_{ji})[1]. If the fundamental 2-form $F_{ih}=g_{jh}F_i^j$ satisfies

$$\nabla_j F_{ih}+\nabla_i F_{jh}=0, \tag{1. 1}$$

where ∇_j denotes the operator of the Riemannian covariant differentiation, then the manifold is called a K-spacr (or almost Tachibana space or nearly Kähler manifold).

It is well known that a Kähler manifold is a K-space but a K-space is not necessarily a Kähler manifold. In the sequel, by a K-space we mean a non-Kähler K-space. A 6-dimensional K-space has been studied by Takamatsu [5], [6], Sato [3], Yamaguchi, Chuman and Matsumoto [8], Tanno [7] and others.

One of the examples of K-spaces is a 6-dimensional sphere S^6 [1]. The following is a conjecture. A 6-dimensional K-space is a space of constant curvature.

The results known up to now which support this conjecture are the following. (See also Remark in §2)

THEOREM A (Takamatsu [5]). *There does not exist a K-space of constant curvature provided that* $n\neq 6$.

THEOREM B (Tanno [7]). *A 6-dimensional K-space of constant holomorphic sectional curvature is a space of constant curvature.*

Now, let $R_{kji}{}^h$, R_{ji} and R be the curvature tensor, the Ricci tensor and the scalar curvature respectively and put $R_{kjih}=g_{ht}R_{kji}{}^t$, $R^*{}_{ji}=\frac{1}{2}F^{ab}R_{absi}F_j{}^s$, $R^*=g^{ji}R^*{}_{ji}$ etc..

The purpose of this note is to prove the following theorem which supports our conjecture.

THEOREM. *If a 6-dimensional K-space M satisfies*

$$\nabla_m R_{kjih}-F_i{}^t F_h{}^s \nabla_m R_{kjts}=0,$$

then M is a space of constant curvature.

1) The Latin indices run over the range 1, 2,..., n.

COROLLARY. *If a 6-dimensional K-space M is locally symmetric, then M is a space of constant curvature.*

2. Preliminaries

We need the following lemmas to prove Theorem.

LEMMA 2.1 (Tachibana [4]). *In a K-space, we have*

$$(2.1)\qquad (\nabla_j F_{ab})\nabla_i F^{ab}=R_{ji}-R^*_{ji},\ (\nabla_j F_{ab})\nabla^j F^{ab}=R-R^*.$$

LEMMA 2.2.(Gray [2]). *In a K-space, we have*

$$(2.2)\qquad R_{kjih}-F_k{}^a F_j{}^b R_{abih}=-(\nabla_k F_j{}^s)\nabla_s F_{ih}.$$

LEMMA 2.3 (Takamatsu [6]). *In a K-space, we have*

$$(2.3)\qquad (R_{ji}-R^*_{ji})(R^{ji}-R^{*ji})=2(R_{kjih}R^{kjih}-F_i{}^t F_h{}^s R_{kjts}R^{kjih}).$$

LEMMA 2.4 (Takamatsu [6]). *In a 6-dimensional K-space, we have*

$$(2.4)\qquad R_{ji}-R^*_{ji}=\frac{1}{6}(R-R^*)g_{ji},$$

$$(2.5)\qquad 5R^*=R.$$

LEMMA 2.5 (Yamaguchi, Chuman and Matsumoto [8]). *A 6-dimensionl K-space is an Einstein space.*

LEMMA 2.6. *In a 6-dimensional K-space, we have*

$$(2.6)\qquad R_{kjih}-F_i{}^t F_h{}^s R_{kjts}=\frac{R}{30}(g_{ji}g_{kh}-g_{ki}g_{jh}-F_{ji}F_{kh}+F_{ki}F_{jh}).$$

(c.f. Yamaguchi, Chuman and Matsumoto [8])

PROOF. To prove (2. 4), Takamatsu used in [6] the following identity:

$$(2.7)\qquad U_{kjih}U^{kjih}=-\frac{3}{4}\left(R_{ji}-R^*_{ji}-\frac{R-R^*}{6}g_{ji}\right)\left(R^{ji}-R^{*ji}-\frac{R-R^*}{6}g^{ji}\right)$$

where $U_{kjih}=\frac{1}{2}(R_{kjih}-F_i{}^t F_h{}^s R_{kjts})-\frac{1}{4}(g_{kh}S_{ji}-g_{jh}S_{ki}+g_{ji}S_{kh}-g_{ki}S_{jh})$ $+\frac{1}{4}(F_i{}^t F_{hk}S_{jt}-F_i{}^t F_{hj}S_{kt}+F_{ij}F_h{}^s S_{ks}-F_{ik}F_h{}^s S_{js})+\frac{1}{16}(R-R^*)(g_{ji}g_{kh}-$ $g_{ki}g_{jh}-F_{ji}F_{kh}+F_{ki}F_{jh})$ and $S_{ji}=R_{ji}-R^*_{ji}$.

From (2. 7), we have

$$R_{ji}-R^*_{ji}=\frac{1}{6}(R-R^*)g_{ji},\quad U_{kjih}=0.$$

Hence, substituting (2. 4) and (2. 5) into $U_{kjih}=0$, we easily have (2. 6).

LEMMA 2. 7. *In a 6-dimensional K-space, we have*

(2. 8) $$F_i{}^t F_h{}^s R_{kjts} R^{kjih} = |R_{kjih}|^2 - \frac{4}{75} R^2$$

where $|R_{kjih}|^2 = R_{kjih} R^{kjih}$.

PROOF. From (2. 3), making use of (2. 4) and (2. 5), we have

$$\begin{aligned} F_i{}^t F_h{}^s R_{kjts} R^{kjih} &= -\frac{1}{2} |R_{ji} - R^*{}_{ji}|^2 + |R_{kjih}|^2 \\ &= |R_{kjih}|^2 - \frac{1}{12}(R - R^*)^2 \\ &= |R_{kjih}|^2 - \frac{4}{75} R^2. \end{aligned}$$

REMARK. (2. 6) can be written as

$$R_{kjih} - \frac{R}{30}(g_{ji}g_{kh} - g_{ki}g_{jh}) - F_i{}^t F_h{}^s \left[R_{kjts} - \frac{R}{30}(g_{jt}g_{ks} - g_{kt}g_{js}) \right] = 0$$

which also supports our conjecture.

3. Proof of Theorem

First of all, applying ∇_m to the both sides of (2. 6) and taking account of the assumption and $\nabla_m R = 0$, we have

(3. 1) $$\begin{aligned} &(\nabla_m F_i{}^t) F_h{}^s R_{kjts} + F_i{}^t (\nabla_m F_h{}^s) R_{kjts} \\ &= \frac{R}{30} \left[(\nabla_m F_{ji}) F_{kh} + F_{ji} \nabla_m F_{kh} - (\nabla_m F_{ki}) F_{jh} - F_{ki} \nabla_m F_{jh} \right]. \end{aligned}$$

Next squaring the both sides of (3. 1) and making use of $F^{ji} \nabla_m F_{ji} = 0$, we have

(3. 2) $$\begin{aligned} &2 |(\nabla_m F_i{}^t) F_h{}^s R_{kjts}|^2 + 2 (\nabla_m F_i{}^t) F_h{}^s (\nabla^m F^{hb}) F^{ia} R_{kjts} R^{kj}{}_{ab} \\ &= \left(\frac{R}{30}\right)^2 \left[4 |(\nabla_m F_{ji}) F_{kh}|^2 - 8 |\nabla_m F_{ji}|^2 \right]. \end{aligned}$$

Making use of Lemmas in §2, we shall calculate the left hand side of (3. 2).

Now, for the first term, by (2. 1), (2. 4) and (2. 5), we have

(3. 3) $$\begin{aligned} |(\nabla_m F_i{}^t) F_h{}^s R_{kjts}|^2 &= (\nabla_m F_i{}^t) F_h{}^s R_{kjts} (\nabla^m F^{ia}) F^{hb} R^{kj}{}_{ab} \\ &= \nabla_m F_i{}^t (\nabla^m F^{ia}) R_{kjt}{}^b R^{kj}{}_{ab} \\ &= (R^{ta} - R^{*ta}) R_{kjt}{}^b R^{kj}{}_{ab} \\ &= \frac{1}{6}(R - R^*) |R_{kjih}|^2 = \frac{2}{15} R |R_{kjih}|^2. \end{aligned}$$

For the second term, by (1. 1), (2. 2) and (2. 6), we have

$$(\nabla_m F_i{}^t) F_h{}^s (\nabla^m F^{hb}) F^{ia} R_{kjts} R^{kj}{}_{ab} = (\nabla^t F_{im}) F_h{}^s (\nabla^b F^{hm}) F^{ia} R_{kjts} R^{kj}{}_{ab}$$

$$= (\nabla^t F^{ia}) F_{im} (\nabla^b F_h{}^s) F^{hm} R_{kjts} R^{kj}{}_{ab} = -\nabla^i F^{at} (\nabla_i F^{bs}) R_{kjts} R^{kj}{}_{ab}$$

$$= (R^{atbs} - F_m{}^b F_c{}^s R^{atmc}) R_{kjts} R^{kj}{}_{ab}$$

$$= \frac{R}{30}(g^{as} g^{tb} - g^{ab} g^{ts} - F^{as} F^{tb} + F^{ab} F^{ts}) R_{kjts} R^{kjab}$$

$$= \frac{R}{30}[-|R_{kjih}|^2 - F_a{}^s F_b{}^t R_{kjst} R^{kjab} + F^{ab} F^{ts} R_{kjts} R^{kjab}].$$

In this place, by the definition of $R^*{}_{ji}$, we have

$$F^{ab} F^{ts} R_{kjts} R^{kj}{}_{ab} = 4 F_k{}^s R^*{}_{sj} F^{kt} R^*{}_t{}^j$$

$$= 4 R^*{}_{kj} R^{*kj}$$

$$= \frac{2}{75} R^2,$$

because by (2. 4), (2. 5) and Lemma 2. 5, we have

$$R^*{}_{ji} = \frac{1}{6} R^* g_{ji} = \frac{1}{30} R g_{ji}.$$

Hence, making use of (2. 8), we have

(3. 4)
$$(\nabla_m F_i{}^t) F_h{}^s (\nabla^m F^{hb}) F^{ia} R_{kjts} R^{kj}{}_{ab}$$

$$= \frac{R}{30}(-|R_{kjih}|^2 - |R_{kjih}|^2 + \frac{4}{75} R^2 + \frac{2}{75} R^2)$$

$$= \frac{R}{15}(-|R_{kjih}|^2 + \frac{3}{75} R^2).$$

For the right hand side of (3. 2), by (2. 1) and (2. 5), we have

(3. 5)
$$|(\nabla_m F_{ji}) F_{kh}|^2 = 6(R - R^*) = \frac{24}{5} R,$$

$$|\nabla_m F_{ji}|^2 = R - R^* = \frac{4}{5} R.$$

Consequently, substituting (3. 3), (3. 4) and (3. 5) into (3. 2), we have

$$2\left[\frac{2}{15} R |R_{kjih}|^2 + \frac{R}{15}\left(-|R_{kjih}|^2 + \frac{3}{75} R^2\right)\right]$$

$$= \left(\frac{R}{30}\right)^2 \left(\frac{96}{5} R - \frac{32}{5} R\right), \quad \text{i.e.}$$

$$|R_{kjih}|^2 = \frac{1}{15} R^2.$$

This equation can be written as

$$\left[R_{kjih}-\frac{R}{30}(g_{ji}g_{kh}-g_{jh}g_{ki})\right]\left[R^{kjih}-\frac{R}{30}(g^{ji}g^{kh}-g^{jh}g^{ki})\right]=0$$

from which we have

$$R_{kjih}=\frac{R}{30}(g_{ji}g_{kh}-g_{jh}g_{ki}). \qquad \text{Q. E. D.}$$

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