The immigration between branching processes

By

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1. Introduction

Let $\{p_k; k=0, 1, 2, \cdots\}$ and $\{q_k; k=0, 1, 2, \cdots\}$ be probability functions; $p_k \geqq 0, \sum p_k = 1$, $q_k \geqq 0$, $\sum q_k = 1$. Let $\{X_n; n=0, 1, 2, \cdots\}$ be a Galton Watson process based on $\{p_k\}$; i.e, a Markov chain with stationary transition probabilities given by

$$p_{ij} = P(X_{n+1}=j \mid X_n=i)$$
$$= \begin{cases} p_j^{*i} & \text{if} \quad i \geqq 1, \ j \geqq 0, \\ \delta_{0j} & \text{if} \quad i=0, \ j \geqq 0, \end{cases}$$

where $\{p_k^{*i}\}$ is the i-fold convolution of $\{p_k\}$ and δ_{0j} is the Kronecker delta.

Similarly we define Galton Watson process $\{Y_n; n=0, 1, 2, \cdots\}$ based on $\{q_k\}$.

This setting may be comprehended that after one unit of time each particle splits independently of others into a random number of offspring according to the probability law $\{p_k\}$ in A-district and according to the probabiltiy law $\{q_k\}$ in B-district.

In some report we see that the birth-rate in the city is smaller than the birth-rate in the country, and that many persons immigrate from the country into the city.

In this paper we consider a simplified type of immigration from B into A, and study the extinction probability and the limit behavior of the number of particles.

2. Immigration from B into A

Now we assume that each particle in A-district and B-district splits as stated in 1, and assume that when every particle in the n-th generation in A-district has no offspring then instantly a particle in the $(n+1)$-th generation in B-district (if exist) immigrate into A-district. That is, let the number of particles in the n-th generation in A-district be $\overline{X}_n$, the number of their offsprings be $\widetilde{X}_{n+1}$, the number of particles in B-district be $\overline{Y}_n$ and the number of their offsprings be $\widetilde{Y}_{n+1}$. Then

$$\overline{X}_{n+1}=1, \ \overline{Y}_{n+1}=\widetilde{Y}_{n+1}-1 \qquad \text{if } \widetilde{X}_{n+1}=0, \ \widetilde{Y}_{n+1}>0,$$
$$\overline{X}_{n+1}=\widetilde{X}_{n+1}, \ \overline{Y}_{n+1}=\widetilde{Y}_{n+1} \qquad \text{otherwise.}$$

If $\widetilde{X}_{n+1}=\widetilde{Y}_{n+1}=0$ for the first time, the process is said to terminate in the $(n+1)$-th step.

This process is denoted formally as follows.

Let $\{Z_n=(\overline{X}_n, \overline{Y}_n);\ n=0,1,2,\cdots\}$ be a Markov chain on the pairs of nonnegative integers. Its transition function

$$\pi(i,j\longrightarrow k,l)=P\{Z_{n+1}=(k,l)\mid Z_n=(i,j)\}$$

is defined in terms of given probability functions $\{p_k\}$ and $\{q_k\}$ as follows.

$$\pi(i,0\longrightarrow k,0)=p_{ik}=p_k^{*i}$$

$$\pi(i,0\longrightarrow k,l)=0 \qquad (l=1,2,\cdots)$$

$$\pi(i,j\longrightarrow 0,0)=p_{i0}q_{j0}=p_0^i q_0^j$$

$$\pi(i,j\longrightarrow 0,l)=0 \qquad (l=1,2,\cdots)$$

$$\pi(i,j\longrightarrow 1,l)=p_{i0}q_{jl+1}+p_{i1}q_{jl}$$

$$\pi(i,j\longrightarrow k,l)=p_{ik}q_{jl} \qquad (j=1,2,\cdots;\ k=2,3,\cdots).$$

The study of Galton-Watson processes has been carried out using generating functions skilfully ([1], [4]).

We assume $X_0=1$, $p_0+p_1<1$, and define the generating function of X_1 by

$$f(s)=\sum_{k=0}^{\infty}p_k s^k, \qquad |s|\leqq 1,$$

and that of X_n by

$$f_{(n)}(s)=\sum_{k=0}^{\infty}P\{X_n=k\}s^k, \qquad |s|\leqq 1,$$

and denote the extinction probability of $\{X_n\}$ by q. Then following properties are well-known.

(1) $f(0)=p_0,\ f(1)=1,\ f'(1)=m$ (The mean of X_1),

(2) $f(s)$ is strictly convex and increasing in $[0,1]$,

(3) if $m\leqq 1$, then $f(t)>t$ for $t\in[0,1)$, and $q=1$,

(4) if $m>1$, then $f(t)=t$ has a unique root q in $[0,1)$,

(5) $f_{(n)}(s)=f[f_{(n-1)}(s)]=f_{(n-1)}[f(s)] \qquad (n=1,2,\cdots)$

where $f_{(0)}(s)=s$,

(6) $\sum_{k=0}^{\infty}p_{ik}s^k=[f(s)]^i$

(7) $\lim_{n\to\infty}P\{X_n=k\}=0 \qquad (k=1,2,\cdots)$,

i.e. all states $k\geqq 1$ are transient.

Using these properties, we study several properties of the generating functions of Z_n and $\overline{X}_n$, and deduce some limit property of $\{\overline{X}_n\}$.

Now we assume that $Z_0=(1,1)$, and denote the n-step transition function by

$$\pi_{(n)}(i,j\longrightarrow k,l)=P\{Z_{m+n}=(k,l)\,|\,Z_m=(i,j)\}\qquad(n=1,2,\cdots).$$

We define the generating function of Z_n by

$$\varphi_{(n)}(s,t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{(n)}(1,1\longrightarrow k,l)s^kt^l\qquad(n=1,2,\cdots),$$

and

$$\varphi_{(0)}(s,t)=st,$$

and denote $\varphi_{(1)}(s,t)$ by $\varphi(s,t)$.

To use in the next theorem we denote the generating function of Y_n, assuming $Y_0=1$, by $g_{(n)}(t)$, i.e.,

$$g_{(n)}(t)=\sum_{l=0}^{\infty}P\{Y_n=l\}t^l,\qquad|t|\leqq1,\ (n=1,2,\cdots)$$

and

$$g_{(0)}(t)=t,\ g(t)=g_{(1)}(t)=\sum_{l=0}^{\infty}q_lt^l.$$

THEOREM 1. $\varphi_{(N)}(s,t)=\varphi(f_{(N-1)}(s),\ g_{(N-1)}(t))+\overline{E}_{(N)}(s,t),\qquad(N=2,3,\cdots),\qquad(1)$

where $$E_{(N)}(s,t)=(s-t)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi_{(N-1)}(1,1\longrightarrow m,n)p_0^m\times\sum_{l=0}^{\infty}q_{n,l+1}t^l,\qquad(N=2,3,\cdots),$$

and $$\begin{aligned}\overline{E}_{(N)}(s,t)&=E_{(2)}(f_{(N-2)}(s),\ g_{(N-2)}(t))\\&\quad+E_{(3)}(f_{(N-3)}(s),\ g_{(N-3)}(t))\\&\quad+\cdots\\&\quad+E_{(N)}(s,t),\qquad(N=2,3,\cdots).\end{aligned}$$

PROOF. $$\begin{aligned}\varphi_{(2)}(s,t)&=\sum_k\sum_l\pi_{(2)}(1,1\longrightarrow k,l)s^kt^l\\&=\sum_k\sum_l\sum_m\sum_n\pi(1,1\longrightarrow m,n)\pi(m,n\longrightarrow k,l)s^kt^l\\&=\sum_m\sum_n\pi(1,1\longrightarrow m,n)\sum_k\sum_l\pi(m,n\longrightarrow k,l)s^kt^l.\end{aligned}$$

and $$\sum_k\sum_l\pi(m,n\longrightarrow k,l)s^kt^l$$

$$=\sum_k\sum_l\sum_{k_1+\cdots+k_n=k}p_{k_1}\cdots p_{k_n}s^k\sum_{l_1+\cdots+l_n=l}q_{l_1}\cdots q_{l_n}t^l$$

$$+\sum_{l=1}^{\infty}\{p_{m0}q_{n,\,l+1}st^l-p_{m0}q_{nl}t^l\}+p_{m0}q_{n1}s$$

$$=f^m(s)g^n(t)+s\sum_{l=0}^{\infty} p_{m0}q_{n,l+1}t^l - t\sum_{l=0}^{\infty} p_{m0}q_{n,l+1}t^l$$

$$=f^m(s)g^n(t)+(s-t)\sum_{l=0}^{\infty} p_{m0}\, q_{n,l+1}t^l.$$

Hence $\varphi_{(2)}(s,t)=\sum_m\sum_n \pi(1,1\longrightarrow m,n)f^m(s)g^n(t)$

$$+(s-t)\sum_m\sum_n \pi(1,1\longrightarrow m,n)\sum_{l=0}^{\infty} p_0^m\, q_{n,l+1}t^l$$

$$=\varphi(f(s),\ g(t))+E_{(2)}(s,t).$$

Thus (1) holds for $n=2$.

Now we assume inductively that (1) holds for $n=r$.

$$\varphi_{(r+1)}(s,t)=\sum_k\sum_l \pi_{(r+1)}(1,1\longrightarrow k,l)s^k t^l$$

$$=\sum_m\sum_n \pi_{(r)}(1,1\longrightarrow m,n)\sum_k\sum_l \pi(m,n\longrightarrow k,l)s^k t^l$$

$$=\sum_m\sum_n \pi_{(r)}(1,1\longrightarrow m,n)\{f^m_{(s)}g^n_{(s)}+(s-t)\sum_{l=0}^{\infty} p_0^m\, q_{n,l+1}t^l\}$$

$$=\varphi_{(r)}(f(s),g(t))+E_{(r+1)}(s,t)$$

$$=\varphi\{f_{(r-1)}(f(s)),g_{(r-1)}(g(t))\}+\overline{E}_{(r)}(f(s),g(t))+E_{(r+1)}(s,t)$$

$$=\varphi\{f_{(r)}(s),\ g_{(r)}(t)\}+\overline{E}_{(r+1)}(s,t).$$

Thus (1) holds for $n=r+1$.

By induction (1) holds for $n=2, 3, \cdots$.

3. Limit behavior of $\{\overline{X}_n\}$

We now consider the limit behavior of $\{\overline{X}_n\}$.
If $\{\overline{Y}_n\}$ is never extinct, our process reduces to a kind of branching process with state dependent immigration studied in [3].

We denote the generating function of $\overline{X}_n$ by $\overline{f}_{(n)}(s)$.

Then, $\overline{f}_{(n)}(s)=\varphi_{(n)}(s,1)$

$$=\varphi(f_{(n-1)}(s),1)+E_{(2)}(f_{(n-2)}(s),1)+\cdots+E_{(n)}(s,1).$$

Denoting $\dfrac{\partial\varphi(s,t)}{\partial s}=\varphi_s(s,t)$ and $\dfrac{\partial E_{(n)}(s,t)}{\partial t}=E_{(n)s}(s,t)$, and differentiating $\overline{f}_{(n)}(s)$ by s, we have

$$\overline{f}'_{(n)}(s)=\varphi_s(f_{(n-1)}(s),1)f'_{(n-1)}(s)+E_{(2)s}(\overline{f}_{(n-2)}(s),1)f'_{(n-2)}(s)$$
$$+\cdots+E_{(n)s}(s,1),$$

$$E_{(N)s}(s,t)=\sum_m\sum_n \pi_{(N-1)}(1,1\longrightarrow m,n)p_0^m\sum_{l=0}^{\infty}q_{n,l+1}t^l$$

hence

$$E_{(N)s}(s,1)=\sum_m\sum_n \pi_{(N-1)}(1,1\longrightarrow m,n)p_0^m(1-q_{n,0})$$

$$=\sum_m\sum_n \pi_{(N-1)}(1,1\longrightarrow m,n)p_0^m$$

$$-\sum_m\sum_n \pi_{(N-1)}(1,1\longrightarrow m,n)p_0^m q_0^n$$

$$=\sum_m P(\overline{X}_{N-1}=m\,|\,Z_0=(1,1))p_0^m-\varphi_{(N-1)}(p_0,q_0)$$

$$=\overline{f}_{(N-1)}(p_0)-\varphi_{(N-1)}(p_0,q_0).$$

Thus we have

$$\begin{aligned}\overline{f}'_{(n)}(s)=&\varphi_s(f_{(n-1)}(s),1)f'_{(n-1)}(s)\\&+\{\overline{f}_{(1)}(p_0)-\varphi(p_0,q_0)\}f'_{(n-2)}(s)\\&+\cdots\cdots\\&+\{\overline{f}_{(n-2)}(p_0)-\varphi_{(n-2)}(p_0,q_0)\}f'(s)\\&+\{\overline{f}_{(n-1)}(p_0)-\varphi_{(n-1)}(p_0,q_0)\}.\end{aligned}\tag{1}$$

Denoting $E(X_1)=m$, we have

$$f'_{(N)}(1)=E(X_N)=m^N.$$

Hence, if we take $s=1$ in (1), then

$$\begin{aligned}E(\overline{X}_n)=&\varphi_s(1,1)m^{n-1}\\&+\{\overline{f}_{(1)}(p_0)-\varphi(p_0,q_0)\}m^{n-2}\\&+\cdots\\&+\{\overline{f}_{(n-2)}(p_0)-\varphi_{(n-2)}(p_0,q_0)\}m\\&+\{\overline{f}_{(n-1)}(p_0)-\varphi_{(n-1)}(p_0,q_0)\}.\end{aligned}$$

Thus by [2, p. 22, lemma A] we have,

THEOREM 2. *If* $m<1$ *and* $\{\overline{f}_{(n)}(p_0)-\varphi_{(n)}(p_0,q_0)\}$ *converges to* α *as* $n\longrightarrow\infty$, *then* $\lim_{n\to\infty}E(\overline{X}_n)=\dfrac{\alpha}{1-m}$.

Now we are in a position to study the limit behavior of $\overline{f}_{(n)}(s)$ for the case $m>1$.

$$\bar{f}_{(n)}(s)=\sum_k\sum_l \pi_{(n)}(1, 1\longrightarrow k, l)s^k$$

$$=\sum_k\sum_l\sum_i\sum_j \pi_{(n-1)}(1, 1\longrightarrow i, j)\pi(i, j\longrightarrow k, l)s^k$$

$$=\sum_i\sum_j \pi_{(n-1)}(1, 1\longrightarrow i, j)\sum_k\sum_l \pi(i, j\longrightarrow k, l)s^k,$$

where

$$\sum_k\sum_l \pi(i, j\longrightarrow k, l)s^k$$

$$=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} p_{ik}q_{jl}s^k+p_{i0}q_{j0}+p_{i0}\sum_{l=1}^{\infty} q_{jl}s$$

$$\leqq \sum_{k=0}^{\infty} p_{ik}s^k=[f(s)]^i.$$

Thus

$$\bar{f}_{(n)}(s)\leqq \sum_i\sum_j \pi_{(n-1)}(1, 1\longrightarrow i, j)[f(s)]^i$$

$$=\bar{f}_{(n-1)}[f(s)], \tag{2}$$

and we can see inductively

$$\bar{f}_{(n)}(s)\leqq \bar{f}_{(n-1)}[f(s)]\leqq \bar{f}_{(n-2)}[f_{(2)}(s)]$$

$$\leqq \cdots\cdots \leqq \bar{f}_{(k)}[f_{(n-k)}(s)]\leqq \cdots \leqq f_{(n)}(s). \tag{3}$$

Since $m>1$, the extinction probability q is smaller than 1. When $s\geqq q$, we have $f(s)\leqq s$ by the convexity of $f(s)$, and then by (2)

$$\bar{f}_{(n)}(s)\leqq \bar{f}_{(n-1)}(s) \qquad \text{for } n=1, 2, \cdots.$$

Hence $\{\bar{f}_{(n)}(s)\}$ converges for $s\geqq q$.

Putting $r=\lim_{n\to\infty}\bar{f}_{(n)}(q)$, we have $r\leqq q$ by (3).

When $s\geqq q$, we have by (3)

$$\bar{f}_{(n)}(s)\leqq \bar{f}_{(k)}[f_{(n-k)}(s)],$$

while

$$f_{(n-k)}(s)\longrightarrow q \qquad (n\longrightarrow\infty), \qquad \text{for } s<1,$$

thus

$$\overline{\lim_{n\to\infty}}\ \bar{f}_{(n)}(s)\leqq \bar{f}_{(k)}(q) \qquad \text{for } q\leqq s<1$$

and accordingly

$$\overline{\lim_{n\to\infty}}\ \bar{f}_{(n)}(s)\leqq r \qquad \text{for } q\leqq s<1.$$

On the other hand for $q \leqq s$

$$\bar{f}_{(n)}(s) \geqq \bar{f}_{(n)}(q) \geqq r.$$

Hence, for $q \leqq s < 1$

$$\lim_{n\to\infty} \bar{f}_{(n)}(s) = r \tag{4}$$

Since $\bar{f}_{(n)}(s)$ is monotone increasing convex function on $[0, 1]$, (4) holds for $0 \leqq s < 1$. This property of generating function implies,

THEOREM 3. *When* $m > 1$,

1°) $P(\bar{X}_n = k) \longrightarrow 0 \quad (n \longrightarrow \infty) \quad for\ k = 1, 2, \ldots$

equivalently

$$\pi_{(n)}(1, 1 \longrightarrow k, l) \longrightarrow 0 \quad (n \longrightarrow \infty) \quad for\ k = 1, 2, \ldots;\ l = 0, 1, 2, \ldots$$

2°) $\pi_{(n)}(1, 1 \longrightarrow 0, 0) \longrightarrow r \quad (n \longrightarrow \infty)$

3°) $P(\bar{X}_n \longrightarrow \infty\ (n \longrightarrow \infty)) = 1 - r.$

When $m \leqq 1$ (accordingly $q = 1$), the limit behavior of $\{X_n\}$ seems to be not so simple. For example, when $m \leqq 1$ and $q_0 = q_1 = 0$,
$P(X_n \longrightarrow 0\ (n \longrightarrow \infty)) = P(X_n \longrightarrow \infty\ (n \longrightarrow \infty)) = 0$ and it will be an interesting problem to seek the limit probability of each state.

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