A generalization of Malliavin's tauberian theorem

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(Received October 29, 1973)

1. Introduction

In this paper we generalize the theorem by P. Malliavin together with the formula of Å. Pleijel to deduce the behaviour of a positive measure μ defined on a sector of the complex plane:

$$\Gamma=\{\lambda:|\arg\lambda|\leqq\theta\},\quad 0\leqq\theta<\pi/4$$

from the behaviour of a transform

$$f(z)=\int_\Gamma\frac{\mu(d\lambda)}{\lambda-z},\quad z\notin\Gamma.$$

P. Malliavin [1] proved the following

THEOREM A. *Let $\sigma(\lambda)$ be a non-decreasing function for $\lambda\geqq 0$. Suppose that*

$$\int_0^\infty\frac{d\sigma(\lambda)}{\lambda-z}=a(-z)^{-\alpha}+O(|z|^{-\beta})$$

as $z\to\infty$ with $|\mathrm{Im}\, z|=|z|$ and $\mathrm{Re}\, z\geqq 0$, where $0<\alpha<\beta<1$, $0<\gamma<1$, $a>0$.

Then as $X\to\infty$

$$\sigma(X)=a\frac{\sin(1-\alpha)\pi}{(1-\alpha)\pi}X^{1-\alpha}+O(X^{\gamma-\alpha})+O(X^{1-\beta}).$$

Å. Pleijel [2] proved this theorem in a very elementary way. His proof uses the following approximate inversion formula for the Stieltjes transform of a positive measure:

THEOREM B. *Let $\sigma(\lambda)$ be a non-decreasing function for $\lambda\geqq 0$ and put*

$$f(z)=\int_0^\infty\frac{d\sigma(\lambda)}{\lambda-z},\quad z\notin(0,\infty).$$

Then for X, $Y>0$

$$\left|\sigma(X)-\sigma(0)-\frac{1}{2\pi i}\int_{L(Z)}f(z)dz+\frac{Y}{\pi}\mathrm{Re}\, f(Z)\right|\leqq Y\,\mathrm{Im}\, f(Z),$$

where $L(Z)$ *is an oriented curve in the complex plane from* $\bar{Z}$ *to* $Z=X+iY$ *not intersecting* $[0, \infty)$.

These theorems have already been used to derive asymptotic formulas with remainder estimates for eigenvalues of elliptic operators by S. Agmon and R. Beals, and recently by K. Maruo and H. Tanabe. But these authors restrict themselves to the case where elliptic operators dealt with are assumed to be formally self-adjoint in some sense or other. In a study of eigenvalues of more general elliptic operators we consider it necessary to generalize the above theorems in such a manner as was mentioned in the beginning. In the present paper we will prove Theorems I and II stated in the next section and leave applications of them to a following paper.

2. Main results

Consider a sector in the complex plane: $\Gamma=\{\lambda: |\arg\lambda|\leqq\theta\}$ and suppose throughout this paper that θ statisfies $0\leqq\theta<\pi/4$.

The following theorem is a generalization of Theorem B:

Theorem I. *Let* μ *be a positive measure defined on* Γ *and put*

$$f(z)=\int_\Gamma \frac{\mu(d\lambda)}{\lambda-z}, \quad z\notin\Gamma.$$

Then, for X, Y such that $Y>\sqrt{\sec 2\theta-1}\,X>0$

$$\left|\mu\{\lambda\in\Gamma: \mathrm{Re}\,\lambda\leqq X\}-\mathrm{Re}\frac{1}{2\pi i}\int_{L(Z)} f(z)dz+\frac{Y}{2\pi}\mathrm{Re}\{f(Z)+f(\bar{Z})\}\right|$$

$$\leqq(2\cos(\theta+\pi/4))^{-1}\,Y\,\mathrm{Im}\{f(Z)-f(\bar{Z})\}$$

as long as $\mu\{\lambda\in\Gamma: \mathrm{Re}\,\lambda=X\}=0$. *Here* $L(Z)$ *is an oriented curve from* $\bar{Z}$ *to* $Z=X+iY$ *not intersecting* Γ *and if* $\theta=0$, $\cos(\theta+\pi/4)$ *can be replaced by* 1.

Remark. If $\theta=0$, then the conclusion of the theorem is written as

$$\left|\mu(0, X]-\frac{1}{2\pi i}\int_{L(Z)} f(z)\,dz+\frac{Y}{\pi}\mathrm{Re}\,f(Z)\right|\leqq Y\,\mathrm{Im}\,f(Z)$$

as long as $\mu\{X\}=0$, from which Theorem B follows.

Applying Theorem I to a function $f(z)=a(-z)^{-\alpha}+O(|z|^{\gamma p-\beta}/d(z,\Gamma)^p)$, we have the following

Theorem II. *Let* μ *be a positive measure defined on* Γ. *Suppose that*

$$\int_\Gamma \frac{\mu(d\lambda)}{\lambda-z}=a(-z)^{-\alpha}+O(|z|^{\gamma p-\beta}/d(z,\Gamma)^p)$$

as $|z|\to\infty$ *with* $d(z,\Gamma)\geqq|z|^\gamma$, *where* $0<\alpha<\beta<1$, $0<\gamma<1$, $p>0$ *and* $\mathrm{Re}\,a\geqq0$. *Then as*

$X\to\infty$

$$\left|\mu\{\lambda\in\Gamma:\ \mathrm{Re}\ \lambda\leqq X\}-\mathrm{Re}\ a\left\{\frac{\sin((1-\alpha)(\pi-\varphi))}{(1-\alpha)\pi}-\frac{\sqrt{2}}{\pi}\sin\theta\cos(\alpha(\pi-\varphi))\right\}\sec^{(1-\alpha)/2}2\theta X^{1-\alpha}\right|$$

$$\leqq\mathrm{Re}\ a\sqrt{2}(\cos(\theta+\pi/4))^{-1}\sin\theta\sin(\alpha(\pi-\varphi))\sec^{(1-\alpha)/2}2\theta X^{1-\alpha}$$

$$+O(X^{1-\beta})\sqrt{\sec 2\theta-1}+O(X^{\gamma-\alpha}),\ \varphi=\tan^{-1}\sqrt{\sec 2\theta-1}$$

as long as $\mu\{\lambda\in\Gamma:\mathrm{Re}\ \lambda=X\}=0$. *Here* $d(z,\Gamma)$ *denotes* $\mathrm{dist}(z,\Gamma)$.

REMARK. If θ satisfies, in addition to the assumption of $0\leqq\theta<\pi/4$,

$$\frac{\sin((1-\alpha)(\pi-\varphi))}{(1-\alpha)\pi}-\sqrt{2}\sin\theta\left\{\frac{\cos(\alpha(\pi-\varphi))}{\pi}-\frac{\sin(\alpha(\pi-\varphi))}{\cos(\theta+\pi/4)}\right\}>0$$

which holds good for sufficiently small θ, then the above estimate may be strengthened to some extent. In this case $\mu\{\lambda\in\Gamma:\ \mathrm{Re}\ \lambda\leqq X\}$ can be estimated not only from above but also from below. Particularly if θ is equal to zero, the above formula implies that as $X\to\infty$

$$\mu(0,X]=\mathrm{Re}\ a\frac{\sin((1-\alpha)\pi)}{(1-\alpha)\pi}X^{1-\alpha}+O(X^{\gamma-\alpha})$$

as long as $\mu\{X\}=0$. Compare this with Theorem A.

On the other hand the term $O(X^{1-\beta})\sqrt{\sec 2\theta-1}$ in the above formula can be dropped when $\beta-\alpha+\gamma\geqq 0$.

3. Proof of Theorem I

Let us consider $f(z)=\int_\Gamma\frac{\mu(d\lambda)}{\lambda-z}$, $z\notin\Gamma$ and put

$$I(Z)=\frac{1}{2\pi i}\int_{L(Z)}f(z)dz.$$

From the assumption of $X>0$ and $Y>\sqrt{\sec 2\theta-1}\,X$, $Z=X+iY$ and $\bar{Z}$ do not belong to Γ and, moreover, $L(Z)$ is included surely in the complement of Γ.

Letting $\nu_1=\nu_1(\lambda,Z)$, $\nu_2=\nu_2(\lambda,Z)$ be the angles between the negative real direction and $Z-\lambda$, $\bar{Z}-\lambda$ for $\lambda\in\Gamma$ respectively, we have

$$\mathrm{Re}\ I(Z)=\frac{1}{2\pi}\int_\Gamma(\nu_1+\nu_2)\mu(d\lambda).\tag{3.1}$$

In fact, by a change of the order of integration $I(Z)$ can be written as

$$I(Z)=\frac{1}{2\pi}\int_\Gamma(\nu_1+\nu_2)\,\mu(d\lambda)+\frac{i}{2\pi}\int_\Gamma\log\left|\frac{\lambda-Z}{\lambda-\bar{Z}}\right|\mu(d\lambda).$$

We next prove that

$$2Y\,\mathrm{Re}\,\{f(Z)+f(Z)\}=\int_{\Gamma}\sin(\nu_1+\nu_2)\frac{\cos^2\nu_1+\cos^2\nu_2}{\cos\nu_1\ \cos\nu_2}\mu(d\lambda), \tag{3. 2}$$

$$2Y\,\mathrm{Im}\{f(Z)-f(\overline{Z})\}=\int_{\Gamma}\sin^2(\nu_1+\nu_2)\frac{\cos(\nu_1-\nu_2)}{\cos\nu_1\ \cos\nu_2}\mu(d\lambda) \tag{3. 3}$$

hold as long as $\mu\{\lambda\in\Gamma:\mathrm{Re}\lambda=X\}=0$.

It is evident that

$$2Y=|\lambda-Z|\ \sin\nu_1+|\lambda-\overline{Z}|\ \sin\nu_2,$$

$$|\lambda-Z|\ \cos\nu_1=|\lambda-\overline{Z}|\ \cos\nu_2.$$

Hence it follows that if Re $\lambda\neq X$,

$$2Y\ \mathrm{Re}\left(\frac{1}{\lambda-Z}+\frac{1}{\lambda-\overline{Z}}\right)=(|\lambda-Z|\ \sin\lambda_1+|\lambda-\overline{Z}|\ \sin\nu_2)\left(\frac{\cos\nu_1}{|\lambda-Z|}+\frac{\cos\nu_2}{|\lambda-\overline{Z}|}\right)$$

$$=\sin(\nu_1+\nu_2)\frac{\cos^2\nu_1+\cos^2\nu_2}{\cos\nu_1\ \cos\nu_2}$$

and

$$2Y\ \mathrm{Im}\left(\frac{1}{\lambda-Z}+\frac{1}{\lambda-\overline{Z}}\right)=(|\lambda-Z|\ \sin\nu_1+|\lambda-\overline{Z}|\ \sin\nu_2)\left(\frac{\sin\nu_1}{|\lambda-Z|}+\frac{\sin\nu_2}{|\lambda-\overline{Z}|}\right)$$

$$=\sin^2(\nu_1+\nu_2)\frac{\cos(\nu_1-\nu_2)}{\cos\nu_1\ \cos\nu_2}.$$

Thus integrating these equalities on Γ, we have (3. 2) and (3. 3).

We observe here the behaviour of ν_1 and ν_2 It is easy to see that

$$\pi/2<\nu_1,\ \nu_2<\pi-\theta,\ \ \pi<\nu_1+\nu_2<2\pi-2\theta; \tag{3. 4}$$

$$-\theta<\nu_1,\ \nu_2<\pi/2,\ \ 0<\nu_1+\nu_2<\pi \tag{3. 5}$$

according as Re $\lambda<X$, Re $\lambda>X$ respectively. Furthermore we have

LEMMA 3. 1. ν_1 and ν_2 satisfy

$$|\nu_1-\nu_2|<\pi/4+\theta \tag{3. 6}$$

for all $\lambda\in\Gamma$.

PROOF. When Re $\lambda=X$, λ_1 and ν_2 are equal to $\pi/2$ and the lemma is trivial. Suppose that Re $\lambda\neq X$. Then combining

$$|\mathrm{Im}\ \lambda|\leqq\tan\theta\ |\mathrm{Re}\ \lambda|$$

and

$$\begin{cases}\mathrm{Re}\ \lambda=\dfrac{2Y\cos\nu_1\cos\nu_2+X\sin(\nu_1+\nu_2)}{\sin(\nu_1+\nu_2)},\\[2ex] \mathrm{Im}\lambda=\dfrac{Y\sin(\nu_1-\nu_2)}{\sin(\nu_1+\nu_2)}\end{cases}$$

which is derived from

$$\tan\nu_1 = \frac{Y-\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda - X}, \qquad \tan\nu_2 = \frac{Y+\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda - X},$$

we have

$$Y|\sin(\nu_1-\nu_2)| \leqq \tan\theta\,|2Y\cos\nu_1\cos\nu_2 + X\sin(\nu_1+\nu_2)|.$$

Noting (3. 4) and (3. 5) we have

$$Y\sin|\nu_1-\nu_2| \leqq \tan\theta\{Y\cos(\nu_1-\nu_2)+\sqrt{X^2+Y^2}\}$$

and hence

$$\sin(|\nu_1-\nu_2|-\theta) \leqq \frac{\sqrt{X^2+Y^2}}{Y}\sin\theta < \sin(\pi/4)$$

because of $Y > \sqrt{\sec 2\theta - 1}\,X$. But (3. 4) and (3. 5) imply that

$$-\theta \leqq |\nu_1-\nu_2|-\theta < \pi/2.$$

Hence $|\nu_1-\nu_2|-\theta < \pi/4$ follows. Q. E. D.

The most important step in the proof of Theorem I is given in the following

LEMMA 3. 2. i) *If* $\mathrm{Re}\lambda < X$,

$$|2\cos\nu_1\cos\nu_2(\nu_1+\nu_2-2\pi) - \sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)|$$
$$\leqq C_\theta \sin^2(\nu_1+\nu_2)\cos(\nu_1-\nu_2). \tag{3. 7}$$

ii) *If* $\mathrm{Re}\,\lambda > X$,

$$|2\cos\nu_1\cos\nu_2(\nu_1+\nu_2) - \sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)|$$
$$\leqq C_\theta \sin^2(\nu_1+\nu_2)\cos(\nu_1-\nu_2). \tag{3. 8}$$

Here C_θ equals to $\pi/\cos(\pi/4+\theta)$ for $\theta\neq 0$ and to π for $\theta=0$.

PROOF. The proof of ii). From (3. 5)

$$|\lambda_1-\nu_2| < \pi-(\nu_1+\nu_2), \quad 0<\nu_1+\nu_2<\pi$$

and hence

$$-\cos(\nu_1+\nu_2) < \cos(\nu_1-\nu_2) \leqq 1$$

follows. Noting this and writing as

$$2\cos\nu_1\cos\nu_2(\nu_1+\nu_2) - \sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)$$
$$=\cos(\nu_1-\nu_2)\{(\nu_1+\nu_2)-\sin(\nu_1+\nu_2)\cos(\nu_1+\nu_2)\} + \{(\nu_1+\nu_2)\cos(\nu_1+\nu_2)-\sin(\nu_1+\nu_2)\},$$

we have

$$|2\cos\nu_1\cos\nu_2(\nu_1+\nu_2) - \sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)|$$

$$\leqq \operatorname{Max}\{(\nu_1+\nu_2-\sin(\nu_1+\nu_2))(1+\cos(\nu_1+\nu_2)),\ \sin^3(\nu_1+\nu_2)\}.$$

Here $(\nu_1+\nu_2-\sin(\nu_1+\nu_2))(1+\cos(\nu_1+\nu_2))$ is dominated by

$$4\int_0^{\nu_1+\nu_2}\sin^2(x/2)dx\ \cos^2\frac{\nu_1+\nu_2}{2}\leqq\pi\sin^2(\nu_1+\nu_2).$$

Thus it follows that for $\operatorname{Re}\lambda>X$

$$|2\cos\nu_1\cos\nu_2(\nu_1+\nu_2)-\sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)|$$

$$\leqq\pi\sin^2(\nu_1+\nu_2). \tag{3. 9}$$

Moreover the right of this inequality is dominated by

$$M\sin^2(\nu_1+\nu_2)\cos(\nu_1-\nu_2)\quad\text{with } M=\sup\{\pi/\cos(\nu_1-\nu_2)\}.$$

Making use of (3. 6) and replacing M with C_θ, we get (3. 8). In fact, if $\theta=0$, then $\nu_1=\nu_2$ and hence (3. 8) reduces to (3. 9).

The proof of i). In view of ii) we have (3. 8) for ν_1 and ν_2 satisfying

$$\theta<\nu_2,\ \nu_1<\pi_2,\ \ 2\theta<\nu_1+\nu_2<\pi,\ \ |\nu_1-\nu_2|<\theta+\pi/4.$$

Replacing ν_1, ν_2 by $\pi-\nu_1$, $\pi-\nu_2$ respectively, we have (3. 7) for ν_1, ν_2 satisfying (3. 4) and (3. 6). Thus i) has been proved. Q. E. D.

We are now in a position to prove Theorem I.

Putting $\Gamma_X=\{\lambda\in\Gamma:\ \operatorname{Re}\lambda\leqq X\}$ and remembering (3. 1), (3. 2), we can write

$$\operatorname{Re} I(Z)-\frac{Y}{2\pi}\operatorname{Re}\{f(Z)+f(Z)\}-\mu(\Gamma_X)$$

$$=\frac{1}{2\pi}\int_{\Gamma_X}\left\{\nu_1+\nu_2-2\pi-\frac{\sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)}{2\cos\nu_1\cos\nu_2}\right\}\mu(d\lambda)$$

$$+\frac{1}{2\pi}\int_{\Gamma-\Gamma_X}\left\{\nu_1+\nu_2-\frac{\sin(\nu_1+\nu_2)(\cos^2\nu_1+\cos^2\nu_2)}{2\cos\nu_1\cos\nu_2}\right\}\mu(d\lambda).$$

Thus assuming $\mu\{\lambda\in\Gamma:\ \operatorname{Re}\lambda=X\}=0$ and using Lemma 3. 2, we conclude that

$$\left|\mu(\Gamma_X)-\operatorname{Re} I(Z)+\frac{Y}{2\pi}\operatorname{Re}\{f(Z)+f(Z)\}\right|$$

$$\leqq\frac{1}{2\pi}\int_\Gamma C_\theta\frac{\sin^2(\nu_1+\nu_2)\cos(\nu_1-\nu_2)}{2\cos\nu_1\cos\nu_2}\mu(d\lambda)=(C_\theta/2\pi)Y\operatorname{Im}\{f(Z)-f(\overline{Z})\}$$

because of (3. 3). Q. E. D.

4. Proof of Theorem II

To prove Theorem II we shall apply Theorem I to the function $f(z)$ such that

$$f(z)=a(-z)^{-\alpha}+O(|z|^{\gamma p-\beta}/d(z,\Gamma)^p)$$

as $|z|\to\infty$ in the rigion of the complex plane: $d(z, \Gamma)\geqq|z|^{\gamma}$. Here constants α, β, γ, p and a have been specified in Section 2. We denote by $(-z)^{-\alpha}$ the analytic branch of the power in the complex plane cut along the positive axis which is positive on the negative axis.

Put $Z=X+iY$, where $X>0$ and $Y=\sqrt{\sec 2\theta-1}\,X+(1+C^2)^{\gamma/2}\sec\theta\, X^{\gamma}$ for a constant C with $C>\sqrt{\sec 2\theta-1}$ and let $L(Z)$ be an oriented curve running from $\overline{Z}$ to Z through

$$\{X+iY:\ Y\leqq|y|\leqq CX\}\cup\{z:\ |z|=\sqrt{1+C^2}\,X,\ \mathrm{Re}\, z\leqq X\}.$$

Evidently $L(Z)$ does not intersect Γ and moreover it holds that

$$d(z, \Gamma)\geqq|z|^{\gamma} \ \text{ for all } \ z\in L(Z)$$

if X is sufficiently large. Noting that

$$\begin{cases} f(Z)=a(-Z)^{-\alpha}+O(|Z|^{-\beta}), \\ |Z|^{-\alpha}=\cos^{\alpha/2}2\theta\, X^{-\alpha}+O(X^{\gamma-\alpha-1}), \\ \arg Z=\tan^{-1}\sqrt{\sec 2\theta-1}+O(X^{\gamma-1}), \end{cases}$$

we have

$$\begin{aligned} f(\overline{Z})+f(Z)&=2a|Z|^{-\alpha}\cos(\alpha(\pi-\arg Z))+O(|Z|^{-\beta}) \\ &=2a\cos^{\alpha/2}2\theta\cos(\alpha(\pi-\varphi))X^{-\alpha}+O(X^{\gamma-\alpha-1})+O(X^{-\beta}). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &Y\,\mathrm{Re}\{f(Z)+f(\overline{Z})\} \\ &=2\sqrt{2}\,\mathrm{Re}\, a\sin\theta\sec^{(1-\alpha)/2}2\theta\cos(\alpha(\pi-\varphi))X^{1-\alpha}+O(X^{1-\beta})\sqrt{\sec 2\theta-1}+O(X^{\gamma-\alpha}) \end{aligned} \tag{4. 1}$$

and similarly

$$\begin{aligned} &Y\,\mathrm{Im}\{f(Z)-f(\overline{Z})\} \\ &=2\sqrt{2}\,\mathrm{Re}\, a\sin\theta\sec^{(1-\alpha)/2}2\theta\sin(\alpha(\pi-\varphi))X^{1-\alpha}+O(X^{1-\beta})\sqrt{\sec 2\theta-1}+O(X^{\gamma-\alpha}) \end{aligned} \tag{4. 2}$$

as $X\to\infty$

On the other hand $I(Z)$ is written as

$$I(Z)=\frac{1}{2\pi i}\int_{L(Z)}\{f(z)-a(-z)^{-\alpha}\}\,dz+\frac{a}{2\pi i}\int_{L(Z)}(-z)^{-\alpha}\,dz=I_1(Z)+I_2(Z).$$

Here $I_1(Z)$ is evaluated as

$$|I_1(Z)|\leqq\text{const.}\int_{L(Z)}|z|^{\gamma p-\beta}/d(z, \Gamma)^{p}\,|dz|$$

$$\leqq \text{const. } X^{\gamma p-\beta+1-p} \leqq \text{const. } X^{\gamma-\beta}$$

because of $d(z, \Gamma) \geqq CX - \tan\theta\, X$ for any $z \in L(Z)$. We next find

$$I_2(Z) = a|Z|^{1-\alpha} \frac{\sin((1-\alpha)(\pi - \arg Z))}{(1-\alpha)\pi}$$

$$= a \sec^{(1-\alpha)/2} 2\theta \frac{\sin((1-\alpha)(\pi-\varphi))}{(1-\alpha)\pi} X^{1-\alpha} + O(X^{\gamma-\alpha})$$

since $|Z|^{1-\alpha} = \sec^{(1-\alpha)/2} 2\theta X^{1-\alpha} + O(X^{\gamma-\alpha})$.

Thus we conclude that as $X \to \infty$

$$\operatorname{Re} I(Z) = \operatorname{Re} a \sec^{(1-\alpha)/2} 2\theta \frac{\sin((1-\alpha)(\pi-\varphi))}{(1-\alpha)\pi} X^{1-\alpha} + O(X^{\gamma-\alpha}). \tag{4. 3}$$

Hence combining (4. 1), (4. 2) and (4. 3) with the formula of Theorem I, we complete the proof of Theorem II.

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References

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