Boundary representations of a tensor product of C*-algebras

By

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1. Introduction

In [1] Arveson gives a non-commutative generalization of Choquet boundary and Silow boundary. We shall study those of a tensor product of C*-algebras.

If E is a subspace of a C*-algebra and M_n is the algebra of $n\times n$ complex matrices, then the algebraic tensor product $E\otimes M_n$ is the set of $n\times n$ matrices with entries in E. If φ: $E\rightarrow F$ is a linear map from one linear space into another, then, for each positive integer n, define φ_n: $E\otimes M_n \longrightarrow F\otimes M_n$ by applying element by element to each matrix over E, i.e. $\varphi_n(T_{ij})=(\varphi(T_{ij}))$. φ is called completely positive (resp., completely isometric) if each φ_n is positive (resp., isometric).

Following Arveson [1], let B be a C*-algebra with unit and A a subspace of B which contains unit and generates B as a C*-algebra.

An irreducible representation π of B is called a boundary representation for A if the restriction $\pi|A$ has a unique completely positive linear extension to B.

A closed two-sided ideal J in B is called a boundary ideal for A if the canonical quotient map q_J: $B\longrightarrow B/J$ is completely isometric on A.

A boundary ideal is called the Silov boundary for A if it contains every other boundary ideal.

A is called an admissible subspace of B if the intersection of the kernels of the boundary representatoins for A is a boundary ideal for A.

Throughout this paper, we use the following notations. Let B_1 and B_2 be C*-algebras, and let for each $i=1, 2$, e_i be unit in B_i, A_i a subspace of B_i which contains e_i and generates B_i as a C*-algebra.

2. Boundary representations

Let $A_1\otimes A_2$ be the algebraic tensor product, and $B_1\underset{\alpha}{\otimes}B_2$ the C*-tensor product [3]. Then $A_1\otimes A_2$ generates $B_1\underset{\alpha}{\otimes}B_2$ as a C*-algebra.

THEOREM 1. *Let π_1 (resp., π_2) be a boundary representation of B_1 (resp., B_2) for A_1 (resp., A_2). Then $\pi_1\otimes\pi_2$ is a boundary representation of $B_1\underset{\alpha}{\otimes}B_2$ for $A_1\otimes A_2$.*

PROOF. Let φ be a completely positive extension to $B_1\underset{\alpha}{\otimes}B_2$ of the restriction $\pi_1\otimes\pi_2|_{A_1\otimes A_2}$. Then there is a representation π of $B_1\underset{\alpha}{\otimes}B_2$ on a Hilbert space H such that

$$\varphi(x)=H_1\otimes H_2\pi(x)H_1\otimes H_2,\quad x\in B_1\underset{\alpha}{\otimes}B_2,$$

where H_1 and H_2 are representation spaces of π_1 and π_2.

Let $L(H_1)$ and $L(H_2)$ be the C*-algebras of all bounded operators on H_1 and H_2. We define the bounded linear map $L_{\xi,\eta}$ of $L(H_1)\underset{\alpha}{\otimes}L(H_2)$ to $L(H_1)$ by

$$L_{\xi,\eta}(x\otimes y)=(y\xi|\eta)x,\quad x\in L(H_1),\ y\in L(H_2),\ \xi,\eta\in H_2.$$

Then $L_{\xi,\xi}$ is a completely positive map. By [1: Theorem 1. 2. 9] it has a completely positive extension to $L(H_1\otimes H_2)$, and is also denoted by $L_{\xi,\xi}$.

Then the map: $a\to L_{\xi,\xi}\varphi(a\otimes e_2)$ is completely positive and we have

$$L_{\xi,\xi}\varphi(a\otimes e_2)=(\xi|\xi)\pi_1(a),\quad a\in A_1.$$

Since π_1 is a boundary representation of B_1 for A_1, we have

$$L_{\xi,\xi}\varphi(a\otimes e_2)=(\xi|\xi)\pi_1(a),\quad a\in B_1.$$

Since $L_{\xi,\eta}$ is a linear combination of maps of the form $L_{\xi,\xi}$, we have

$$L_{\xi,\eta}\varphi(a\otimes e_2)=(\xi|\eta)\pi_1(a),\quad a\in B_1.$$

Hence we have

$$\varphi(a\otimes e_2)=\pi_1(a)\otimes I_{H_2},\quad a\in B_1.$$

Consequently, by [1; p. 174], $H_1\otimes H_2$ is a invariant subspace for $\pi(B_1\underset{\alpha}{\otimes}B_2)$.

Similarly, we have $\varphi(e_1\otimes b)=I_{H_1}\otimes\pi_2(b)$, $b\in B_2$, and $H_1\otimes H_2$ is a invariant subspace for $\pi(e_1\otimes B_2)$.

Hence we have

$$\begin{aligned}\varphi(a\otimes b)&=H_1\otimes H_2\pi(a\otimes b)H_1\otimes H_2\\&=H_1\otimes H_2\pi(a\otimes e_2)\pi(e_1\otimes b)H_1\otimes H_2\\&=\pi_1(a)\otimes\pi_2(b),\quad a\in B_1,\ b\in B_2.\end{aligned}$$

Consequently, we have $\varphi=\pi_1\otimes\pi_2$. This completes the proof.

In [2] Hopenwasser proved the following result.

Let B be a C-algebra with unit e_b. Let S be a linear subspace of $B\otimes M_n$ which generares $B\otimes M_n$ and which contains the set of matrix units $e_b\otimes e_{ij}$, $i, j=1,\dots, n$. Let T be the set of operators in B which appear as a matrix entry in some element of S. Then an irreducible representation π of B is a unique completely positive extension of $\pi|_T$ to B if and only if $\pi\otimes I_n$*

is a boundary representation for S.

We shall give the proof of the "if" part in a slightly different way.

PROOF. Let π be a boundary representation for T, acting on the Hilbert space H, and let φ be a completely positive extension to $B\otimes M_n$.

Then, by [1: p. 146], we have a representation π_b of $B\otimes M_n$ such that

$$\varphi(x\otimes y)=H\otimes H_n\pi_b(x\otimes y)H\otimes H_n,\quad x\in B, \text{ and } y\in M_n,$$

where H_n is n-dimensional Hilbert space.

Since $e_b\otimes e_{ij}\in S$,

$$\varphi(e_b\otimes e_{ij})=P\pi_b(e_b\otimes e_{ij})P=I_H\otimes e_{ij},$$

where P is the projection on $H\otimes H_n$.

Hence the map: $x\longrightarrow\varphi(e_b\otimes x)$ is a representation of M_n, and so P is invariant for $\pi_b(e_b\otimes M_n)$.

Now, we have

$$\varphi(x\otimes e_{ij})=P\pi_b(x\otimes e_n)\pi_b(e_b\otimes e_{ij})P$$
$$=P\pi_b(x\otimes e_n)PI_H\otimes e_{ij},$$

where e_n is unit of M_n.

On the other hand, we have

$$\varphi(x\otimes e_{ij})=P\pi_b(e_b\otimes e_{ij})\pi_b(x\otimes e_n)$$
$$=I_H\otimes e_{ij}P\pi_b(x\otimes e_n)P.$$

Hence, we have $P\pi_b(x\otimes e_n)P\in(I_H\otimes L(H_n))'$, and so there is a positive linear map ρ such that

$$P\pi_b(x\otimes e_n)P=\rho(x)\otimes I_{H_n}.$$

Since we have for each $s\in S$, $\varphi\otimes I_n(s)=\pi\otimes I_n(s)$, we have $\rho=\pi$ on T.

On the other hand, the map: $x\longrightarrow\varphi(x\otimes e_n)$ is completely positive, and π is a boundary representation for T we have $\pi=\rho$ on B.

Then P is invariant for $\pi_b(B\otimes e_n)$.

Consequently, we have $\varphi=\rho\otimes I_n=\pi\otimes I_n$. This completes the proof.

3. Boundary ideals

We assume throughout this section, for each $i=1, 2$, B_i acts on a Hilbert space H_i.

THEOREM 2. *Let J_i be a boundary ideal for A_i of B_i. Then ker $(q_{J_1}\otimes q_{J_2})$ is a boundary ideal of $B_1\underset{\alpha}{\otimes}B_2$ for $A_1\otimes A_2$.*

PROOF. The map $q_{J_1}(a)\longrightarrow a$ is completely isometric on $q_{J_1}(A_1)$ by [1: Theorem 1.

2. 9], this map has a completely positive linear extension to B_1/J_1. There are a representation π_1 of B_1/J_1 and a linear isometric map V_1 from H_1 into a representation space H_{π_1} of π_1 such that

$$a=V_1^*\pi_1(q_{J_1}(a))V_1, \quad a\in A_1.$$

Similary, there are a representation π_2 of B_2/J_2 and a linear isometric map V_2 from H_2 into a representation space H_{π_2} of π_2 such that

$$b=V_2^*\pi_2(q_{J_2}(b))V_2, \quad b\in A_2.$$

We have for $a\in A_1$ and $b\in A_2$

$$a\otimes b=(V_1\otimes V_2)^*\pi_1\otimes\pi_2(q_{J_1}(a)\otimes q_{J_2}(b))V_1\otimes V_2.$$

Hence the map: $q_{\ker(q_{J_1}\otimes q_{J_2})}(x)\longrightarrow x$ is completely contractive.

Consequently, $\ker(q_{J_1}\otimes q_{J_2})$ is a boundary ideal.

THEOREM 3. *Let A_1 (resp., A_2) be an admissible subspace of B_1(resp., B_2), and K_1(resp., K_2) be the intersection of all kernels of boundary representations of B_1(resp., B_2) for A_1(resp., A_2). Then $A_1\otimes A_2$ is an admissible subspace of $B_1\underset{\alpha}{\otimes}B_2$, and $\ker(q_{K_1}\otimes q_{K_2})$ is the Silov boundary for $A_1\otimes A_2$.*

PROOE. Let $\mathcal{B}_i$ denote the set of boundary representations of B_i for A_i, and let $\rho_i=\sum_{\pi_{ij}\in\mathcal{B}_i}\oplus\pi_{ij}$ be the direct sum of boundary representations of B_i. Let J be the intersection of the kernels of representations of the form $\pi_{1m}\otimes\pi_{2n}$ where π_{1m} and π_{2n} are boundary representations of B_1 and B_2. Since $q_{K_1}\otimes q_{K_2}$ $(B_1\underset{\alpha}{\otimes}B_2)$ is $*$-isomorphic to $\rho_1\otimes\rho_2(B_1\underset{\alpha}{\otimes}B_2)$, we have

$$\ker(q_{K_1}\otimes q_{K_2})=J.$$

Let K be the intersection of all kernels of boundary representations of $B_1\underset{\alpha}{\otimes}B_2$ for $A_1\otimes A_2$.

By Theorem 1, $\pi_{1m}\otimes\pi_{2n}$ is a boundary representation, then we have

$$J\supset K.$$

On the other hand, by Theorm 2, $\ker(q_{K_1}\otimes q_{K_2})$ is a boundary ideal. Therefore, K is a boundary ideal, and so $A_1\otimes A_2$ is admissible. Then K is the Silov boundary ideal [1: Theorem 2. 2. 3], hence we have

$$K\supset\ker(q_{K_1}\otimes q_{K_2}).$$

Consequently, we have

$$K=\ker(q_{K_1}\otimes q_{K_2}).$$

This completes the proof.

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References

[1] W. ARVESON: *Subalgebras of C*-algebras.* Acta Math., 123 (1969), 141–224.
[2] A. HOPENWASSER: *Boundary representations on C*-algebras with marix units.* Trans. Amer. Math. Soc., 177 (1973), 483–490.
[3] A. WULFSOHON: *Produit tensoriel de C*-algèbres.* Bull. Sci. Math., 87 (1963), 13–27.