A sequential procedure with finite memory for some statistical problem

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1. Introduction

In this paper we shall give a sequential procedure with finite memory for the following statistical problem, so that the limiting probability of making the incorrect choice is made zero: find a normal population with the same mean as $N(\theta, \sigma_1^2)$(θ and σ_1^2 are unknown to us) from m normal populations $N(\theta_i, \sigma_2^2)$(θ_i and σ_2^2 are unknown to us for $i=1, \cdots, m$). Here, it is assumed that there exists only one normal population with the same mean as $N(\theta, \sigma_1^2)$. Statistical problems like this, for example, problems of testing hypotheses with finite memory, were investigated by T. M. Cover [1] and [2]. Let $N(\theta, \sigma_1^2)$ be denoted by Π and $N(\theta_i, \sigma_2^2)$by $\Pi_i(i=1,\cdots, m)$. After the preceding experiment let it be assumed that Π_i is decided to have the same mean as Π. Then we draw independently a sample X from Π and X_i from Π_i and make $|X-X_j|$. Comparing $|X-X_i|$ with a preassigned positive number l, we decide whether or not Π_i has the same mean as Π. If Π_i is decided not to have the same mean, we draw independently $m-1$ samples X from Π and a sample X_j from each Π_j except Π_i, respectively and make $|X-X_j|$ ($j=1, \cdots, m, j\neq i$). By comparing them with l, decide which population has the same mean as Π. If Π_i is decided to have the same mean, we proceed with the next experiment. Now we shall state finite memory. Here, there are m specified memories $T_i(i=1, \cdots, m)$. According to comparison described above, one of m memories is used. If memory T_i is used, Π_i is decided to have the same mean. That is, "memory T_i is used" is equal to "Π_i is decided to have the same mean." Hence at each experiment memory is changed.

Next, we shall describe a process of the experiments. The nth stage of the experiments consists of the d_n experiments described above, where d_n tends to infinity as $n\longrightarrow\infty$. We call "Π_i is favorable at the nth stage" if after the d_n experiments memory T_i is used. Therefore in this statistical problem we use only m memories. Let $\bar{P}_i(d_n)$ denote the probability of memory T_i at the nth stage, that is, the probability of Π_i being decided to have the same mean after the d_n experiments. We denote by $P_i(n)$ the stationary probability that Π_i is favorable at the nth stage by using a Markov chain $M(n)$ described in the next section. When Π_1 has truly the same mean as Π, according to the sequential procedure stated in the next section, it can be shown that $\sum_{n=1}^{\infty} \overline{P_1}(d_n) = \infty$ and $\sum_{n=1}^{\infty} \overline{P_i}(d_n) < \infty$ for $i = 2, \cdots, m$. Therefore by the Borel zero-one law it is found that with probability one memory T_1 is used an infinite number of times and memory T_i $(i = 2, \cdots, m)$ are used only a finite number of times, that is, Π_1 is decided to have the same mean an infinite number of times and Π_i $(i = 2, \cdots, m)$ are decided to have the same mean only a finite number of times. This shows that the limiting probability of making the incorrect choice is made zero.

This paper consists of three sections. In Section 2 we shall describe the procedure with finite memory. In Section 3 we shall prove several lemmas, and then by using them a theorem will be established.

2. The procedure with finite memory

First we shall state the experiments. As described in Section 1, we make the d_n experiments at the nth stage. After the experiments at the nth stage we go on to the $(n+1)$th stage and successively continue these stages. Now we shall describe the nth stage in detail. It is assumed that Π_i is decided to have the same mean at the rth experiment on the nth stage. Then at the $(r+1)$th experiment we draw independently a sample X_n and X_{in} from Π and Π_i, respectively. If $|X_n - X_{in}| \leqslant l_n$, Π_i is favorable. If $|X_n - X_{in}| > l_n$, we draw $(m-1)$ independent samples X_n from Π and a sample X_{jn} from each Π_j except Π_i, respectively. Thus random variables $\{X_{kn}\}$ are mutually independent for all values of k and n, $k = 1, \cdots, m$ and $n = 1, 2, \cdots$. If there exist $j_1, \cdots, j_h$ such that $|X_n - X_{j_t n}| \leqslant l_n$ for $t = 1, \cdots, h$ and $|X_n - X_{kn}| > l_n$ for $k \neq j_1, \cdots j_h$, where l_n is a positive real number such that $\sum_{n=1}^{\infty} 1/l_n^2 < \infty$, e.g. $l_n = n$, Π_{j_t} $(t = 1, \cdots, h)$ are favorable with equal probability $1/h$, that is, Π_{j_t} $(t = 1, \cdots, h)$ are decided to have the same mean with equal probability $1/h$. Otherwise Π_i is favorable. We set $A_i(n) = \Pr(|X_n - X_{in}| > l_n)$. A random variable X_n is normally distributed with mean θ and variance σ_1^2, and a random variable X_{in} is normally distributed with mean θ_i and variance σ_2^2, so a random variable $X_n - X_{in}$ is normally distributed with mean $\theta - \theta_i$ and variance $\sigma_1^2 + \sigma_2^2$. The following figure shows a state transition of the memories from the rth experiment to the $(r+1)$th experiment at the nth stage.

$$T_i \nearrow T_{j_1},\ \frac{1}{h}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{\substack{k\neq j_t\\ k=i}}A_k(n),$$

$$\vdots$$

$$T_i \searrow T_{j_h},\ \frac{1}{h}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{\substack{k\neq j_t\\ k=i}}A_k(n),$$

if $|X_n-X_{in}|>l_n$ and $|X_n-X_{j_tn}|\leqslant l_n$ $(t=1,\cdots,h)$ and $|X_n-X_{kn}|>l_n$ $(k\neq j_1,\cdots,j_h,i)$

$T_i \longrightarrow T_i$, otherwise

Fig. A state transition

We note that at the first experiment on each stage the experiment will be done, T_1 assuming to be used. This precaution yields independence of each stage. Now let the transition probability matrix of the Markov chain at the nth stage be denoted by $M(n)=(P_{ij}(n))$, where $P_{ij}(n)$ is a transition probability from memory T_i to memory T_j being used for i, $j=1,\cdots,m$. Therefore the experiments at the nth stage turn out that the experiment is done d_n times by using this matrix $M(n)$. Let T_k be denoted by $k(k=1,\cdots,m)$. From the figure we get

$$P_{ij}(n)=\sum_{h=1}^{m-1}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{\substack{k\neq j_t\\ k=i}}A_k(n)$$

for $j\neq i$, where $R_h=(j_1,\cdots,j_h)$, $j_t\in\{1,\cdots,m\}$ $(t=1,\cdots,h)$ and $\sum\limits_{\substack{R_h\not\ni i\\ R_h\ni j}}$ means the summation of all combinations of R_h such that $j_t=j$ for some $t(t=1,\cdots,h)$ and $j_t\neq i$ for all $t(t=1,\cdots,h)$, and $\prod\limits_{\substack{k\neq j_t\\ k=i}}$ means the multiplication of all values of k such that $k\neq j_t$ for all $t(t=1,\cdots,h)$ and $k=i$. Let $\overline{P_i}(d_n)$ and $P_i(n)$ denote the same notations as in Section 1. For sufficiently large d_n, $\overline{P_i}(d_n)$ is nearly equal to $P_i(n)$, so $\overline{P_i}(d_n)$ is nearly equal to $P_i(n)$ for sufficiently large n because of $d_n\to\infty$ as $n\to\infty$. Thus for sufficiently large n we may regard the probability of memory T_i being used at the nth stage as $P_i(n)$. Hence when Π_1 has truly the same mean as Π, to show that $\sum_{n=1}^{\infty}\overline{P_1}(d_n)=\infty$ and $\sum_{n=1}^{\infty}\overline{P_i}(d_n)<\infty$ for $i=2,\cdots,m$, it suffices to show that $\sum_{n=1}^{\infty}P_1(n)=\infty$ and $\sum_{n=1}^{\infty}P_i(n)<\infty$ for $i=2,\cdots,m$. In Section 3 we shall assume that Π_1 has truly the same mean as Π. The properties of the stationary probabilities $P_i(n)$ will be stated in the next section.

3. Proof of lemmas and a theorem

Without loss of generality we may assume that

$$0=|\theta-\theta_1|<|\theta-\theta_2|\leqslant\cdots\cdots\leqslant|\theta-\theta_m|.$$

From the preservation of probabilities at each stage, we obtain

$$P_1(n) = 1 - \sum_{i=2}^{m} P_i(n) \qquad \text{for } n = 1, 2, \cdots.$$

If $\sum_{n=1}^{\infty} P_i(n) < \infty$ for $i = 2, \cdots, m$, then $\sum_{n=1}^{\infty} P_1(n) = \infty$. Thus by the Borel zero-one law it follows that with probability one Π_1 is decided to have the same mean as Π an infinite number of times and $\Pi_i (i = 2, \cdots, m)$ are decided to have the same mean as Π only a finite number of times. This shows that the limiting probability of making the incorrect choice is made zero.

THEOREM. *We assume that*

$$0 = |\theta - \theta_1| < |\theta - \theta_2| \leqslant \cdots\cdots \leqslant |\theta - \theta_m|.$$

Then we obtain

$$\sum_{n=1}^{\infty} P_i(n) < \infty \qquad \textit{for} \quad i = 2, \cdots, m$$

and

$$\sum_{n=1}^{\infty} P_1(n) = \infty.$$

The proof of this theorem will be given afterward.

First, we mention without proof the lemmas 1 and 2 given by K. TANAKA and E. ISOGAI [5].

LEMMA 1.

$$P_i(n) = \frac{\bar{\alpha}_i(n)}{\sum_{k=1}^{m} \bar{\alpha}_k(n)} \qquad \textit{for} \quad i = 1, \cdots\cdots, m-1,$$

where

$$\bar{\alpha}_i(n) = \begin{cases} (-1) \sum_{j=1}^{m-1} \bar{P}_{mj}(n) D_{ij}(n) & \textit{for} \quad i = 1, \cdots\cdots, m-1 \\ \det(\bar{P}(n)) & \textit{for} \quad i = m \end{cases}$$

and $\bar{P}(n) = (\bar{P}_{ij}(n))$,

$$\bar{P}_{ij}(n) = \begin{cases} -(P_{i1}(n) + \cdots + P_{ii-1}(n) + P_{ii+1}(n) + \cdots + P_{im}(n)) & \text{for } i = j \\ P_{ij}(n) & \textit{for} \quad i \neq j \end{cases}$$

$(i, j = 1, \cdots, m-1)$,

$$\Delta_{ij}(n) = \begin{vmatrix} \bar{P}_{11}(n) \cdots\cdots & \overset{\widehat{j}}{\cdots} \bar{P}_{m-11}(n) \\ \vdots & \vdots \\ \bar{P}_{1m-1}(n) \cdots & \cdots \bar{P}_{m-1m-1}(n) \end{vmatrix} (\,\widehat{i}\,,$$

a symbol "$\wedge$" denotes an exception of the row or column of the corresponding number and $D_{ij}(n)=(-1)^{i+j}\Delta_{ij}(n)$.

LEMMA 2. *For every i, j (i, j=1,⋯, m−1), we have*

$$D^m_{ij}(n)>0 \text{ and } \det(\overline{P}^m(n))<0 \text{ if } m \text{ is even,}$$

and

$$D^m_{ij}(n)<0 \text{ and } \det(\overline{P}^m(n))>0 \text{ if } m \text{ is odd,}$$

where $D^m_{ij}(n)$ *denotes the dependence of* $D_{ij}(n)$ *on m.*
Next, we shall prove the following lemma.

LEMMA 3. *We assume that*

$$0=|\theta-\theta_1|<|\theta-\theta_2|\leqslant\cdots\cdots\leqslant|\theta-\theta_m|.$$

Let A_i ($i=1,\cdots,m$) *be the same notations as in Section 2. Then, there exists a positive integer* N_0 *such that for every* $n\geqslant N_0$

$$A_1(n)<A_2(n)\leqslant\cdots\cdots\leqslant A_m(n).$$

PROOF. A random variable X_n-X_{in} is normally distributed with mean $\theta-\theta_i$ and variance $\sigma_1^2+\sigma_2^2$, so

$$\begin{aligned}
A_j(n)&=Pr(|X_n-X_{jn}|>l_n)\\
&=\int_{l_n}^{\infty}(2\pi)^{-\frac{1}{2}}(\sigma_1^2+\sigma_2^2)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2(\sigma_1^2+\sigma_2^2)}(x-(\theta-\theta_j))^2\right\}dx\\
&+\int_{-\infty}^{-l_n}(2\pi)^{-\frac{1}{2}}(\sigma_1^2+\sigma_2^2)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2(\sigma_1^2+\sigma_2^2)}(x-(\theta-\theta_j))^2\right\}dx\\
&=\int_{\frac{l_n-(\theta-\theta_j)}{\sqrt{\sigma_1^2+\sigma_2^2}}}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx+\int_{-\infty}^{\frac{-l_n-(\theta-\theta_j)}{\sqrt{\sigma_1^2+\sigma_2^2}}}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx\\
&=\int_{\frac{l_n-(\theta-\theta_j)}{\sqrt{\sigma_1^2+\sigma_2^2}}}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx+\int_{\frac{l_n+(\theta-\theta_j)}{\sqrt{\sigma_1^2+\sigma_2^2}}}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx\\
&=\int_{B_j^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx+\int_{\widetilde{B}_j^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx,
\end{aligned}$$

where $C_j=\theta-\theta_j$, $\widetilde{C}_j=\theta_j-\theta$, $B_j^n=\dfrac{C_j+l_n}{\sqrt{\sigma_1^2+\sigma_2^2}}$ and

$\widetilde{B}_j^n=\dfrac{\widetilde{C}_j+l_n}{\sqrt{\sigma_1^2+\sigma_2^2}}$.

By the assumption of the lemma, we get

$$|C_j|=|\widetilde{C}_j| \text{ and } 0=|C_1|<|C_2|\leqslant\cdots\cdots\leqslant|C_m|.$$

We put $\bar{B}_j^n=\dfrac{|C_j|+l_n}{\sqrt{\sigma_1^2+\sigma_2^2}}$ and $\bar{\bar{B}}_j^n=\dfrac{-|C_j|+l_n}{\sqrt{\sigma_1^2+\sigma_2^2}}$.

Then we obtain

$$A_j(n)=\int_{\bar{B}_j^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx+\int_{\bar{\bar{B}}_j^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx. \tag{3. 1}$$

Since $\sum_{n=1}^{\infty}1/l_n^2<\infty$, $l_n\longrightarrow\infty$ as $n\longrightarrow\infty$. Therefore, there exists a positive integer N_0 such that for every $n\geqslant N_0$ $-|C_m|+l_n>0$. From this, we get easily that

$$0<\bar{\bar{B}}_m^n\leqslant\bar{\bar{B}}_{m-1}^n\leqslant\cdots\cdots\leqslant\bar{\bar{B}}_2^n<\bar{\bar{B}}_1^n=\bar{B}_1^n<\bar{B}_2^n\leqslant\bar{B}_3^n\leqslant\cdots\cdots\leqslant\bar{B}_m^n \tag{3. 2}$$

for every $n\geqslant N_0$.

Now we shall show that $A_{j+1}(n)\geqslant A_j(n)$.

$$A_{j+1}(n)-A_j(n)$$

$$=\int_{\bar{B}_{j+1}^n}^{\bar{B}_j^n}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx-\int_{\bar{\bar{B}}_j^n}^{\bar{\bar{B}}_{j+1}^n}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx \qquad \text{(from (3. 1))}$$

$$\begin{cases}\geqslant 0 & \text{for } j=2,\cdots,m-1\\ >0 & \text{for } j=1 \text{ (by (3. 2)).}\end{cases}$$

This shows the proof of the lemma.

(Q. E. D.)

LEMMA 4. *We assume that*

$$0=|\theta-\theta_1|<|\theta-\theta_2|\leqslant\cdots\cdots\leqslant|\theta-\theta_m|.$$

Then for every $n\geqslant N_0$, *where* N_0 *is the same as in Lemma* 3, *we obtain*

$$P_{i1}(n)>P_{i2}(n)\geqslant\cdots\cdots\geqslant P_{ii-1}(n)\geqslant P_{ii+1}(n)\geqslant\cdots\cdots\geqslant P_{im}(n)$$

for $1\leqslant i\leqslant m$.

PROOF. For simplicity, we denote $P_{ij}(n)$ and $A_i(n)$ by P_{ij} andA_i, respectively. For $j\neq i$,

$$P_{ij}=\sum_{h=1}^{m-1}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t})\prod_{\substack{k\neq j_t\\ k=i}}A_k.$$

We set $(E) = \sum_{\substack{R_h \not\ni i \\ R_h \ni j}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k.$

$$(E) = \sum_{\substack{R_h \not\ni i, j+1 \\ R_h \ni j}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$+ \sum_{\substack{R_h \not\ni i \\ R_h \ni j, j+1}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k.$$

Let us consider the first part of the right hand side in the above equation.

$$\sum_{\substack{R_h \not\ni i, j+1 \\ R_h \ni j}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$= (1-A_j) \sum_{R_h \not\ni i, j+1, j} \prod_{t=1}^{h-1} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$\geqslant (1-A_{j+1}) \sum_{R_h \not\ni i, j+1, j} \prod_{t=1}^{h-1} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$= \sum_{\substack{R_h \not\ni i, j \\ R_h \ni j+1}} \prod_{i=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k,$$

because of $1-A_{j+1} \leqslant 1-A_j$.

Hence

$$(E) \geqslant \sum_{\substack{R_h \not\ni i, j \\ R_h \ni j+1}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$+ \sum_{\substack{R_h \not\ni i \\ R_h \ni j, j+1}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k$$

$$= \sum_{\substack{R_h \not\ni i \\ R_h \ni j+1}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k.$$

Therefore we obtain

$$P_{ij} \geqslant \sum_{h=1}^{m-1} \frac{1}{h} \sum_{\substack{R_h \not\ni i \\ R_h \ni j+1}} \prod_{t=1}^{h} (1-A_{j_t}) \prod_{\substack{k \neq j_t \\ k=i}} A_k = P_{ij+1}.$$

For $j=1$, we can replace "$\geqslant$" by "$>$", because of $A_1 < A_2$.
This proves Lemma 4.

(Q.E.D.)

Now, we shall consider the evaluation of P_i ($i=1,\cdots, m$),
By putting $\alpha_i=(-1)^{m-1}\bar{\alpha}_i$,

$$P_i=\frac{\bar{\alpha}_i}{\sum_{k=1}^{m}\bar{\alpha}_k} \qquad \text{(by Lemma 1)}$$

$$=\frac{(-1)^{m-1}\bar{\alpha}_i}{(-1)^{m-1}\sum_{k=1}^{m}\bar{\alpha}_k}$$

$$=\frac{\alpha_i}{\sum_{k=1}^{m}\alpha_k}, \qquad 1\leqslant i\leqslant m.$$

For $1\leqslant i\leqslant m-1$,

$$\alpha_i=(-1)^{m-1}\bar{\alpha}_i$$

$$=(-1)^{m-1}(-1)\sum_{j=1}^{m-1}P_{mj}D_{ij}^{m} \qquad \text{(by Lemma 1)}$$

$$=\sum_{j=1}^{m-1}P_{mj}(-1)^{m}D_{ij}^{m}.$$

From Lemma 2, we have $(-1)^m D_{ij}^m>0$.
Thus we get $\alpha_i>0$ for $1\leqslant i\leqslant m-1$. By Lemma 2, we have

$$\alpha_m=(-1)^{m-1}\det(\bar{P}^m)>0.$$

Therefore, we obtain

$$P_i=\frac{\alpha_i}{\sum_{k=1}^{m}\alpha_k} \quad \text{and } \alpha_i>0 \text{ for } i=1,\cdots, m.$$

LEMMA 5. *It is assumed that*

$$0=|\theta-\theta_1|<|\theta-\theta_2|\leqslant\cdots\cdots\leqslant|\theta-\theta_m|.$$

Then for every $n\geqslant N_0$, where N_0 is the same as in Lemma 3, we obtain

$$\alpha_i(n)\geqslant K_m P_{12}(n)P_{21}(n)\cdots P_{i-11}(n)P_{i+11}(n)\cdots P_{m1}(n)$$

for $i=1,\cdots, m$, and

$$\alpha_1(n)\geqslant P_{21}(n)P_{31}(n)\cdots P_{m1}(n),$$

where K_m is a constant depending on m.

PROOF. We shall show the outline of the lemma. For simplicity, we omit n. First we shall show the first part of the lemma.

$$0<(-1)^m D_{ii}$$
$$\leqslant \sum_{\tau} |\overline{P}_{\tau(1)}| \cdots |\overline{P}_{\tau(i-1)i-1}| \ |\overline{P}_{\tau(i+1)i+1}| \cdots |\overline{P}_{\tau(m-1)m-1}|,$$

where τ is a permutation on a set $\{1, 2, \cdots, i-1, i+1, \cdots, m-1\}$ and the sum is taken over all permutations on the set. According to Lemma 4, we obtain

$$|\overline{P}_{\tau(j)j}| \leqslant \begin{cases} (m-1)P_{\tau(j)1} & \text{for } 2\leqslant\tau(j)\leqslant m-1 \\ (m-1)P_{12} & \text{for } \tau(j)=1. \end{cases}$$

Hence, we get

$$(-1)^m D_{ii} \leqslant \sum_{\tau} (m-1)^{m-2} P_{12}P_{21}\cdots P_{i-11} P_{i+11}\cdots P_{m-11}\,.$$

Setting $K_m^1 = \sum_{\tau}(m-1)^{m-2}$, we have

$$0<(-1)^m D_{ii} \leqslant K_m^1 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m-11}$$

for $1\leqslant i\leqslant m-1$.

Since in the case $i<j$, we can prove

$$(-1)^m D_{ij} \leqslant K_m^1 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m-11}$$

in the same way as in the case $j<i$, we assume that $j<i$ $(2\leqslant i \leqslant m-1)$.

$$0<(-1)^m D_{ij}$$
$$\leqslant \sum_{\tau} |\overline{P}_{\tau(1)1}| \cdots |\overline{P}_{\tau(j-1)j-1}| \ |\overline{P}_{\tau(j)j+1}| \cdots |\overline{P}_{\tau(i-1)i}| \ |\overline{P}_{\tau(i+1)i+1}| \cdots |\overline{P}_{\tau(m-1)m-1}|.$$

A set $\{\tau(1), \cdots, \tau(i-1), \tau(i+1), \cdots, \tau(m-1)\}$ coincides with a set $\{1, 2, \cdots, i-1, i+1, \cdots, m-1\}$. According to Lemma 4, we have

$$|\overline{P}_{\tau(k)k'}| \leqslant \begin{cases} (m-1)P_{\tau(k)1} & \text{for } \tau(k)\neq 1 \\ (m-1)P_{12} & \text{for } \tau(k)=1 \end{cases}$$

for every $k' \in \{1, \cdots, j-1, j+1, \cdots, m-1\}$.

Hence, we obtain

$$(-1)^m D_{ij}$$
$$\leqslant \sum_{\tau}(m-1)^{m-2} P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m-11}$$
$$= K_m^1 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m-11}.$$

Thus

$$0 < (-1)^m D_{ij}$$

$$\leqslant K_m^1 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m-11}$$

for $1\leqslant i,\ j\leqslant m$.
Therefore for $1\leqslant i\leqslant m-1$,

$$\alpha_i \leqslant \sum_{j=1}^{m-1} K_m^1 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m1}.$$

Putting $K_m^2=\sum_{j=1}^{m-1} K_m^1$, we obtain

$$\alpha_i \leqslant K_m^2 P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m1}$$

for $i=1,\cdots,m-1$.
From the definition of α_m,

$$\alpha_m=(-1)^{m-1}\det(\overline{P})$$
$$\leqslant \sum_\tau |\overline{P}_{\tau(1)1}|\cdots|\overline{P}_{\tau(m-1)m-1}|$$
$$\leqslant \sum_\tau (m-1)^{m-1} P_{12}P_{21}\cdots P_{m-11}.$$

Setting $K_m^3=\sum_\tau (m-1)^{m-1}$, we obtain

$$\alpha_m\leqslant K_3{}^m P_{12}P_{21}\cdots P_{m-11}.$$

Therefore putting $K_m=\max(K_m^2,K_m^3)$, we have

$$\alpha_i\leqslant K_m P_{12}P_{21}\cdots P_{i-11}P_{i+11}\cdots P_{m1}$$

for $i=1,\cdots,m$.
This proves the first part in the lemma.

Next, we shall show the remaining part in the lemma.
Since $(-1)^m D_{1j}>0$ for $2\leqslant j\leqslant m-1$, we get

$$\alpha_1=\sum_{j=1}^{m-1} P_{mj}\,(-1)^m D_{1j}\geqslant P_{m1}(-1)^m D_{11}.$$

$$(-1)^m D_{11}=(-1)^m\sum_\tau \operatorname{sgn}(\tau)\prod_{k=2}^{m-1}\overline{P_{\tau(k)k}}$$
$$=(-1)^m\operatorname{sgn}(\tau_0)\prod_{k=2}^{m-1}\overline{P_{\tau_0(k)k}}+(-1)^m\sum_{\tau\neq\tau_0}\operatorname{sgn}(\tau)\prod_{k=2}^{m-1}\overline{P_{\tau(k)k}}$$
$$=(-1)^m\prod_{k=2}^{m-1}(-\beta_{kk})+(-1)^m\sum_{\tau\neq\tau_0}\operatorname{sgn}(\tau)\prod_{k=2}^{m-1}\overline{P_{\tau(k)k}}$$
$$\geqslant\prod_{k=2}^{m-1}\beta_{kk}-\sum_{\tau\neq\tau_0}\prod_{k=2}^{m-1}|\overline{P_{\tau(k)k}}|,$$

where τ_0 is an identical permutation.

We denote by G_1 and G_2 collections of all terms obtained by expanding $\prod_{k=2}^{m-1} \beta_{kk}$ and $\sum_{\tau \neq \tau_0} \prod_{k=2}^{m-1} |P_{\tau(k)k}|$, respectively. Then we can show that all terms in G_2 are mutually different and each term in G_2 belongs to G_1. Furthermore we can also prove that a term $P_{12}P_{21} \cdots P_{m-11}$ belongs to G_1, but not to G_2. So, we have

$$\prod_{k=2}^{m-1} \beta_{kk} - \sum_{\tau \neq \tau_0} \prod_{k=2}^{m-1} |\overline{P_{\tau(k)k}}| \geqslant P_{12}P_{21} \cdots P_{m-11}.$$

Therefore, we get $\alpha_1 \geqslant P_{12}P_{21} \cdots P_{m1}$. Thus the proof of the lemma is completed.

(Q.E.D.)

Since $$P_i(n) = \frac{\alpha_i(n)}{\sum_{k=1}^{m} \alpha_k(n)} \quad \text{for } i=1, \cdots, m, \text{ by Lemma 5 we have}$$

$$P_i(n) \leqslant \frac{\alpha_i(n)}{\alpha_1(n)}$$

$$\leqslant K_m \frac{P_{12}(n)}{P_{i1}(n)} \qquad (i=2, \cdots, m) \tag{3. 3}$$

for every $n \geqslant N_0$.

Now we shall consider the stationary probabilities $P_i(n)$ $(i = 2, \cdots, m)$. Since the probabilities $P_i(n)(i=2, \cdots, m)$ are stationary, they have the properties in the lemmas from Lemma 1 to Lemma 5.

LEMMA 6. *It is assumed that*

$$0 = |\theta - \theta_1| < |\theta - \theta_2| \leqslant \cdots\cdots \leqslant |\theta - \theta_m|.$$

Then, for $i = 2, \cdots, m$,

$$\frac{P_{12}(n)}{P_{i1}(n)} \sim \frac{A_1(n)}{A_i(n)},$$

where "$\sim$" denotes asymptotic equality.

PROOF. Since $X_n - X_{jn}$ is normally distributed with mean $\theta - \theta_j$ and variance $\sigma_1^2 + \sigma_2^2$, and $l_n \longrightarrow \infty$ as $n \longrightarrow \infty$, it is easily found that

$$A_j(n) = Pr(|X_n - X_{jn}| > l_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3. 4}$$

For $j \neq i$, we have

$$P_{ij}(n) = A_j(n) \Big[\frac{1}{m-1} \prod_{k \neq i} (1 - A_k(n))$$

$$+\sum_{h=1}^{m-2}\frac{1}{h}\sum_{\substack{R_h \not\ni i \\ R_h \ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{k\neq j_t, i}A_k(n)\Big].$$

Let us denote the inside of the above brackets by $W_{ij}(n)$.

By (3. 4), we have $W_{ij}(n)\longrightarrow\frac{1}{m-1}$ as $n\longrightarrow\infty$.

Hence

$$P_{ij}(n)\longrightarrow\frac{1}{m-1}A_i(n) \quad \text{as } n\longrightarrow\infty.$$

Therefore we obtain

$$\frac{P_{12}(n)}{P_{i1}(n)}=\frac{A_1(n)}{A_i(n)}\cdot\frac{W_{12}(n)}{W_{i1}(n)}\sim\frac{A_1(n)}{A_i(n)},$$

which concludes the proof of Lemma 6.

(Q.E.D.)

LEMMA 7. *We assume that*

$$0=|\theta-\theta_1|<|\theta-\theta_2|\leqslant\cdots\cdots\leqslant|\theta-\theta_m|.$$

For $i=2,\cdots,m$, we obtain

$$\sum_{n=1}^{\infty}\frac{A_1(n)}{A_i(n)}<\infty.$$

PROOF.

$$\frac{A_1(n)}{A_i(n)}=\frac{2\int_{\bar{B}_1^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx}{\int_{\bar{B}_i^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx+\int_{\bar{\bar{B}}_i^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx}$$

$$\leqslant\frac{2\int_{\bar{B}_1^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx}{\int_{\bar{\bar{B}}_i^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx}$$

By an inequality

$$\int_y^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx\sim\frac{1}{y}\exp\left(-\frac{1}{2}y^2\right) \quad \text{for sufficiently large } y,$$

and

$$\frac{\bar{\bar{B}}_i^n}{\bar{B}_1^n}=\frac{-|C_i|+l_n}{l_n}\longrightarrow 1 \quad \text{as } n\longrightarrow\infty,$$

we have

$$\frac{2\int_{\overline{B}_1^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx}{\int_{\overline{\overline{B}}_i^n}^{\infty}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}dx} \sim 2\exp\left[-\frac{1}{2}\left\{(\overline{B}_1^n)^2-(\overline{\overline{B}}_i^n)^2\right\}\right].$$

$$(\overline{B}_1^n)^2-(\overline{\overline{B}}_i^n)^2$$

$$=\left(l_n(\sigma_1^2+\sigma_2^2)^{-\frac{1}{2}}\right)^2-\left((-|C_i|+l_n)\cdot(\sigma_1^2+\sigma_2^2)^{-\frac{1}{2}}\right)^2$$

$$=-(\sigma_1^2+\sigma_2^2)^{-1}(C_i^2-2|C_i|l_n).$$

Therefore,

$$\exp\left[-\frac{1}{2}\left\{(\overline{B}_1^n)^2-(\overline{\overline{B}}_i^n)^2\right\}\right]$$

$$=\exp\left[\frac{C_i^2}{2(\sigma_1^2+\sigma_2^2)}\right]\cdot\exp\left[-\frac{|C_i|}{(\sigma_1^2+\sigma_2^2)}\cdot l_n\right]$$

$$\leqslant\left(\exp\left[\frac{C_i^2}{2(\sigma_1^2+\sigma_2^2)}\right]\right)\cdot\frac{2(\sigma_1^2+\sigma_2^2)^2}{C_i^2}\cdot\frac{1}{l_n^2},$$

where the above inequality follows from a simple inequality

$$e^{-x}<\frac{2}{x^2} \qquad \text{for every } x>0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{l_n^2}<\infty$, it is found that

$$\sum_{n=1}^{\infty}\exp\left[-\frac{1}{2}\left\{(\overline{B}_1^n)^2-(\overline{\overline{B}}_i^n)^2\right\}\right]<\infty.$$

Therefore, we obtain

$$\sum_{n=1}^{\infty}\frac{A_1(n)}{A_i(n)}<\infty \qquad \text{for } i=2,\cdots,m.$$

Thus, the proof of the lemma is completed.

(Q.E.D.)

Now we shall prove the theorem. From the preservation of probabilities at each stage, we may show that $\sum_{n=1}^{\infty}P_i(n)<\infty$ for $i=2,\cdots,m$.

From (3. 3), we get

$$P_i(n) \leqslant K_m \frac{P_{12}(n)}{P_{i1}(n)} \qquad \text{for } i=2,\cdots, m.$$

According to Lemmas 6 and 7, we obtain

$$\sum_{n=1}^{\infty} \frac{P_{12}(n)}{P_{i1}(n)} < \infty \qquad \text{for } i=2,\cdots, m.$$

Therefore, we have

$$\sum_{n=1}^{\infty} P_i(n) < \infty \qquad \text{for } i=2,\cdots, m,$$

which concludes the proof of the theorem.

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