On a sequential procedure with finite memory for testing statistical hypotheses

By

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1. Summary and Introduction

Many statistical procedure on testing hypotheses about the mean of a normal distribution with an unknown variance has been investigated by many people. In this paper we shall discuss the problem of the testing statistical hypotheses by using a sequential procedure with finite memory, so that the limiting probability of selecting the incorrect hypotheses is made zero. Now let a population have a normal distribution $N(\theta, \sigma^2)$, where θ and σ^2 are unknown to us. We denote the hypotheses: $\theta=\theta_i$ by H_i, $i=1, 2,\dots m$. At the preceding experiment the hypothesis H_i is assumed to be acceptable, where "we accept the hypothesis H_i" is called "the hypothesis H_i is acceptable". Then a sample X_i is drawn from $N(\theta, \sigma^2)$ and we make $|X_i-\theta_i|$. Comparing $|X_i-\theta_i|$ with a preassigned positive number l, we decide which hypothesis is acceptable. If we reject the hypothesis H_i, we draw $(m-1)$ mutually independent samples X_j from $N(\theta, \sigma^2)$ and make $|X_j-\theta_j|$, $j=1, 2,\dots, m$, and $j\neq i$. By comparing them with l, we decide which hypothesis is acceptable. Next, we shall describe finite memory. There are now m specified memories T_i, $i=1, 2,\dots, m$. According to the procedure described above, one of m memories is used. If memory T_i is used, we accept the hypothesis H_i. Hence at each experiment memory is changed.

Now we shall state a process of the experiments. The nth stage of the experiments consists of d_n experiments described above, where d_n tends to infinity as $n\to\infty$. When after d_n experiments memory T_i is used, it is said that the hypothesis H_i is acceptable at the nth stage. When after rth experiment at the nth stage memory T_i is used, it is said that the hypothesis H_i is acceptable at the rth experiment on the nth stage. Therefore, in this paper, we use only m memories in the procedure of testing statistical hypotheses. Let $\overline{P}_i(d_n)$ denote the probability that the hypothesis H_i is acceptable and $P_i(n)$ denote the stationary probability that the hypothesis H_i is acceptable on the nth stage by using a specified Markov chain $M(n)$. When the hypothesis H_1 is true, according to the sequential procedure specified in next section, it can be shown that $\sum_{n=1}^{\infty} \overline{P}_1(d_n)=\infty$ and $\sum_{n=1}^{\infty}$

$\bar{P}_i(d_n) < \infty$ for $i=2,\ldots, m$. Thus by the Borel zero-one law it is found that with probability one the hypotheses H_i $(i=2,\ldots, m)$ are acceptable only a finite number of times and the hypothesis H_1 is acceptable an infinite number of times. This shows that by using the procedure the limiting probability of selecting the incorrect hypotheses is made zero.

This paper consists of three sections. In Section 2 we shall describe the procedure with finite memory. In Section 3 several lemmas will be proved, and by using them a theorem will be established.

2. The statistical procedure with finite memory

The experiments are carried out as follows. As described in Section 1, we make d_n experiments at the nth stage. We go on to the $(n+1)$th stage after the nth stage and successively continue these stages. Now we shall describe the nth stage in detail. The memory T_i is assumed to be used after the rth experiment, that is, the hypothesis H_i is assumed to be acceptable at the rth experiment. Then at the $(r+1)$th experiment we draw a sample X_{ni} from $N(\theta, \sigma^2)$ and make $|X_{ni}-\theta_i|$. If it holds that $|X_{ni}-\theta_i| \leqslant l_n$, memory T_i is used again. Here, l_n is a positive number such that $\sum_{n=1}^{\infty} 1/l_n^2 < \infty$, for example $l_n=n$. If it holds that $|X_{ni}-\theta_i| > l_n$, furthermore we draw $(m-1)$ samples X_{nj} independently from $N(\theta, \sigma^2)$ and make $|X_{nj}-\theta_j|$, $j=1, 2,\ldots, m$, and $j \neq i$. If there exist $j_1,\ldots, j_h$ such that $|X_{nj_t}-\theta_{j_t}| \leqslant l_n$ for $t=1,\ldots, h$, and $|X_{nk}-\theta_k| > l_n$ for $k \neq j_1,\ldots, j_h$, and $k \neq i$, memory $T_{j_t}(t=1,\ldots, h)$ are used with equal probability $1/h$. Otherwise memory T_i is used. We set $A_j(n)=Pr(|X_{nj}-\theta_j| > l_n)$, $j=1, 2,\ldots, m$. The following figure shows a state transition of the memories from the rth experiment to the $(r+1)$th experiment on the nth stage.

$$
T_i \to
\begin{cases}
T_{j_1},\ \dfrac{1}{h}\prod_{t=1}^{h}(1-A_{j_t}(n)) \prod_{\substack{k \neq j_t \\ k=i}} A_k(n) \\
\quad \vdots \qquad\qquad \vdots \\
T_{j_h},\ \dfrac{1}{h}\prod_{t=1}^{h}(1-A_{jt}(n)) \prod_{\substack{k \neq j_t \\ k=i}} A_k(n)
\end{cases}
\quad \text{if} \quad
\begin{cases}
|X_{ni}-\theta_i| > l_n \\
\text{and} \\
|X_{nj_t}-\theta_{j_t}| \leqslant l_n \\
(t=1,\ \ldots, h) \\
\text{and} \\
|X_{nk}-\theta_k| > l_n \\
(k \neq j_1,\ldots, j_h, i)
\end{cases}
$$

$$
T_i \to T_i, \qquad \text{otherwise}
$$

Fig. A state transition

We note that at the first experiment on each stage the experiment will be done, T_1 assuming to be used, that is, the hypothesis H_1 assuming to be acceptable. This precaution yields independence of each stage. Now let the transition probability matrix of the Markov chain at the nth stage be denoted by $M(n)=(P_{ij}(n))$, where $P_{ij}(n)$ is a transition probability from memory T_i to memory T_j, $i, j=1, 2,\ldots, m$. Therefore the experiment at the nth stage turn out that the experiments are done d_n times by using this transition matrix $M(n)$. From now, we denote T_k by k $(k=1, 2,\dots, m)$. From the figure we get

$$P_{ij}(n)=\sum_{h=1}^{m-1}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{k\neq j_t}A_k(n),\quad j\neq i,$$

where $R_h=\{j_1,\dots, j_h\}$, $j_k\in\{1,\dots, m\}$ $(k=1,\dots, h)$, and $\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}$ means the summation of all combinations of R_h, such that $R_h\not\ni i$ and $R_h\ni j$, and $\prod_{k\neq j_t}$ the multiplication of all values of k such that $k\neq j_t$ $(t=1, 2,\dots, h)$. Let $\overline{P}_i(d_n)$ be the same notation as in Section 1. For sufficiently large d_n, $\overline{P}_i(d_n)$ is nearly equal to $P_i(n)$, so from the nature of d_n, $\overline{P}_i(d_n)$ is nearly equal to $P_i(n)$ for sufficiently large n. Hence for sufficiently large n, we may regard the probability that the memory T_i is used after the d_n experiments as $P_i(n)$. The properties of the stationary probabilities $P_i(n)$ will be described in the next section.

3. Proof of lemmas and a theorem

Our problem will be solved as follows. Without loss of generality we may assume that the hypothesis H_1 is true and $0<|\theta_1-\theta_2|\leqslant|\theta_1-\theta_3|\leqslant\cdots\leqslant|\theta_1-\theta_m|$. From the preservation of probabilities at each stage, we obtain

$$P_1(n)=1-\sum_{i=2}^{m}P_i(n)\qquad \text{for } n=1, 2,\dots.$$

If $\sum_{n=1}^{\infty}P_i(n)<\infty$ for $i=2,\dots m$, then $\sum_{n=1}^{\infty}P_1(n)=\infty$. Thus by Borel zero-one law it is found that with probability one the hypotheses H_i $(i=2,\dots, m)$ are acceptable only a finite number of times and the hypothesis H_1 is acceptable an infinite number of times. Therefore we obtain the following theorem.

THEOREM. *We assume that the hypothesis H_1 is true and* $0<|\theta_1-\theta_2|\leqslant|\theta_1-\theta_3|\leqslant\cdots\leqslant|\theta_1-\theta_m|$. *Then it holds that*

$$\sum_{n=1}^{\infty}P_i(n)<\infty\qquad \textit{for } i=2,\dots, m$$

and

$$\sum_{n=1}^{\infty}P_1(n)=\infty.$$

The proof of this theorem will be given afterward. The stationary probabilities $P_i(n)(i=1, 2,\dots, m)$ satisfy the following relations:

$$(3.\ 1)\qquad P_i(n)=\sum_{j=1}^{m}P_j(n)P_{ji}(n),\quad i=1, 2,\dots, m$$

and

$$\sum_{i=1}^{m} P_i(n)=1 \tag{3. 2}$$

We set $\overline{P}_{ij}(n)=P_{ij}(n)$ for $i\neq j$ and $\overline{P}_{ii}(n)=P_{ii}(n)-1$. In the same way as K. TANAKA, K. INADA and S. IWASE [4], we have

$$P_i(n)=\frac{\alpha_i(n)}{\sum_{j=1}^{m}\alpha_j(n)} \tag{3. 3}$$

where

$$\alpha_i(n)=(-1)^{m-1}\begin{vmatrix} \overline{P}_{11}(n)\cdots\cdots-\overset{i}{\overline{P}_{m1}}(n)\cdots\cdots\ \overline{P}_{m-11}(n) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \overline{P}_{1m-1}(n)\cdots-\overline{P}_{mm-1}(n)\cdots\ -\overline{P}_{m-1m-1}(n)\end{vmatrix}>0$$

for $i=1,\ldots, m-1$, and

$$\alpha_m(n)=(-1)^{m-1}\begin{vmatrix} \overline{P}_{11}(n)\cdots\cdots\overline{P}_{m-11}(n) \\ \vdots \qquad \vdots \\ \overline{P}_{1m-1}(n)\cdots\overline{P}_{m-1m-1}(n)\end{vmatrix}>0$$

LEMMA 1. *We assume that the hypothesis H_1 is true and* $0<|\theta_1-\theta_2|\leqslant|\theta_1-\theta_3|\leqslant\cdots\leqslant|\theta_1-\theta_m|$.
Then

$$A_1(n)<A_2(n)\leqslant\cdots\leqslant A_m(n).$$

PROOF.

$$A_j(n)=P_r(|X_{nj}-\theta_j|>l_n)$$

$$=\left(\int_{\theta_j+l_n}^{\infty}+\int_{-\infty}^{\theta_j-l_n}\right)\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x-\theta_1)^2\right]dx$$

$$=\left(\int_{\frac{\theta_j-\theta_1+l_n}{\sigma}}^{\infty}+\int_{-\infty}^{\frac{\theta_j-\theta_1-l_n}{\sigma}}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx$$

$$=\left(\int_{B_{j1}^n}^{\infty}+\int_{B_{1j}^n}^{\infty}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx$$

where $B_{ij}^n=\dfrac{c_{ij}+l_n}{\sigma}$ and $c_{ij}=\theta_i-\theta_j$.

By assumption

$$|c_{ij}|=|c_{ji}| \quad\text{and}\quad 0<|c_{12}|\leqslant|c_{13}|\leqslant\cdots\cdots\leqslant|c_{1m}|.$$

We put

$$\overline{B}_{ij}^{n}=\frac{|c_{ij}|+l_n}{\sigma} \quad \text{and} \quad \overline{\overline{B}}_{ij}^{n}=\frac{-|c_{ij}|+l_n}{\sigma}.$$

Since $l_n \longrightarrow \infty$, without loss of generality we may assume that $-|c_{1m}|+l_n>0$ for $n=1, 2, \ldots$.
Then

$$0<\overline{\overline{B}}_{1m}^{n}\leqslant\cdots\cdots\leqslant\overline{\overline{B}}_{12}^{n}<\overline{\overline{B}}_{11}^{n}=\overline{B}_{11}^{n}<\overline{B}_{12}^{n}\leqslant\cdots\cdots\leqslant\overline{B}_{1m}^{n}.$$

Since

$$A_j(n)=\left(\int_{\overline{B}_{1j}^{n}}^{\infty}+\int_{\overline{\overline{B}}_{1j}^{n}}^{\infty}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx$$

and

$$A_{j+1}(n)=\left(\int_{\overline{B}_{1j+1}^{n}}^{\infty}+\int_{\overline{\overline{B}}_{1j+1}^{n}}^{\infty}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx,$$

we have

$$A_{j+1}(n)-A_j(n)=\left(\int_{\overline{\overline{B}}_{1j+1}^{n}}^{\overline{\overline{B}}_{1j}^{n}}-\int_{\overline{B}_{1j}^{n}}^{\overline{B}_{1j+1}^{n}}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx\geqslant 0.$$

In fact

$$\overline{\overline{B}}_{1j}^{n}-\overline{\overline{B}}_{1j+1}^{n}=\overline{B}_{1j+1}^{n}-\overline{B}_{1j}^{n} \quad \text{and}$$

$$0<\overline{\overline{B}}_{1j+1}^{n}\leqslant\overline{\overline{B}}_{1j}^{n}\leqslant\overline{B}_{1j}^{n}\leqslant\overline{B}_{1j+1}^{n}.$$

In the case $j=1$, "$\geqslant$" is replaced by "$>$". (Q.E.D.)

LEMMA 2. *For $i=1, 2, \ldots, m$,*

$$P_{i1}(n)>P_{i2}(n)\geqslant\cdots\cdots\geqslant P_{ii-1}(n)\geqslant P_{ii+1}(n)\geqslant\cdots\cdots\geqslant P_{im}(n).$$

PROOF.

$$P_{ij}(n)=\sum_{h=1}^{m-1}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{k\neq j_t}A_k(n),\ j\neq i.$$

Let $j<l\ (j\neq i,\ l\neq i)$ be satisfied. Then by Lemma 1, we have

$$(1-A_j(n))A_l(n)\geqslant(1-A_l(n))A_j(n).$$

Now we put

$$E_{ij}=\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t}(1-A_{j_t}(n))\prod_{k\neq j_t}A_k(n)$$

$$=\Big[\sum_{\substack{R_h \not\ni i \\ R_h \ni j, l}} + \sum_{\substack{R_h \not\ni i, l \\ R_h \ni j}}\Big] \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t} A_k(n)$$

Let us consider the second part of the right hand side in the above equation.

$$\sum_{\substack{R_h \not\ni i, l \\ R_h \ni j}} \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t} A_k(n)$$

$$=(1-A_j(n))A_l(n) \sum_{R_h \not\ni i, j, l} \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t, j, l} A_k(n)$$

$$\geqslant(1-A_l(n))A_j(n) \sum_{R_h \not\ni i, j, l} \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t, j, l} A_k(n)$$

$$= \sum_{\substack{R_h \not\ni i, j \\ R_h \ni l}} \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t} A_k(n)$$

Thus, it holds that

$$E_{ij} \geqslant \Big[\sum_{\substack{R_h \not\ni i \\ R_h \ni j, l}} + \sum_{\substack{R_h \not\ni i, j \\ R_h \ni l}}\Big] \prod_t (1-A_{j_t}(n)) \prod_{k \neq j_t} A_k(n) = E_{il}$$

Therefore we obtain

$$P_{ij}(n) \geqslant P_{il}(n), \quad \text{for} \quad j > l \; (j \neq i, \; l \neq i).$$

(In the case $j=1$, "$\geqslant$" is replaced by "$>$".)

This proves Lemma 2. (Q.E.D.)

Now we shall consider the following expression of $P_i(n)$:

$$P_i(n) = \frac{\alpha_i(n)}{\sum_{j=1}^{m} \alpha_j(n)}, \quad \alpha_i(n) > 0, \quad \text{for} \quad i=1, 2, \ldots, m. \tag{3. 3}$$

Using Lemma 2, in the same way as K. TANAKA, K. INADA and S. IWASE [4], we can obtain the following evaluation:

$$\begin{aligned} &\alpha_i(n) \leqslant K_m P_{12}(n) P_{21}(n) \cdots\cdots P_{i-11}(n) P_{i+11}(n) \cdots\cdots P_{m1}(n) \\ &\text{for} \quad i=2, \ldots, m, \quad \text{and} \\ &\alpha_1(n) \geqslant P_{21}(n) P_{31}(n) \cdots\cdots P_{m1}(n) \end{aligned} \tag{3. 4}$$

where K_m is a positive constant such that it depends on only m.

Therefore for $i=2, \ldots, m$, it holds that

$$P_i(n) = \frac{\alpha_i(n)}{\sum_{j=1}^{m} \alpha_j(n)} \leqslant \frac{\alpha_i(n)}{\alpha_1(n)} \leqslant K_m \frac{P_{12}(n)}{P_{i1}(n)}. \tag{3. 5}$$

LEMMA 3. *For $i=2, \ldots, m$,*

$$\frac{P_{12}(n)}{P_{i1}(n)} \sim \frac{A_1(n)}{A_i(n)}.$$

REMARK. "$a_n \sim b_n$ means "$a_n/b_n \to 1$ as $n\to\infty$".

PROOF. For every j,

$$A_j(n)=P_r(|X_{nj}-\theta_j|>l_n)\to 0 \quad \text{as} \quad n\to\infty.$$

From the definition of $P_{ij}(n)$, we have

$$P_{ij}(n)=A_i(n)\Big[\sum_{h=1}^{m-1}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{k\neq j_t, i}A_k(n)\Big]$$

$$=A_i(n)\Big[\frac{1}{m-1}\prod_{k\neq i}(1-A_k(n))$$

$$+\sum_{h=1}^{m-2}\frac{1}{h}\sum_{\substack{R_h\not\ni i\\ R_h\ni j}}\prod_{t=1}^{h}(1-A_{j_t}(n))\prod_{k\neq j_t, i}A_k(n)\Big]$$

Since the above brackets tend to $\frac{1}{m-1}$ as $n\to\infty$, it holds that

$$\frac{P_{12}(n)}{P_{i1}(n)}\sim\frac{A_1(n)}{A_i(n)}. \qquad \text{(Q.E.D.)}$$

LEMMA 4. *For $i=2,\dots, m$,*

$$\sum_{n=1}^{\infty}\frac{A_1(n)}{A_i(n)}<\infty.$$

PROOF. Using the same notation as in Lemma 1,

$$\frac{A_1(n)}{A_i(n)}=\frac{2\int_{B_{11}^n}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx}{\left(\int_{\bar{B}_{1i}^n}^{\infty}+\int_{\bar{\bar{B}}_{1i}^n}^{\infty}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx}$$

$$\leqslant\frac{2\int_{B_{1i}^n}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx}{\int_{\bar{\bar{B}}_{1i}^n}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx}$$

By the next inequality

$$\int_{y}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx \sim \frac{1}{y}\exp\left(-\frac{1}{2}y^2\right) \qquad \text{for sufficiently large } y>0,$$

we have

$$\frac{A_1(n)}{A_i(n)}\sim 2\left(\frac{\bar{\bar{B}}_{1i}^n}{B_{11}^n}\right)\cdot\exp\left[-\frac{1}{2}\left\{(B_{11}^n)^2-(\bar{\bar{B}}_{1i}^n)^2\right\}\right].$$

Since

$$\frac{\overline{\overline{B}}{}^n_{1i}}{B^n_{11}}=\frac{-c_{1i}+l_n}{l_n}\longrightarrow 1 \quad \text{as} \quad n\longrightarrow\infty,$$

we obtain

$$\frac{A_1(n)}{A_i(n)}\sim 2\exp\left[-\frac{1}{2}\left\{(B^n_{11})^2-(\overline{\overline{B}}{}^n_{1i})^2\right\}\right].$$

Furthermore we get

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left\{(B^n_{11})^2-(\overline{\overline{B}}{}^n_{1i})^2\right\}\right]\\
&=\exp\left[-\frac{1}{2}\left\{\left(\frac{l_n}{\sigma}\right)^2-\left(\frac{-|c_{1i}|+l_n}{\sigma}\right)^2\right\}\right]\\
&=\exp\left[\frac{c_{1i}{}^2}{2\sigma^2}-\frac{|c_{1i}|}{\sigma^2}l_n\right]\\
&=\exp\left[\frac{c_{1i}{}^2}{2\sigma^2}\right]\cdot\exp\left[-\frac{|c_{1i}|}{\sigma^2}l_n\right]\\
&\leqslant\exp\left[\frac{c_{1i}{}^2}{2\sigma^2}\right]\cdot\frac{\sigma^4}{c_{1i}{}^2}\cdot\frac{1}{l_n{}^2},
\end{aligned}$$

where the above inequality follows from a simple inequality:

$$e^{-x}<\frac{2}{x^2} \qquad \text{for every} \quad x>0.$$

As l_n is a positive number such that $\sum_{n=1}^{\infty} 1/l_n{}^2<\infty$, it holds that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\exp\left[-\frac{1}{2}\left\{(B^n_{11})^2-(\overline{\overline{B}}{}^n_{1i})^2\right\}\right]\\
&\leqslant\exp\left[\frac{c_{1i}{}^2}{2\sigma^2}\right]\cdot\frac{\sigma^4}{c_{1i}{}^2}\cdot\sum_{n=1}^{\infty}\frac{1}{l_n{}^2}
\end{aligned}$$

Therefore we have

$$\sum_{n=1}^{\infty}\frac{A_1(n)}{A_i(n)}<\infty \qquad \text{for } i=2,\dots, m. \qquad \text{(Q.E.D.)}$$

Now we shall prove the theorem. From (3. 5),

$$P_i(n)\leqslant K_m\frac{P_{12}(n)}{P_{i1}(n)} \qquad \text{for} \quad i=2,\dots, m.$$

According to Lemma 3 and Lemma 4, we obtain

$$\sum_{n=1}^{\infty} P_i(n) \leqslant K_m \sum_{n=1}^{\infty} \frac{P_{12}(n)}{P_{i1}(n)} < \infty \qquad \text{for } i=2,\dots, m.$$

As the preservation of probabililies at each stage, we have

$$P_1(n) = 1 - \sum_{i=2}^{m} P_i(n).$$

Therefore we have

$$\sum_{n=1}^{\infty} P_1(n) = \infty.$$

Thus, the proof of the theorem is completed.

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