Some 3-dimensional Riemannian manifolds with constant scalar curvature

By

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1. Introduction

Let (M, g) be Riemannian manifold. By R we denote the Riemannian curvature tensor. By $T_x(M)$ we denote the tangent space to M at x. Let $X, Y \in T_x(M)$. Then $R(X, Y)$ operates on the tensor algebra as a derivation at each point x. In a locally symmetric space ($\nabla R=0$), we have

(*) $\qquad R(X, Y) \cdot R=0$ for any point $x \in M$ and $X, Y \in T_x(M)$.

We consider the converse under some additional conditions.

THEOREM A (S. Tanno [7]). *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold. If (M, g) satisfies (*) and the scalar curvature S is positive and bounded away from 0 on M, then (M, g) is of positive constant curvature.*

THEOREM B (K. Sekigawa [5]). *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold of class C^∞ satisfying (*). If the rank of the Ricci tensor R_1 is non-zero on M, then (M, g) is of constant curvature.*

In this note, we shall prove the followings

THEOREM C *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold satisfying (*). If the scalar curvature S is constant, then (M, g) is of constant curvature.*

THEOREM D *Let (M, g) be a 3-dimensional homogeneous Riemanian manifold satisfying (*). Then (M, g) is either*

(1) *a space of constant curvature, or*

(2) *a locally product Riemannian manifold of a 2-dimensional space of constant curvature and a real line.*

It may be noticed that (*) is equivalent to (**) $R(X, Y) \cdot R_1=0$. In this note, (M, g) is assumed to be connected and of class C^∞.

2. Preliminaries

Let (M, g) be a 3-dimensional Riemannian manifold. Assume (*). dim $M=3$ implies that

(2. 1) $$R(X, Y)=R^1X\wedge Y+X\wedge R^1Y-(S/2)X\wedge Y,$$

where $g(R^1X, Y)=R_1(X, Y)$ and $(X\wedge Y)Z=g(Y, Z)Y-g(X, Z)Y$.

Let (K_1, K_2, K_3) be eigenvalues of the Ricci transformation R^1 at a point x. Then (*) is equivalent to

(2. 2) $$(K_i-K_j)(2(K_i+K_j)-S)=0.$$

Therefore we have only three cases: (K, K, K), $(K, K, 0)$ and $(0, 0, 0)$ at each point. First, if (K, K, K), $K\neq 0$, holds at some point x, then it folds on some open neighborhood U of x. Hence U is an Einstein space, and K is constant on U and on M. Therefore (M, g) is of constant curvature (cf. Takagi and Sekigawa [6]). From now we assume that rank $R^1\leqq 2$ on M. Let $W=\{x\in M;\ \text{rank } R^1=2 \text{ at } x\}$. By W_0 we denote one component of W. On W_0 we have two C^∞-distributions T_1 and T_0 such that

$$T_1=\{X: R^1X=KX\},\ T_0=\{Z: R^1Z=0\}.$$

For X, $Y\in T_1$ and $Z\in T_0$, by (2. 1) we have

(2. 3) $$R(X, Y)=KX\wedge Y,$$
$$R(X, Z)=0.$$

This shows that T_0 is the nullity distribution. Since the index of nullity at each point of M is 1 or 3, the nullity index of M is 1. Thus integral curves of T_0 are geodesic (and complete if (M, g) is complete) (cf. Clifton and Maltz [2], Abe [1], etc.).

Let $(E_1, E_2, E_3)=(E)$ be a local field of orthonormal frame such that $E_3\in T_0$ (consequently, E_1, $E_2\in T_1$) and

$$\nabla_{E_3}E_i=0 \qquad i=1, 2, 3.$$

We call this (E) an adapted frame field. If we put $\nabla_{E_i}E_j=\sum_{k=1}^{3}B_{ijk}E_k$, then we get $B_{ijk}=-B_{ikj}$ and

(2. 4) $$B_{3ij}=0\ i, j=1, 2, 3.$$

The second Bianchi identity and (2. 3) give

(2. 5) $$E_3K+K(B_{131}+B_{232})=0.$$

By (2. 4) and $R(E_i, E_3)E_3=\nabla_{E_i}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_i}E_3-\nabla_{[E_i, E_3]}E_3=0$, we get

(2. 6) $$E_3B_{1\,31}+(B_{1\,31})^2+B_{1\,32}B_{2\,31}=0,$$
$$E_3B_{1\,32}+B_{1\,31}B_{1\,32}+B_{1\,32}B_{2\,32}=0,$$

$$E_3B_{2\,31}+B_{2\,31}B_{1\,31}+B_{2\,32}B_{2\,31}=0,$$
$$E_3B_{2\,32}+(B_{2\,32})^2+B_{2\,31}B_{1\,32}=0.$$

(2. 5) and $(2.\ 6)_2$, (2. 5) and $(2.\ 6)_3$, (2. 5) and $(2.\ 6)_{1,4}$ imply

(2. 7) $\qquad B_{1\,32}=C_1(E)K,\quad B_{2\,31}=C_2(E)K,$

(2. 8) $\qquad B_{1\,31}-B_{2\,32}=D(E)K,$

where $C_1(E)$, $C_2(E)$ and $D(E)$ are functions defined on the same domain as (E) such that $E_3C_1(E)=E_3C_2(E)=E_3D(E)=0$. By (2. 5) and (2. 8), we get

(2. 9) $\qquad 2B_{1\,31}=D(E)K-E_3K/K.$

Now, let $\gamma_x^3(s)$ be an integral curve of T_0 through $x=\gamma_x^3(0)$ with arc-length parameter s. Then $(2.\ 6)_1$, (2. 7) and (2. 9) give

(2. 10) $$\frac{1}{2}\frac{d}{ds}\left(\frac{1}{K}\frac{dK}{ds}\right)=HK^2+\frac{1}{4}\left(\frac{1}{K}\frac{dK}{ds}\right)^2,$$

where $H=H(E)=D(E)^2/4+C_1(E)C_2(E)$. (2. 10) implies that H is independent of the choice of the adapted frame fields (E). Solving (2. 10), we get

(2. 11) $\qquad K=\gamma(\text{for } H=0),\qquad$ or

(2. 12) $\qquad K=\pm 1/((\alpha s-\beta)^2-H\alpha^2)\quad (\text{for } H\neq 0),$

where α, β, and γ are constant along $\gamma_x^3(s)$.

With respect to our problem, without loss of essentiality, we may assume that M is orientable. Let (E) ae any adapted frame field which is compatible with the orientation. We call it an oriented adapted frame field. Then we see that $f=(C_1(E)-C_2(E))K$ is independent of the choice of oriented adapted frame fields, and hence f is a C^∞-function on W_0. $f=0$ holds on an open set $U\subset W_0$, if and only if T_1 is integrable on U. This is a geometric meaning of f.

3. Proofs of theorems C, D

In the proofs we can assume that M is orientable. By the arguments of §2, we assume that rank $R^1\leqq 2$ on M. The assumptions in theorems C, D, follow that $S=2K$ is constant. Then we see that rank $R^1=2$ on M and $W=W_0=M$. f is defined on M. Since K is constant on M, by (2. 11) and (2. 12), we have $H=0$. If $f\neq 0$, that is, there exists a point $x_0\in M$ such that $f(x_0)\neq 0$. We put $V=\{x\in M;\ f(x)\neq 0\}$. Let V_0 be one component of V. $H=H(E)=0$ implies $D(E)^2=-4C_1(E)C_2(E)$. Put $\cos 2\theta(E)=K(C_1(E)+C_2(E))/f$ and $\sin 2\theta(E)=KD(E)/f$. Define (E^*) by $E_3^*=E_3$ and

$$E_1^*=\cos\theta(E)E_1-\sin\theta(E)E_2,\ E_2^*=\sin\theta(E)E_1+\cos\theta(E)E_2,$$

Then we have $D(E^*)=0$. Furthermore, for (E) and (E'), we have $E_1^*(E')$ and

$E_2^*(E)=\pm E_1^*(E)=\pm E_2^*(E')$. $H=0$ and $D(E^*)=0$ imply $C_1(E^*)C_2(E^*)=0$. So we can assume that $C_2(E^*)=0$ (otherwise, change $(E_1^*, E_2^*, E_3^*)\longrightarrow(E_2^*, -E_1^*, E_3^*)$). Then we get

(3. 1) $$B^*{}_{1\,32}\neq 0,\ B^*{}_{2\,31}=B^*{}_{1\,31}=B^*{}_{2\,32}=0.$$

$R(E_1^*, E_2^*)E_3^*=0$ implies $B^*{}_{2\,21}=0$ and

(3. 2) $$E_2^*B^*{}_{1\,32}+B^*{}_{1\,32}B^*{}_{1\,21}=0.$$

$R(E_1^*, E_2^*)E_1^*=-KE_2^*$ implies

(3. 3) $$E_2^*B^*{}_{1\,21}+(B^*{}_{1\,21})^2=-K.$$

By $B^*{}_{2ij}=0$, each trajectory of E_2^* is a geodesic in V_0. Let $\gamma_x{}^2(t)$ be a trajectory of E_2^* through x and parametrized by arc-length parameter t such that $\gamma_x{}^2(0)=x$. Put $B_{1\,21}{}^*=h$ on V_0. From (3. 2) and (3. 3), we have

(3. 4) $$df/dt+h(t)f(t)=0,$$

(3. 5) $$dh/dt+h(t)^2=-K.$$

From (3. 4) and (3. 5), we have

(3. 6) $$d^2(1/f)/dt^2+K(1/f)=0.$$

By the fact of theorem A, in the proof of theorem C, it is sufficient to deal with the case where K is negative. Then, solving (3. 6), we get

(3. 7) $$f(t)=1/(c_1\exp(\sqrt{-K}\,t)+c_2\exp(-\sqrt{-K}\,t)),$$

where c_1 and c_2 are certain real numbers.

We put $L_x{}^2=\{\gamma_x{}^2(t)\in M;\ -\infty<t<\infty\}$. Then, from (3. 7), we can see that $L_x{}^2\subset V_0$, for any $x\in V_0$. Moreover, by the similar arguments as in [5], we can see that, for each point $x\in V_0$, $L_x{}^2$ is a closed subset of M and is a compact subset of M, since M is compact. Thus, there exist two different real numbers t_1, t_2 such that $(df/dt)(t_a)=0$, $a=1, 2$. Thus, from (3. 7), we get

$$c_1\exp(\sqrt{-K}\,t_1)-c_2\exp(-\sqrt{-K}\,t_1)=0,$$

$$c_2\exp(\sqrt{-K}\,t_2)-c_2\exp(-\sqrt{-K}\,t_2)=0.$$

Since $\begin{vmatrix}\exp(\sqrt{-K}\,t_1), -\exp(-\sqrt{-K}\,t_1)\\ \exp(\sqrt{-K}\,t_2), -\exp(-\sqrt{-K}\,t_2)\end{vmatrix}=\exp(\sqrt{-K})(t_2-t_1)-\exp(\sqrt{-K})(t_1-t_2)$ $\neq 0$, we have $c_1=c_2=0$. But, this is a contradiction. Therefore, we can conclude that f is identically 0 on M. This completes the proof of theorem C.

Next, we shall prove theorem D. Let (M, g) be Riemannian homogeneous. Then, the scalar curvature S is constant on M. Of course, (M, g) is complete. We assume that

(M, g) satisfies (**). Then, by the previous arguments, in this paper and the construction of f, we can see that f is constant on M.

If $f \neq 0$, then, from (3. 4), we have $h(t)=0$ for all t. Thus, from (3. 5), it must follow that $K=0$. But, this is a contradiction. Therefore, f must be 0 on M. This completes the proof of theorem D.

4. A remark

Let (M, g) be a 3-dimensional non-compact, complete, non-homogeneous, irreducible Riemannian manifold with constant scalar curvature S satisfying (*) (or (**)). Then, (M, g) is not always locally symmetric. Because, the following Riemannian manifold (M, g) is an example of such a Riemannian manifold (cf. K. Sekigawa [4]):

$M=\mathbb{R}^3$ (3-dimensional real number space),

$$(g);\ \begin{pmatrix} 1/\lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ with respect to the canonical coordinate system (u_1, u_2, u_3) on $\mathbb{R}^3$, where

$$1/\lambda=\exp\left(\sqrt{-S/2}\, t\right), \quad t=(\cos u_1)u_2+(-\sin u_1)u_3,$$

S is a negative real number.

The scalar curvature of the above Riemannian manifold (M, g) is S, and $\nabla R \neq 0$ for (M, g).

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