A remark on linear mappings on Banach *-algebras

By

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1. Introduction

Let A and B be complex Banach algebras with an identity. A linear map $\phi : A \rightarrow B$ is called a Jordan homomorphism if $\phi(ab+ba)=\phi(a)\phi(b)+\phi(b)\phi(a)$ for all a and b in A, (equivalently, $\phi(a^2)=(\phi(a))^2$). It is well known that such maps preserve the power structure, that is $\phi(a^n)=(\phi(a))^n$, for every positive integer n. But the following proposition is valid.

PROPOSITION 1. *Let A and B be complex Banach algebras with an identity e_A, e_B respectively and ϕ be a continuous linear map from A into B such that $\phi(e_A)=e_B$. Suppose that there exists a positive integer $k(\geqq 2)$ such that $\phi(a^k)=(\phi(a))^k$ for all element a in A. Then ϕ is a Jordan homomorphism.*

PROOF. We shall use the vector-valued exponential functions. For each element a of a Banach algebra C, $\exp(a)$ is defined by $\exp(a)=e_C+\sum_{n=1}^{\infty}\frac{1}{n!}a^n$ where e_C denotes the identity element of C. Then it is well known that $\exp(a)=\lim_{n\to\infty}\left(e_C+\frac{1}{n}a\right)^n$. Now, by induction, we have

$$\phi(a^{k^n})=(\phi(a))^{k^n} \qquad \text{for } n=1, 2, 3, \cdots \text{ and } a \in A.$$

Thus,

$$\begin{aligned}\phi(\exp(a)) &= \lim_{n\to\infty}\phi\left(\left(e_A+\frac{1}{k^n}a\right)^{k^n}\right) \\ &= \lim_{n\to\infty}\phi\left(e_B+\frac{1}{k^n}\phi(a)\right)^{k^n} \\ &= \exp\phi(a) \qquad \text{for each } a \text{ in } A.\end{aligned}$$

Replace a by λa with complex number λ, expand in power of λ, and equate coefficients of λ to obtain $\phi(a^n)=(\phi(a))^n$ $(n=1, 2, 3, \cdots)$. We completes the proof.

In the next section, we shall specialize the above results to the case of Banach *-algebras.

2. A specialization to Banach *-algebras

Throughout this section, we consider complex *-Banach algebras with an identity (namely, complex Banach *-algebras with an isometric involution and an identity of norm one). By a C*-homomorphism of one*-Banach algebra into another, we mean a self-adjoint linear map preserving squares of self-adjoint elements in A.

Let A be a complex *-Banach algebra, we recall that A^+ is the subset of H_A consisting of all finite sums of elements of A, and that an element of A^+ is said to be positive.

A linear map of one *-Banach algebra into another is said to be positive ıf it carries positive elements into positive elements (See [5]). Such a map is self-adjoint ($\phi(a^*)=(\phi(a)^*)$.

Several authors have studied the condition that a linear maps of a C*-algebra becomes C*-homomorphism. For example, let ϕ be a self-adjoint linear mapping of a Von Neumann algebra A into a C*-algebra B with an identity e_B which preserves invertible operators and $\phi(e_A)=e_B$ then ϕ is a C*-homomorphism (Russo [3]).

Proposition 2. *Let A and B be two complex *-Banach algebras with an identity e_A, e_B respectively, and $\phi : A \to B$ be a positive linear map such that $\phi(e_A)=e_B$. Moreover suppose B is commutative and*-semi-smple.*

Then the following statements are equivalent.

(1) *ϕ is C*-homomorphism.*

(2) *There exists a positive integer k ($\geqq 2$) such that*

$$\phi(h^k)=(\phi(h))^k \text{ for each self-adjoint } h \in A.$$

(3) $\phi(\exp(-h))=(\phi(\exp(h)))^{-1}$ *for each self-adjoint $h \in A$.*

(4) $\operatorname{Sup}\{\|\phi(\exp(\xi h))\ \phi(\exp(-\xi h))\| - \infty < +\infty\} < +\infty$ *for each self-adjoint $h \in A$.*

Remark. We should remark that when B is a C*-algebra, sup $\{\|\phi(\exp\xi h)\|\ \|\phi(\exp(-\xi h)\|; -\infty < +\infty\}$ is always divergent for self-adjoint element $\phi(h)$ whose spectrum contains more than two points.

We need some lemmas. For the moment, let A and B be C*-algebras. Then a positive linear map such as $\phi(e_A)=e_B$ is bounded and $\|\phi\|=1$. For each self-adjoint h in A, $\exp(h)$ is positive element. Suppose that the identity element of a C*-algebra acting on a Hilbert space H is the identity operator on H.

Lemma 3. *Let A and B be C*-algebras and $\phi : A \to B$ be a posiitve linear map such as $\phi(e_A)=e_B$. Suppose B is commutative. Then $\phi(\exp h) \geqq \exp \phi(h)$ for each self-adjoint element* *h of A.*

PROOF It follows from "generalized Schwartz inequality" and boundedness of ϕ.

LEMMA 4. *Let A and B be C*-algebras. Suppose that B acts on some Hilbert space and ϕ is completely positive.*

Then $\phi(x^*)\,\phi(x)\leqq\phi(x^*x)$ for each $x\epsilon A$.

PROOF Let the canonical expression of ϕ be $V^*\pi V$, where π is a *-representation of A on some Hilbert space K and V is a bounded linear operator from H into K such that $\pi(A)VH$ generates K.

Since $(e_A)=V^*\pi(e_A)V=V^*V=e_B$, V is an isometry. Thus VV^* is a projection. We have

$$\begin{aligned}\phi(x^*)\phi(x)&=V^*\pi(x^*)VV^*\pi(x)V\\&\leqq V^*\pi(x^*x)V\\&=\phi(x^*x)\qquad\text{for each } x\epsilon A.\end{aligned}$$

q. e. d.

Now we proceed to proof of proposition 2.

(1)→(2) It is well known.

(2)→(3) Since A has an identity and B is *-semi-simple, ϕ is continuous. [5]. Hence it is contained in proposition 1.

(3)→(4) Since ϕ is continuous, it is clear.

(4)→(1) Since B is *-semi-simple, we may assume that B is a C*-algebra. Let h be a self-adjoint element of A. We consider the complex variable B-valued entire function $\Psi(\lambda)=\exp(\lambda\phi(h))\phi(\exp\text{-}\lambda h)$.

Then

$$\begin{aligned}\|\Psi(\lambda)\|^2&=\|\phi(\exp(-\bar{\lambda}h))\exp(\bar{\lambda}\phi(h))\exp(\lambda\phi(h))\,\phi(\exp(-\lambda h))\|\\&=\|\phi(\exp(-\bar{\lambda}h))\exp(2Re\lambda\phi(h))\phi(\exp(-\lambda h))\|\\&\leqq\|\phi(\exp(-\bar{\lambda}h))\phi(\exp(2Re\lambda(h)\phi(\exp(-\lambda h))\|\\&=\|\phi(\exp(-\bar{\lambda}h))\phi(\exp(-\lambda h)\phi(\exp(2\mathrm{Re}\lambda(h))\|\end{aligned}$$

Since ϕ is positive and $\exp\,(2\mathrm{Re}\lambda h)\geqq0$, there exists a positive square root $(\phi(\exp 2\mathrm{Re}\lambda.h))^{\frac{1}{2}}$

$$\begin{aligned}\|\Psi(\lambda)\|^2&\leqq\|(\phi(\exp 2\mathrm{Re}\,\lambda.h))^{\frac{1}{2}}\phi(\exp(-2\mathrm{Re}\lambda)h)(\phi(\exp 2\mathrm{Re}\lambda h))^{\frac{1}{2}}\|\\&=\|\phi(\exp(-2\mathrm{Re}\lambda)h)\phi(\exp(2\mathrm{Re}\lambda)h\|\end{aligned}$$

Consequently $\Psi(\lambda)$ is bounded in the whole plane. Thus $\Psi(\lambda)$ is contant by Liouville's theorem for vector-valued entire functions. Since $\Psi(0)=e_B$, we have $\exp\lambda\phi(h)=\phi(\exp\lambda h)$. Equate coefficients of λ to obtain $\phi(h^2)=(\phi(h))^2$. q. e. d.

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