On infinitesimal transformations of K-contact and normal contact metric spaces

By

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§ 1. Introduction

An odd dimensional differentiable manifold with contact metric structure is called a normal contact metric space if the Nijenhuis' tensor vanishes identically. This space corresponds to an even dimensional Kählerian space. So it is interesting to discuss analogues of some theorems which are valid in Kählerian spaces. Several problems of this type have been solved in [4]. Further, recently a K-contact metric space has been defined in [2]. In this paper, we shall add some results concerning infinitesimal transformations in a K-contact and a normal contact metric space. In § 2 some preliminary identities and notions are given for the later use. In § 3 we shall prove that in a normal contact metric space an infinitesimal conformal transformation is an infinitesimal isometry if the space is of constant scalar curvature. It will be shown in § 4 that for an infinitesimal projective transformation, the analogue of the theorem in § 3 holds good. In § 5, we discuss an infinitesimal transformation which leaves $\varphi_j{}^i$ invariant, and some tensors which are left invariant by this transformation. In the last section § 6, we shall show that an infinitesimal affine contact transformation is necessarily an automorphism.

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§ 2. Preliminaries.

An $n(=2m+1)$ dimensional differentiable manifold M of class C^∞ with (φ, ξ, η, g) - structure (or an almost contact metric structure [6]) has been defined by S. Sasaki [5]. By definition it is a manifold with tensor fields $\varphi_j{}^i, \xi^i, \eta_i$ and so called an associated Riemannian metric tensor g_{ji} defined over M which satisfy the following relations

(2. 1) $\qquad \xi^i\eta_i=1,$

(2. 2) $\qquad rank|\varphi_j{}^i|=n-1,$

(2. 3) $$\varphi_j{}^i\xi^j=0,$$

(2. 4) $$\varphi_j{}^i\eta_i=0,$$

(2. 5) $$\varphi_j{}^r\varphi_r{}^i=-\delta_j^i+\xi^i\eta_j,$$

(2. 6) $$g_{ji}\xi^j=\eta_i,$$

(2. 7) $$g_{ji}\varphi_k{}^j\varphi_h{}^i=g_{kh}-\eta_k\eta_h.$$

On the other hand let M be a differentiable manifold with a contact structure [1]. If we put

(2. 8) $$2g_{ir}\varphi_j{}^r=2\varphi_{ji}=\partial_j\eta_i-\partial_i\eta_j{}^{1)}$$

then we can find four tensors $\varphi_j{}^i$, ξ^i, η_i and g_{ji} so that they define an (φ,ξ,η,g) -structure. Such a structure is called a contact metric structure [6].

A manifold with a (an almost) contact metric structure is called a (an almost) contact metric space.

In an almost contact metric space there are four tensor fields $N_{ji}{}^h, N_{ji}, N_j{}^i$ and N_j which are the analogue of the Nijenhuis tensor in an almost complex structure [6].

In a contact metric space, $N_{ji}=0$ and $N_j{}^i=0$ hold good, $N_j{}^i=0$ is equivalent to the fact that ξ^i is a Killing vector field and $N_{ji}{}^h=0$ yields $N_j{}^i=0$.

A contact metric space with $N_j{}^i=0$ or $N_{ji}{}^h=0$ is called a K-cotact metric space [2] or a normal contact metric space respectively. Of course a normal contact metric space is a K-contact metric space and a K-contact metric space is a contact metric space. In the following we consider mainly a K-contact and a normal contact metric space and use a notation η^i instead of ξ^i.

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and put[2)]

(2. 9) $$\begin{cases} R_{ji}=g^{kh}R_{kjih}, & \widetilde{R}_{ji}=\varphi_j{}^rR_{ri}, & R=g^{ji}R_{ji}, \\ H_{ji}=\varphi^{kh}R_{kjih}, & R^*{}_{ji}=-\varphi_j{}^rH_{ri}, & R^*=g^{ji}R^*{}_{ji}. \end{cases}$$

We see that

(2. 10) $$H_{ji}=-\frac{1}{2}\varphi^{kh}R_{khji}.$$

In a contact metric space, φ_{ji} is a closed skew symmetric tensor and

(2. 11) $$\nabla_r\varphi_j{}^r=(n-1)\eta_j$$

holds good [7], where ∇_i denotes the covariant differentiation with respect to the Riemannian connection.

In a K-contact metric space, since η_i is a Killing vector, we have

1) This definition is slightly different from S. Sasaki's.

2) The notations of H_{ji} and $\widetilde{R}_{ji}$ are different from M. Okumura's.

(2. 12) $$\nabla_j \eta_i = \varphi_{ji},$$

(2. 13) $$\nabla_k \varphi_{ji} + R_{rkji}\eta^r = 0.$$

Using the closedness of φ_{ji}, we get from (2. 13) and (2. 9)

(2. 14) $$H_{ir}\eta^r = 0, \qquad \widetilde{R}_{ir}\eta^r = 0, \qquad R^*{}_{ir}\eta^r = 0,$$

and

(2. 15) $$R_{kjih}\eta^k\eta^j = 0,$$

(2. 16) $$R_{kjih}\eta^k\eta^h = g_{ji} - \eta_j\eta_i.$$

Transvecting (2. 13) with g^{ki} and making use of (2. 11), we get

(2. 17) $$R_{ir}\eta^r = (n-1)\eta_i.$$

In a normal contact metric space, the formula

(2. 18) $$\nabla_k \varphi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}$$

is fundamental, from which and (2. 13), we have

(2. 19) $$\eta_r R_{kji}{}^r = \eta_k g_{ji} - \eta_j g_{ki}.$$

Operating ∇_l to (2. 18) and making use of Ricci's identity and (2. 9), we obtain

(2. 20) $$\widetilde{R}_{ji} - H_{ji} = (n-2)\varphi_{ji},$$

(2. 21) $$R_{ji} - R^*{}_{ji} = (n-2)g_{ji} + \eta_j\eta_i,$$

(2. 22) $$R - R^* = (n-1)^2.$$

We see that $\widetilde{R}_{ji}$ is a skew symmetric and $R^*{}_{ji}$ is a symmetric tensor.

Let $\underset{v}{\pounds}$ be the Lie derivative with respect to a vector v^i, then the following identities hold good [8].

(2. 23) $$\underset{v}{\pounds}\left\{{h \atop ji}\right\} = \nabla_j\nabla_i v^h + R_{rji}{}^h v^r,$$

(2. 24) $$\nabla_j \underset{v}{\pounds}\eta_i - \underset{v}{\pounds}\nabla_j\eta_i = \eta_r \underset{v}{\pounds}\left\{{r \atop ji}\right\}$$

(2. 25) $$\underset{v}{\pounds} R_{kji}{}^h = \nabla_k \underset{v}{\pounds}\left\{{h \atop ji}\right\} - \nabla_j \underset{v}{\pounds}\left\{{h \atop ki}\right\},$$

(2. 26) $$\underset{v}{\pounds}\left\{{h \atop ji}\right\} = \frac{1}{2}g^{hr}(\nabla_j \underset{v}{\pounds} g_{ri} + \nabla_i \underset{v}{\pounds} g_{rj} - \nabla_r \underset{v}{\pounds} g_{ji}).$$

A vector field v^i is called an infinitesimal isometry or a Killing vector if $\underset{v}{\pounds} g_{ji} = 0$; an infinitesimal affine transformation if $\underset{v}{\pounds}\left\{{h \atop ji}\right\} = 0$; an infinitesimal (strict if $\sigma = 0$) contact transformation if $\underset{v}{\pounds}\eta_i = \sigma\eta_i$ where σ is a scalar and σ is called an associated function of v^i.

A vector field v^i satisfying

(2. 27) $$\underset{v}{\pounds}\, g_{ji}=2\rho g_{ji}$$

where ρ is a scalar, is called an infinitesimal conformal transformation, then v^i satisfies

(2. 28) $$\underset{v}{\pounds}\, \left\{ {h \atop ji} \right\}=\delta_j^h\rho_i+\delta_i^h\rho_j-g_{ji}\rho^h,\quad \rho_i=\partial_i\rho,$$

(2. 29) $$\underset{v}{\pounds}\, R_{ji}=-(n-2)\nabla_j\rho_i-g_{ji}\nabla_r\rho^r,$$

(2. 30) $$\underset{v}{\pounds}\, R=-2\rho R-2(n-1)\nabla_r\rho^r.$$

A vector field v^i satisfying

(2. 31) $$\underset{v}{\pounds}\, \left\{ {h \atop ji} \right\}=\delta_j^h\rho_i+\delta_i^h\rho_j$$

is called an infinitesimal projective transformation and

(2. 32) $$\underset{v}{\pounds}\, R_{ji}=-(n-1)\nabla_j\rho_i$$

is valid.

A vector field v^i is called an automorphism, if v^i leaves four structure tensors invariant, that is $\underset{v}{\pounds}\,\varphi_j{}^i=0$, $\underset{v}{\pounds}\,\eta_i=0$, $\underset{v}{\pounds}\,\eta^i=0$ and $\underset{v}{\pounds}\,g_{ji}=0$.

A normal contact metric space in which the Ricci tensor takes the form

(2. 33) $$R_{ji}=ag_{ji}+b\eta_j\eta_i$$

is called an η-Einstein space, where a and b become constant $(n>3)$ and

(2. 34) $$a+b=n-1,\quad R=an+b$$

hold good [3]. But we can extend the above definition of an η-Einstein space to the case which the underlying space is K-contact. In this case the properties that a and b are constant and (2. 34) also hold good. We shall call such a space an η-Einstein K-space.

§ 3. Infinitesimal conformal transformations.

THEOREM 3. 1. *Let M be a normal contact metric space of constant scalar curvature $R\neq n(n-1)$, $(n>3)$. Then an infinitesimal conformal transformation in M is necessarily an infinitesimal isometry.*

To prove this theorem, we shall use the following theorem which has been proved by M. Okumura [4].

LEMMA. *If a normal contact metric space $(n>3)$ admits an infinitesimal conformal transformation v^i, then v^i is decomposed into*

(3. 1) $$v^i=w^i-\rho^i$$

where w^i is a Killing vector and ρ_i is a gradient vector defining an infinitesimal conformal transformation.

Proof of the theorem. Substituting (3. 1) into

$$\underset{v}{\pounds}\, g_{ji}=\nabla_j v_i+\nabla_i v_j=2\rho g_{ji}.$$

we have

$$\nabla_j \rho_i+\nabla_i \rho_j=-2\rho g_{ji}$$

from which we get $\nabla_r \rho^r=-n\rho$, substituting this into (2. 30), it follows that

$$\underset{v}{\pounds}\, R=-2[R-n(n-1)]\rho. \tag{3. 2}$$

If we assume $R=$const. and $R\neq n(n-1)$, then we obtain $\rho=0$. Hence we have $\underset{v}{\pounds}\, g_{ji}=0$.

COROLLARY. *In an η-Einstein space ($n>3$) with $b\neq 0$, any infinitesimal conformal transformation is necessary an infinitesimal isometry* [4].

§ 4. Infinitesimal projective transformations.

LEMMA 4. 1. [4]. *If a normal contact metric space admits an infinitesimal projective transformation v^i, then v^i is decomposed into*

$$v^i=w^i-\frac{1}{2}\rho^i \tag{4. 1}$$

where w^i is a Killing vector and ρ^i is the associated vector and ρ^i is also an infinitesimal projective transformation whose associated vector is $-2\rho_i$.

LEMMA 4. 2. *In a normal contact metric space of constant scalar curvature, the relation*

$$\nabla^i H_{ji}=R^*\eta_j \tag{4. 2}$$

holds good.

Proof. By virtue of (2. 9), (2. 14) and (2. 18), we get

$$\begin{aligned}\nabla^i H_{ji}&=\nabla^i(\varphi_j{}^r R^*{}_{ri})=(\eta_j g^{ir}-\eta^r\delta_j{}^i)R^*{}_{ri}+\varphi_j{}^r\nabla^i R^*{}_{ri}\\ &=R^*\eta_j+\varphi_j{}^r\nabla^i R^*{}_{ri}.\end{aligned} \tag{4. 3}$$

On the other hand if we operate ∇^i to (2. 21), then we have

$$\nabla^i R_{ji}=\nabla^i R^*{}_{ji}.$$

Since $\nabla_j R=2\nabla^i R_{ji}=0$, we get $\nabla^i R^*{}_{ji}=0$. Hence (4. 2) follows from (4. 3).

LEMMA 4. 3. *Let M be a normal contact metric space of constant scalar curvature*

$R \neq n(n-1)$. *Then the associated vector* ρ_i *of an infinitesimal projective transformation in M belongs to the distribution orthogonal to* η^i.

Proof. By virtue of Lemma 4. 1, we obtain

$$\nabla_j \nabla_i \rho_h + R_{rjih} \rho^r = -2(g_{jh} \rho_i + g_{ih} \rho_j). \tag{4. 4}$$

Taking the alternative part of (4. 4) with respect to i and h, we get

$$R_{rjih} \rho^r = g_{ji} \rho_h - g_{jh} \rho_i.$$

Transvecting this with φ^{ji}, we have

$$H_{rh} \rho^r = \varphi_{rh} \rho^r.$$

Operating ∇^h to the last equation and making use of Lemma 4. 2, we see that

$$[R^* - (n-1)] \eta^r \rho_r = 0,$$

from which we have

$$[R - n(n-1)] \eta^r \rho_r = 0.$$

because of (2. 22). If we assume that $R \neq n(n-1)$, then we have $\eta^r \rho_r = 0$.

THEOREM 4. 1. *Let M be a normal contact metric space of constant scalar curvature* $R \neq n(n-1)$. *Then an infinisimal projective transformation in M whose associated vector does not belong to the distribution orthogonal to* η^i *is necessary an infinitesimal isometery.*

Proof. Lemma 4. 3 shows that there exists no non-trivial associated vector ρ^i if R=const. $R \neq n(n-1)$ and $\eta^r \rho_r \neq 0$. Hence Theorem 4. 1 follows from Lemma 4. 1.

As an application of Lemma 4. 1. we have the following.

THEOREM 4. 2. *In a normal contact metric space, if an infinitesimal projective transformation* v^i *is an infinitesimal contact transformation whose associated function is a constant, then* v^i *is an automorphism.*

In the first place we shall give a lemma.

LEMMA 4. 4. *In an almost contact metric space, if an infinitesimal transformation* v^i *satisfies*

$$\underset{v}{\pounds} \varphi_j{}^i = \sigma \varphi_j{}^i \tag{4. 5}$$

where σ *is a scalar, then we have* $\sigma = 0$.

Proof. Operating $\underset{v}{\pounds}$ to (2. 5) and making use of (4. 5), we have

$$2\sigma(-\delta_j{}^i + \eta^i \eta_j) = \underset{v}{\pounds} (\eta^i \eta_j).$$

Contracting the last equation with respect to i and j, we find $\sigma = 0$.

Proof of the theorem. By assumption, we put

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\}=\delta_j{}^h\rho_i+\delta_i{}^h\rho_j$$

$$\underset{v}{\pounds}\eta_i=\sigma\eta_i, \qquad \sigma=\text{const.}$$

Substituting these equations into (2. 24), we get

$$\sigma\varphi_{ji}-\underset{v}{\pounds}\varphi_{ji}=\eta_j\rho_i+\eta_i\rho_j$$

from which we have

$$\eta_j\rho_i+\eta_i\rho_j=0, \tag{4. 6}$$

$$\underset{v}{\pounds}\varphi_{ji}=\sigma\varphi_{ji}. \tag{4. 7}$$

Transvecting (4. 6) with g^{ji} and η^j respectively, we get $\rho_i=0$
Applying Lemma 4. 1, it follows that

$$\underset{v}{\pounds}g_{ji}=0. \tag{4. 8}$$

From (4. 7) and (4. 8), we have

$$\underset{v}{\pounds}\varphi_j{}^i=\sigma\varphi_j{}^i$$

Applying Lemma 4. 4. we obtain $\underset{v}{\pounds}\varphi_j{}^i=0$, $\underset{v}{\pounds}\eta_i=0$ and $\underset{v}{\pounds}\eta^i=0$.

§ 5. **Infinitesimal φ-transformations.**

In this section we shall discuss an infinitesimal transformation which leaves $\varphi_j{}^i$ invariant. We call it an *infinitesimal φ-transformation* for brevity. We shall show some results concerning mainly to the curvature of the space in consideration.

THEOREM 5. 1. *In a contact metric space, an infinitesimal φ-transformation satisfies*

$$\underset{v}{\pounds}\eta_i=\sigma\eta_i, \tag{5. 1}$$

$$\underset{v}{\pounds}g_{ji}=\sigma(g_{ji}+\eta_j\eta_i), \tag{5. 2}$$

and σ is a constant. Conversely if an infinitesimal transformation v^i satisfies (5. 1) *and* (5. 2), *then v^i is an infinitesimal φ-transformation and consequently σ is a constant.*

Proof. We assume that

$$\underset{v}{\pounds}\varphi_j{}^i=0. \tag{5. 3}$$

Operating $\underset{v}{\pounds}$ to (2. 5), we get

$$\eta^i\underset{v}{\pounds}\eta_j+\eta_j\underset{v}{\pounds}\eta^i=0. \tag{5. 4}$$

Transvecting (5. 4) with η_i and η^i respectively, we have

$$\underset{v}{\pounds}\eta_j=\sigma\eta_j, \tag{5. 5}$$

and

$$(5.\ 6)\qquad \mathop{\pounds}_{v} \eta^i = -\sigma\eta^i$$

where we have put

$$\sigma = \eta^r \mathop{\pounds}_{v} \eta_r = -\eta_r \mathop{\pounds}_{v} \eta^r.$$

Substituting (5. 5) into (2. 24), it follows that

$$(5.\ 7)\qquad \sigma_j\eta_i + \sigma_i\eta_j - \mathop{\pounds}_{v} \nabla_j\eta_i = \eta_h \mathop{\pounds}_{v} \left\{ {h \atop ji} \right\}, \qquad \sigma_j = \partial_j\sigma,$$

from which we get

$$(5.\ 8)\qquad \sigma_j\eta_i - \sigma_i\eta_j + 2\sigma\varphi_{ji} - 2\mathop{\pounds}_{v} \varphi_{ji} = 0$$

by virtue of (2. 8).

Transvecting (5. 8) with η^j, we have

$$(5.\ 9)\qquad \sigma_j = (\eta^r\sigma_r)\eta_j$$

by virtue of (2. 3) and (5. 6). Hence (5. 8) turns to

$$(5.\ 10)\qquad \mathop{\pounds}_{v} \varphi_{ji} = \sigma\varphi_{ji}.$$

On the other hand, operating $\mathop{\pounds}_{v}$ to

$$\varphi_j{}^r\varphi_{ri} = -g_{ji} + \eta_j\eta_i$$

and making use of (5. 3), (5. 5) and (5. 10), we get (5. 2).

In the next place well shall prove that σ is a constant. If we substitute (5. 2) into the identity (2. 26), then we have

$$(5.\ 11)\qquad \mathop{\pounds}_{v} \left\{ {h \atop ji} \right\} = \frac{1}{2}\left[\begin{array}{l} \sigma_j(\delta_i{}^h + \eta^h\eta_i) + \sigma_i(\delta_j{}^h + \eta^h\eta_j) \\ -\sigma^h(g_{ji} + \eta_j\eta_i) + 2\sigma(\varphi_j{}^h\eta_i + \varphi_i{}^h\eta_j) \end{array} \right]$$

From (5. 7), (5. 10) and (5. 11), it follows that

$$\eta^r\sigma_r = g^{ji}\eta_h \mathop{\pounds}_{v} \left\{ {h \atop ji} \right\} = \frac{3-n}{2}\eta^r\sigma_r$$

from which

$$\eta^r\sigma_r = 0.$$

Hence (5. 9) shows that $\sigma_j = 0$, $\sigma = \text{const.}$

Therefore (5. 11) becomes

$$(5.\ 12)\qquad \mathop{\pounds}_{v} \left\{ {h \atop ji} \right\} = \sigma(\varphi_j{}^h\eta_i + \varphi_i{}^h\eta_j).$$

To prove the converse, we assume (5. 1) and (5. 2), then we see that (5. 6) and (5. 10) follow, because

$$\underset{v}{\pounds}\eta^i=g^{ji}\underset{v}{\pounds}\eta_j+\eta_j\underset{v}{\pounds}g^{ji}$$
$$=\sigma\eta^i-\sigma\eta_j(g^{ji}+\eta^j\eta^i)=-\sigma\eta^i,$$

and transvecting (5. 8) with η^i, we get

$$\sigma_j-(\eta^r\sigma_r)\eta_j-2\eta^i\underset{v}{\pounds}\varphi_{ji}=0$$

from which and making use of (2. 3) and (5. 6), we get $\sigma_j=(\eta^r\sigma_r)\eta_j$. Hence (5. 8) yields (5. 10). It is easily verified that $\underset{v}{\pounds}\varphi_j{}^i=0$ from (5. 2) and (5. 10). Q. E. D.

COROLLARY. *In a compact contact metric space, an infinitesimal φ-transformation is an automorphism.* [7].

Proof. From (5. 2)

$$\underset{v}{\pounds}g_{ji}=\nabla_j v_i+\nabla_i v_j=\sigma(g_{ji}+\eta_j\eta_i),\qquad \sigma=\text{const.}$$

from which it follows that

$$\nabla_r v^r=\frac{n+1}{2}\sigma.$$

Applying Green's theorem we have $\sigma=0$ which means v^i is an automorphism.

THEOREM 5. 2.[3] *In an Einstein contact metric space ($R\neq 0$), an infinitesimal φ-transformation is an automorphism.*

Proof. Substituting (5. 12) into the identity (2. 25), we have

$$(5.\ 13)\qquad \underset{v}{\pounds}R_{kji}{}^h=\sigma[\nabla_k(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j)-\nabla_j(\varphi_k{}^h\eta_i+\varphi_i{}^h\eta_k)]$$

from which it follows that

$$(5.\ 14)\qquad \underset{v}{\pounds}R_{ji}=\sigma\nabla_r(\varphi_j{}^r\eta_i+\varphi_i{}^r\eta_j)$$

$$(5.\ 15)\qquad \underset{v}{\pounds}R=-\sigma(R+R_{ji}\eta^j\eta^i),$$

because of (5. 2). By assumption

$$R_{ji}=\frac{R}{n}g_{ji}.$$

(5. 15) reduced to

$$\underset{v}{\pounds}R=-\frac{(n+1)\sigma}{n}R$$

and we have $\sigma=0$.

THEOREM 5. 3.[4] *In a K-contact metric space of constant scalar curvature $R\neq -(n-1)$, an infinitesimal φ-transformation is an automorphism.*

Proof. Using (2. 17), (5. 15) becomes

3, 4) These theorem recentry have been proved for a global φ-transformation by S. Tanno.

$$\underset{v}{\pounds} R=-\sigma(R+n-1)$$

which implies $\sigma=0$ if $R=$const. and $R\neq-(n-1)$.

THEOREM 5. 4. *In an contact metric space, an infinitesimal φ-transformation v^i satisfies*

$$\underset{v}{\pounds} N_j{}^i=-\sigma N_j{}^i, \tag{5. 16}$$

$$\underset{v}{\pounds} N_{kj}{}^i=0 \tag{5. 17}$$

where

$$N_j{}^i\equiv\eta^h(\nabla_h\varphi_j{}^i-\nabla_j\varphi_h{}^i)-\varphi_j{}^h\nabla_h\eta^i, \tag{5. 18}$$

$$N_{kj}{}^i\equiv\varphi_k{}^h(\nabla_h\varphi_j{}^i-\nabla_j\varphi_h{}^i)-\varphi_j{}^h(\nabla_h\varphi_k{}^i-\nabla_k\varphi_h{}^i) \\ +(\nabla_j\eta^i)\eta_k-(\nabla_k\eta^i)\eta_j. \tag{5. 19}$$

Proof. For an infinitesimal φ-transformation, the following equations hold good

$$\underset{v}{\pounds}\varphi_j{}^i=0, \quad \underset{v}{\pounds}\eta^i=-\sigma\eta^i \tag{5. 20}$$

$$\underset{v}{\pounds}\left\{{h \atop ji}\right\}=\sigma(\varphi_j{}^h\eta_i+\varphi_i{}^h\eta_j), \quad \sigma=\text{const.} \tag{5. 21}$$

Substituting (5. 20) and (5. 21) into the identity

$$\underset{v}{\pounds}\nabla_h\varphi_j{}^i-\nabla_h\underset{v}{\pounds}\varphi_j{}^i=\varphi_j{}^r\underset{v}{\pounds}\left\{{i \atop hr}\right\}-\varphi_r{}^i\underset{v}{\pounds}\left\{{r \atop hj}\right\},$$

$$\underset{v}{\pounds}\nabla_h\eta^i-\nabla_h\underset{v}{\pounds}\eta^i=\eta^r\underset{v}{\pounds}\left\{{i \atop hr}\right\},$$

we obtain

$$\underset{v}{\pounds}\nabla_h\varphi_j{}^i=\sigma(\delta_h^i-\eta^i\eta_h)\eta_j, \tag{5. 22}$$

$$\underset{v}{\pounds}\nabla_h\eta^i=\sigma(\varphi_h{}^i-\nabla_h\eta^i). \tag{5. 23}$$

Operating $\underset{v}{\pounds}$ to (5. 18) and (5. 19) and making use of (5. 20), (5. 22) and (5. 23), we obtain (5. 16) and (5. 17).

This theorem shows that in a contact metric space, $N_{ji}{}^h$ is an invariant tensor by an infinitesimal φ-transformation. From now on, we shall seek for other invariant tensors by this transformation. In the first place we shall give the following.

LEMMA. *In a K-contact metric space, for an infinitesimal φ-transformation v^i, the following equations hold good.*

$$\underset{v}{\pounds} R_{kji}{}^h=\sigma\left[\begin{array}{c}(\nabla_k\varphi_j{}^h-\nabla_j\varphi_k{}^h)\eta_i+(\nabla_k\varphi_i{}^h)\eta_j-(\nabla_j\varphi_i{}^h)\eta_k \\ +2\varphi_{kj}\varphi_i{}^h+\varphi_j{}^h\varphi_{ki}-\varphi_k{}^h\varphi_{ji}\end{array}\right], \tag{5. 24}$$

(5. 25) $$\underset{v}{\pounds} R_{ji}=2\sigma(-g_{ji}+n\eta_j\eta_i),$$

(5. 26) $$\underset{v}{\pounds} H_{ji}=-n\sigma\varphi_{ji}.$$

Proof. (5. 24) follows from (5. 13). (5. 25) is obtained from (5. 24) and (2. 11). Transvecting (5. 24) with $\varphi^k{}_h$ we get (5. 26).

Now, by virtue of (5. 13), (5. 22) and the relation

$$\underset{v}{\pounds} \nabla_j\eta_i=\sigma\nabla_j\eta_i$$

which is analogous to (5. 23), we can verify that, after some computation, in a contact metric space

(5. 27) $$\underset{v}{\pounds} P_{kji}{}^h=0$$

holds good, where $P_{kji}{}^h$ is defined by

(5. 28) $$P_{kji}{}^h=R_{kji}{}^h-\frac{1}{\sigma}\underset{v}{\pounds} R_{kji}{}^h+(\delta_k^h\eta_j-\delta_j^h\eta_k)\eta_i.$$

We can obtain easily the following identities

(5. 29) $$P_{(kj)i}{}^h=0, \qquad P_{[kji]}{}^h=0. \qquad P_{kjr}{}^r=0,$$

(5. 30) $$P_{ji}\equiv P_{rji}{}^r=R_{ji}-\frac{1}{\sigma}\underset{v}{\pounds} R_{ji}+(n-1)\eta_j\eta_i,$$

(5. 31) $$Q_{ji}\equiv\varphi^k{}_h P_{kji}{}^h=H_{ji}-\frac{1}{\sigma}\underset{v}{\pounds} H_{ji}.$$

In a K-contact metric space, by virtue of (5. 25) and (5. 26), we see

(5. 32) $$P_{ji}=R_{ji}+2g_{ji}-(n+1)\eta_j\eta_i,$$

(5. 33) $$Q_{ji}=H_{ji}+n\varphi_{ji}.$$

Moreover we have

$$\eta^k\eta_h P_{kji}{}^h=0, \qquad \varphi_h{}^i P_{kji}{}^h=2Q_{kj}.$$

In a normal contact metric space we see that

$$\eta_h P_{kji}{}^h=0,$$

(5. 34) $$P_{ji}=\varphi_j{}^r Q_{ir}, \qquad Q_{ji}=\varphi_j{}^r P_{ir}.$$

because of (2. 20) and (2. 21).

In a normal contact metric space (5. 28) reduces to

(5. 35) $$\begin{aligned}P_{kji}{}^h=R_{kji}{}^h&+(g_{ki}\eta_j-g_{ji}\eta_k)\eta^h-(\delta_k^h\eta_j-\delta_j^h\eta_k)\eta_i\\ &-\varphi_j{}^h\varphi_{ki}+\varphi_k{}^h\varphi_{ji}-2\varphi_i{}^h\varphi_{kj}.\end{aligned}$$

Thus we have obtained three invariant tensors $P_{kji}{}^h$, P_{ji} and Q_{ji}. We shall call $P_{kji}{}^h$ the φ-curvature tensor.

In a K-contact metric space, if $P_{kji}{}^{h}=0$, then we get $P_{ji}=0$, that is, the Ricci tensor takes the form

$$R_{ji}=-2g_{ji}+(n+1)\eta_j\eta_i$$

which implies the space is an η-Einstein one with $R=-(n-1)$.
Conversely if an η-Einstein K-space admits an infinitesimal φ-transformation, then taking the Lie derivative of

$$R_{ji}=ag_{ji}+b\eta_j\eta_i$$

and making use of (5. 25), (5. 1) and (5.2) we obtain $P_{ji}=0$. Thus we have

THEOREM 5. 5. *In order that a K-contact metric space be an η-Einstein one it is necessary and sufficient that the tensor P_{ji} vanishes identically.*

Next, in a normal contact metric space, we can verify H_{ji} is a closed tensor by means of (2. 10), (2. 18) and Bianchi's identity. So $Q_{ji}=H_{ji}+n\varphi_{ji}$ is also closed. On the other hand from (5. 33) and (5. 34), we find

$$\begin{aligned}\nabla^i Q_{ji}&=\nabla^i(\varphi_j{}^r P_{ir})\\&=(R+n-1)\eta_j+\frac{1}{2}\varphi_j{}^r\nabla_r R.\end{aligned}$$

Thus, if $R=-(n-1)$, then we have $\nabla^i Q_{ji}=0$ which shows that Q_{ji} is a harmonic tensor. Conversely, if Q_{ji} be harmonic, we get $R=-(n-1)$. Hence we have

THEOREM 5. 6. *In a normal contact metric space, we have $R=-(n-1)$ if and only if the tensor Q_{ji} is harmonic.*

Combining Theorem 5. 3 and 5. 6, we have the following

THEOREM 5. 7. *If a normal contact metric space of constant scalar curvature has no harmonic tensor Q_{ji}, then an infinitesimal φ-transformation is an automorphism.*

§ 6. Infinitesimal affine transformations.

We shall prove the following

THEOREM 6. 1. *In a K-contact metric space, if an infinitesimal contact transformation v^i is affine, then v^i is an automorphism.*

Proof. By assumption, we shall put

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\}=0, \tag{6. 1}$$

$$\underset{v}{\pounds}\eta_i=\sigma\eta_i. \tag{6. 2}$$

Substituting (6. 1) and (6. 2) into the identity (2. 24), we get

$$\sigma_j\eta_i+\sigma\varphi_{ji}-\underset{v}{\pounds}\varphi_{ji}=0, \qquad \sigma_j=\partial_j\sigma, \tag{6. 3}$$

from which we have

$$\sigma_j\eta_i+\sigma_i\eta_j=0.$$

If we transvect this with η^i and g^{ji} respectively, then we get

$$\sigma_j=0.$$

Hence (6. 3) turns to

$$\underset{v}{\pounds}\varphi_{ji}=\sigma\varphi_{ji}, \qquad \sigma=\text{const.} \tag{6. 4}$$

from which we have

$$\underset{v}{\pounds}\eta^i=-\sigma\eta^i. \tag{6. 5}$$

Next, from (2. 16) it follows that

$$R_{kji}{}^h\eta^k\eta_h=g_{ji}-\eta_j\eta_i.$$

Operating $\underset{v}{\pounds}$ to the last equation and using (6. 2) and (6. 5), we get

$$\underset{v}{\pounds}g_{ji}=2\sigma\eta_j\eta_i \tag{6. 6}$$

because of $\underset{v}{\pounds}R_{kji}{}^h=0$.

On the other hand, taking the Lie derivative of (2. 7) and using (6. 2), (6. 4) and (6. 6), it follows that

$$\varphi_{ri}(\sigma\varphi_j{}^r+\underset{v}{\pounds}\varphi_j{}^r)=0$$

from which we obtain

$$\underset{v}{\pounds}\varphi_j{}^i=-\sigma\varphi_j{}^i.$$

By virtue of Lemma 4. 4, we have $\sigma=0$, so v^i is an automorphism.

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