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REMARKS ON n-TH ROOT CLOSEDNESS FOR COMMUTATIVE C^*-ALGEBRAS

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ABSTRACT. Let $C(X)$ be the algebra of all complex-valued continuous functions defined on a compact Hausdorff space X. For a given natural number n with $n \geq 2$, the algebra $C(X)$ is said to be n-th root closed if each function in $C(X)$ is the n-th power of another. We will show that n-th root closedness is equivalent to square-root closedness, and to algebraic closedness of $C(X)$ when X is first countable.

1. INTRODUCTION AND RESULTS

Let X be a compact Hausdorff space, and let $C(X)$ be the algebra of all complex-valued continuous functions on X. Suppose that X is locally connected, and let A be a uniform algebra on X. Čirka [1] proved that if each function in A is the square of another, then A equals $C(X)$. We say that $C(X)$ is square-root closed if for every $f \in C(X)$ there exists $g \in C(X)$ such that $f = g^2$. This algebraic property is closely related to the topological structure of X. Indeed, for locally connected X, Hatori and the second author [4] proved that $C(X)$ is square-root closed if and only if $\dim X \leq 1$ and $\check{H}^1(X;\mathbb{Z}) = 0$. Here $\dim X$ denotes the covering dimension of X (cf. [11]) and $\check{H}^1(X;\mathbb{Z})$ denotes the first Čech cohomology group of X with the integer coefficients (cf. [5]). Let $P(x,z)$ be a monic polynomial over $C(X)$ with respect to z, that is, for a positive integer n and $f_0, f_1, \cdots, f_{n-1} \in C(X)$,

$$P(x,z) = z^n + f_{n-1}(x)z^{n-1} + \cdots + f_1(x)z + f_0(x)$$

for $x \in X$. The algebra $C(X)$ is said to be algebraically closed if for every monic polynomial $P(x,z)$ over $C(X)$ there exists $g \in C(X)$ such that $P(x, g(x)) = 0$ for every $x \in X$. By definition, $C(X)$ is square-root closed whenever $C(X)$ is algebraically closed. For first countable X, Countryman, Jr. [2] proved that $C(X)$ is algebraically closed if and only if X is almost locally connected and hereditarily unicoherent. Moreover, his result states that square-root closedness is equivalent to algebraic closedness when X is first countable. The second author and Niijima [10] showed that this equivalence is also true for locally connected X. In [6], the authors considered the following condition:

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(K) For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \notin \mathbb{N}$ and $g^p = f^q$.

Here and after, $\mathbb{N}$ denotes the set of all positive integers. The property (K) was inspired by a result of Karahanjan [8], which is an extension of the result of Čirka mentioned above. From definition, if $C(X)$ is algebraically closed, then $C(X)$ has the property (K). The authors [6] proved that the converse holds if X is first countable or locally connected. Let $n \in \mathbb{N}$ with $n \geq 2$. We say that $C(X)$ is n-th root closed if for every $f \in C(X)$ there exists $g \in C(X)$ such that $f = g^n$. Clearly, $C(X)$ is n-th root closed for every $n \in \mathbb{N}$ whenever $C(X)$ is algebraically closed. In this paper, we will make up some results obtained in [6] with respect to n-th root closedness.

Theorem. *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent:*

(1) *$C(X)$ is n-th root closed for some $n \in \mathbb{N}$ with $n \geq 2$*
(2) *$C(X)$ is n-th root closed for every $n \in \mathbb{N}$ with $n \geq 2$*
(3) *$C(X)$ is algebraically closed.*

Corollary. *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent:*

(1) *$C(X)$ is n-th root closed for some $n \in \mathbb{N}$ with $n \geq 2$*
(2) *$C(X)$ is n-th root closed for every $n \in \mathbb{N}$ with $n \geq 2$*
(3) *$C(X)$ is algebraically closed*
(4) *$C(X)$ is square-root closed*
(5) *For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \neq \mathbb{N}$ and $g^p = f^q$*
(6) *X is almost locally connected and hereditarily unicoherent*
(7) *X is almost locally connected, $\dim X \leq 1$ and $\check{H}^1(X; \mathbb{Z}) = 0$.*

2. A PROOF OF RESULTS

We say that a topological space T is *almost locally connected* if and only if T does not contain mutually disjoint connected closed subsets C_n ($n \in \mathbb{N}$), which are also open in the closure of $\cup_{n \in \mathbb{N}} C_n$ in T, with the following property: there exist $x_n, y_n \in C_n$ such that $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ converge to distinct points. The closure of $\cup_{n \in \mathbb{N}}(\{1/n\} \times [0,1])$ in $\mathbb{R}^2$ is not almost locally connected. But the closure of $\cup_{n \in \mathbb{N}}(\{1/n\} \times [0,1/n])$ in $\mathbb{R}^2$ is almost locally connected. A topological space T is said to be *hereditarily unicoherent* if $M \cap N$ is connected for every pair of closed connected subsets M and N of T with $M \cap N \neq \emptyset$. The unit circle S^1 is a typical example that is not hereditarily unicoherent.

To prove our theorem, we need the following result.

Theorem A ([6, Corollary 1.3]). *Let X be a first countable compact Hausdorff space. Then the following conditions are equivalent:*

(i) *For every $f \in C(X)$ there exist $g \in C(X)$ and $p, q \in \mathbb{N}$ such that $q/p \neq \mathbb{N}$ and $g^p = f^q$*
(ii) *$C(X)$ is algebraically closed*
(iii) *$C(X)$ is square-root closed*
(iv) *X is almost locally connected and hereditarily unicoherent*
(v) *X is almost locally connected, $\dim X \leq 1$ and $\check{H}^1(X;\mathbb{Z}) = 0$.*

Proof of Theorem. By definition, the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are true. So, it is enough to show that (1) $\Rightarrow$ (3) holds. By definition, the implication (1) $\Rightarrow$ (i) of Theorem A is true. Since X is first countable, Theorem A shows that (i) is equivalent to the condition that $C(X)$ is algebraically closed. So we have that (1) implies (3), and this completes the proof. □

Proof of Corollary. This immediately follows from Theorem and Theorem A. □

Example 1. Let X be the closure of $\cup_{n\in\mathbb{N}}(\{1/n\} \times [0, 1/n])$ in $\mathbb{R}^2$. Then it is easy to see that X is hereditarily unicoherent and almost locally connected. Thus Theorem A shows that $C(X)$ is algebraically closed. In particular, $C(X)$ is n-th root closed for every $n \in \mathbb{N}$.

Example 2. Let X be the closure of $\cup_{n\in\mathbb{N}}(\{1/n\} \times [0, 1])$ in $\mathbb{R}^2$. Then it is easy to verify that X is not almost locally connected. So, by Corollary, $C(X)$ is not n-th root closed for any $n \geq 2$.

Example 3. Let S^1 be the unit circle. Then S^1 is not hereditarily unicoherent. So, by Corollary, $C(S^1)$ is not n-th root closed for any $n \geq 2$.

Remark 2.1. By Theorem and Corollary, if X is first countable, then n-th root closedness is equivalent to square-root closedness and to algebraic closedness. The same holds for locally connected X (see [6, Corollary 1.2 and Remark]).

Remark 2.2. Let us consider the following two conditions:

(a) The set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for *every* $n \in \mathbb{N}$
(b) The set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for *some* $n \in \mathbb{N}$ with $n \geq 2$.

The implications (2) in Theorem $\Rightarrow$ (a) $\Rightarrow$ (b) is obviously true for any compact Hausdorff space X. In [6] it was shown that if X is locally connected, then the implication (b) $\Rightarrow$ (2) holds. For first countable X, however, the situation is different. For example, let X be the closure of $\cup_{n\in\mathbb{N}}(\{1/n\} \times [0,1])$ in $\mathbb{R}^2$. Then, from [4], we see that $\dim X = 1$ and $\check{H}^1(X;\mathbb{Z}) = 0$. So, by Lemma 2.5 in [6], the set $\{f^n : f \in C(X)\}$ is uniformly dense in $C(X)$ for every $n \geq 2$. But, as in Example 2, $C(X)$ is not n-th root closed for any $n \geq 2$.

Remark 2.3. From Theorem, we see that n-th root closedness for some $n \geq 2$ implies m-th root closedness for all $m \in \mathbb{N}$ whenever X is first countable. The implication is true for locally connected X (see [6, Corollary 1.2 and Remark]). So, a natural question arises: is this implication true for all compact Hausdorff spaces? Countryman, Jr. [2, Remarks (2)] noted that there exists a compact Hausdorff space X, which is not first countable nor locally connected, with the following property: there exists $f \in C(X)$ such that f has a continuous 2^n-th root in $C(X)$ for every $n \in \mathbb{N}$ but that no continuous fifth root. Here, continuous k-th root of f means a continuous function in $C(X)$ whose k-th power equals f. Recently, Kawamura and the second author [9] gave a negative answer to the question above. More explicitly, they showed that for each pair of relatively prime positive integers m and n, there exists a compact Hausdorff space X such that $C(X)$ is n-th root closed but not m-th root closed.

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