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HOLOMORPHICS FUNCTIONS
TAKING VALUES IN QUOTIENTS OF FRECHET b-SPACES

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Abstract. *We define two spaces of holomorphic function taking values in a quotient bornological space, and we establish a sufficient condition under which these two spaces coincide.*

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1. Introduction and notations

To establish a version of Bartle and Graves Theorem for approximatively surjective mappings between b-spaces, we introduced in [2], what we called the class of Fréchet b-spaces. In fact, we observed that the boundedness of a Fréchet space has a property that a general bornology does not have and we said that a b-space E is a Fréchet b-space if for all sequences of bounded subsets $(B_n)_n$ of E, there exists a sequence of positive real numbers $(\lambda_n)_n$ such that $\cup_n \lambda_n B_n$ is bounded in E.

If U is an open connected subset of $\mathbb{C}^n$ and $E \mid F$ is a quotient bornological space, we let $O(U, E \mid F) \simeq O(U, E) \mid O(U, F)$ be a space of holomorphic function taking values in a quotient bornological space $E \mid F$. We define a space of holomorphic function taking values in a quotient bornological space $E \mid F$ as the space $O_1(U, E \mid F) = \varprojlim_V O(V, E \mid F)$ where V ranges over the relatively compact open subsets of U. The objective of this paper is to show that if $E \mid F$ is a quotient bornological space such that E and F are Fréchet b-spaces, then $O_1(U, E \mid F) \simeq O(U, E) \mid O(U, F)$.

Also, we consider the space of holomorphic functions near to a compact set X defined by $O([X]) \simeq \varinjlim_U O(U)$ where U is a neigbourhood of X (see for example

Gunning and Rossi [4]) and we define the space $O_1([X], E \mid F) \simeq \varinjlim_U O_1(U, E \mid F)$ where U is a neighbourhood of X. We will show that if $E \mid F$ is a quotient bornological space and X is a compact space, then the quotient bornological spaces $O_1([X], E \mid F)$ and $O([X], E \mid F)$ are isomorphic.

Let us fix some notations and recall some definitions that will be used in this paper. Let **EV** be the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$, and **Ban** the subcategory of Banach spaces and bounded linear mappings.

1- Let $(E, \|\|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace endowing with a Banach norm $\|\|_F$ such that the inclusion map $(F, \|\|_F) \longrightarrow (E, \|\|_E)$ is bounded. Observe that the norm $\|\|_F$ of F is not necessary the same as the norm induced by $\|\|_E$ on F, and then the Banach subspace F is not necessary closed in E. A quotient Banach space $E \mid F$ is a vector space E/F, where E is a Banach space and F a Banach subspace. It is clear that $E \mid F$ is not necessary an object of the category of Banach spaces **Ban**, but is one if F is closed in E. If $E \mid F$ and $E_1 \mid F_1$ are two quotient Banach spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a linear mapping $u : x + F \longmapsto u_1(x) + F_1$, where $u_1 : E \longrightarrow E_1$ is a bounded linear mapping such that $u_1(F) \subseteq F_1$. We shall say that u_1 induces u. Two bounded linear mappings u_1, $u_2 : E \longrightarrow E_1$ both inducing a strict morphism, induce the same strict morphism iff the linear mapping $u_1 - u_2 : E \longrightarrow F_1$ is bounded. For more information about quotient Banach spaces we refer the reader to [8] and [9].

2- Let E be a real or complex vector space, and let B be an absolutely convex set of E. Let E_B be the vector space generated by B i.e. $E_B = \cup_{\lambda > 0} \lambda B$. The Minkowski functional of B, $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$ is a semi-norm on E_B. It is a norm if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of "bounded" subsets of E with the following properties :

Every finite subset of E is bounded; **2)** every union of two bounded subsets is bounded; **3)** every subset of a bounded subset is bounded; **4)** a set homothetic to a bounded subset is bounded; **5)** each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if the subspace $F \cap E_B$ is closed in E_B for every completant bounded subset B of E.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \longrightarrow F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u : E \longrightarrow F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$. Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. We denote by $\mathbf{b}(E_1, E_2)$ the space of all bounded linear mappings $E_1 \longrightarrow E_2$ and by $\mathbf{b}$ the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [3] and [7].

3- Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. We note that the boundedness β_F of F is not necessary the same as the boundedness induced by β_E on F, and then the b-subspace F is not necessary bornologically closed in E. A quotient bornological space $E \mid F$ is a vector space E/F, where E is a b-space and F a b-subspace of E. Observe that $E \mid F$ is not necessary an object of the category of b-spaces $\mathbf{b}$, but is one if F is bornologically closed in E. If $E \mid F$ and $E_1 \mid F_1$ are quotient bornological spaces, a strict morphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ whose restriction to F is a bounded linear mapping $F \longrightarrow F_1$. Two bounded linear mappings u_1,$v_1 : E \longrightarrow E_1$, both inducing a strict morphism, induce the same strict morphism $E \mid F \longrightarrow E_1 \mid F_1$ iff the linear mapping $u_1 - v_1 : E \longrightarrow F_1$ is bounded.

We call $\tilde{\mathbf{q}}$ the category of quotient bornological spaces and strict morphisms. A pseudo-isomorphism $u : E \mid F \longrightarrow E_1 \mid F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \longrightarrow E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F_1}$.

The category $\tilde{\mathbf{q}}$ is not abelian. In fact, if E is a Banach space and F a closed subspace of E, it would be very nice if the quotient Banach space $E \mid F$ where isomorphic to the quotient $(E/F) \mid \{0\}$. This is not the case in $\tilde{\mathbf{q}}\mathbf{Ban}$ unless F is complemented in E. In [10], Waelbroeck introduced an abelian category $\mathbf{q}$ generated by $\tilde{\mathbf{q}}$ and inverses of pseudo-isomorphims i.e. has the same objects as $\tilde{\mathbf{q}}$ and every morphism u of $\mathbf{q}$ can be expressed as $u = v \circ s^{-1}$, where s is a pseudo-isomorphism and v is a strict morphism.

4. The ε-product of two Banach spaces E and F is the Banach space $E\varepsilon F$ of linear mappings $E' \longrightarrow F$ whose restrictions to the closed unit ball $B_{E'}$ of E' are continuous for the topology $\sigma(E', E)$. It follows from Proposition 2 of [6] that the ε-product is symmetric i.e. the Banach spaces $E\varepsilon F$ and $F\varepsilon E$ are isometrically isomorphic. If E_i et F_i are Banach spaces and $u_i : E_i \longrightarrow F_i$ are bounded linear mappings, $i = 1, 2$, the ε-product of u_1 and u_2 is the bounded linear mapping $u_1 \varepsilon u_2 : E_1 \varepsilon E_2 \longrightarrow F_1 \varepsilon F_2$, $f \longmapsto u_2 \circ f \circ u_1'$, where u_1' is the dual mapping of u_1. It is clear that $u_1 \varepsilon u_2$ is injective whenever u_1 and u_2 are injectives. If G is a Banach space and F is a Banach subspace of a Banach space E, then $G\varepsilon F$ is a Banach subspace of $G\varepsilon E$. For more information about the ε-product the reader is refered to [6].

5. A b-space G is nuclear if all bounded completant subset B of G is included in a bounded completant subset A of G such that the inclusion $i_{AB} : G_B \longrightarrow G_A$ is a nuclear mapping. For more information about nuclear b-spaces we refer the reader to [3].

2. Main result.

If E is a b-space, then $E = \cup_B E_B$ where B ranges over the bounded completant subsets of E. The ε-product of a nuclear b-space N by E is a b-space (is bornologically isomorphic to a b-space), and is isomorphic to $\cup_{i,B} N_i\,(E_B) \simeq N\varepsilon E$ where $i \in I$ and B ranges over the bounded completant subsets of E (we let $N_i\,(E_B) \simeq l^p\,(E_B)$ or $c_o\,(E_B)$ [3]).

We remember that $O\,(U)$ is a nuclear Fréchet space. If we put on $O\,(U)$ its von Neumann boundedness (i.e. a subset B is bounded in the von Neumann boundedness of $O\,(U)$ if it is absorbed by all neighbourhoods of the origin for the Fréchet topology of $O\,(U)$), this space can also be seen as a nuclear b-space.

It is clear that $O\,(U)\,\varepsilon E = \cup_B O\,(U, E_B)$ when E is a b-space. And when E is a Fréchet space, then $O\,(U)\,\varepsilon E = O\,(U, E)$ where ε is the ε-product in the category of locally convex spaces [5].

On the other hand, recall that the boundedness of a Fréchet space has a property that a general bornology does not have. The b-spaces whose boundedness have this property were called [2] "Fréchet b-spaces".

Recall from [2] that a b-space E is a Fréchet b-space if for all sequences of bounded subsets $(B_n)_n$ of E, there exists a sequence of positive real numbers $(\lambda_n)_n$ such that $\cup_n \lambda_n B_n$ is bounded in E.

If E is a Fréchet b-space and $(B_n)_n$ is a sequence of bounded subsets of E, there exists a completant bounded subset B' of E which absorbs all the B_n. In fact, let $(B_n)_n$ be a sequence of bounded subsets of E, there exists a sequence $(\lambda_n)_n$ such that, for all $n \in \mathbb{N}$, $\lambda_n \in \mathbb{R}^+$, $\lambda_n \neq 0$, and the subset $\cup_n \lambda_n B_n$ is bounded in E. The subset $\sum_n 2^{-n-1} \lambda_n B_n$ is completant, contained in the completant hull of $\cup_n \lambda_n B_n$ and absorbs all the the bounded subsets B_n.

For examples of Fréchet b-spaces.

i- Let E be a Fréchet space, we design by E_b, the space E with its von Neumann boundedness (i.e. a subset B is bounded in the von Neumann boundedness of E if it is absorbed by all neighbourhoods of the origin). The b-space E_b is a Fréchet b-space.

ii- If we consider the b-space of all operators on a Fréchet space, its boundedness is not of type Fréchet.

Let U be an open connected subset of $\mathbb{C}^n$ and $E \mid F$ be a quotient bornological space. We define a space of holomorphic function taking values in a quotient bornological space $E \mid F$ as the space $O_1(U, E \mid F) = \varprojlim_V \left(O\left(V\right)\varepsilon(E \mid F)\right)$ where V ranges over the relatively compact open subsets of U.

Also, recall from [2] that if (E, β_E) and (F, β_F) are b-spaces, a bounded linear mapping $u : E \longrightarrow F$ is approximatively surjective if for every $B \in \beta_F$, completant, there exist $B_1 \in \beta_F$ and $C \in \beta_E$, both B_1 and C are completant such that $B \subset B_1$, $u\left(C\right) \subset B_1$ and for every $\varepsilon > 0$ we have $B_1 \subset \varepsilon B_1 + \cup_{M \in \mathbb{R}} Mu(C)$.

We observe that in the Banach case it is clear that a mapping is approximatively surjective if and only if it has a dense range.

In [2], we proved the following result which is a bornological version of Mittag-Leffler Lemma in the category **b**.

Theorem 2.1. [2] *For each $n \in \mathbb{N}$, let E_n be a b-space and let F_n be a closed subspace of E_n with a Fréchet boundedness. For all $n \in \mathbb{N}$, let $u_{n+1} : E_{n+1} \longrightarrow E_n$ be a bounded linear mapping whose restriction $v_{n+1} = u_{n+1|_{F_{n+1}}} : F_{n+1} \longrightarrow F_n$ is an approximatively surjective bounded linear mapping. Then $\varprojlim_n(E_n/F_n) \simeq (\varprojlim_n E_n)/(\varprojlim_n F_n)$.*

As consequence, we obtain the following result which is a bornological version of Mittag-Leffler Lemma in the category **q**.

Corollary 2.2. *For each $n \in \mathbb{N}$, let E_n be a b-space and let F_n be a b-subspace of E_n with a Fréchet boundedness. For all $n \in \mathbb{N}$, let $u_{n+1} : E_{n+1} \longrightarrow E_n$ be a bounded linear mapping whose restriction $v_{n+1} = u_{n+1|_{F_{n+1}}} : F_{n+1} \longrightarrow F_n$ is an approximatively surjective bounded linear mapping. Then $\underleftarrow{\lim}_n(E_n \mid F_n) \simeq (\underleftarrow{lim}_n E_n) \mid (\underleftarrow{\lim}_n F_n)$.*

Proof. It follows from Theorem 2.1 that the functor $\underleftarrow{lim}_n\,(.) : \mathbf{b} \longrightarrow \mathbf{b}$ is exact. Hence, Theorem 4.1 of [10] implies that this functor admits an unique and exact extension to the category $\mathbf{q}$ that we design also by $\underleftarrow{lim}_n\,(.) : \mathbf{q} \longrightarrow \mathbf{q}$. As consequence from ([1], Theorem ...), we obtain

$$\underleftarrow{\lim}_n(E_n \mid F_n) \simeq (\underleftarrow{lim}_n E_n) \mid (\underleftarrow{\lim}_n F_n).$$

Our first result is the following :

Theorem 2.3. *Let $E \mid F$ be a quotient bornological space such that E and F are Fréchet b-spaces. Then $O_1(., E \mid F) \simeq O\,(.)\,\varepsilon(E \mid F)$.*

Proof. If E is a Fréchet b-space and U is a manifold, then $O\,(U, E) \simeq \cup_B O\,(U, E_B)$ where B ranges over the bounded completant subsets of E. If E is a b-space, then $O\,(U, E) = \underleftarrow{\lim}_V\,(\cup_B O\,(V, E_B))$ where B ranges over the bounded completant subsets of E and V ranges over the relatively compact subsets of U.

To end the proof, we shall use the Mittag Leffler Lemma and Runge's Theorem. Let us state the Runge Theorem. Let U be a complex Stein manifold, let V be a holomorphically convex of U, and let E be a Banach space. Then the restriction $O\,(U, E) \longrightarrow O\,(V, E)$ has a dense range. If E is a b-space, then the restriction mapping $O\,(U, E) \longrightarrow O\,(V, E)$ is approximatively surjective.

Let U be a complex manifold and $\widetilde{U}$ be its associated domain of holomorphy, $\widetilde{U}$ is the union of a sequence of V_n where each V_n is relatively compact in the interior of V_{n+1} and is holomorphically convex in $\widetilde{U}$. The Mittag Leffler Lemma uses an infinite commutative diagram part of which is

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O\left(V_{n+1}\right)\varepsilon F & \longrightarrow & O\left(V_{n+1}\right)\varepsilon E & \longrightarrow & \left(O\left(V_{n+1}\right)\varepsilon E\right) \mid \left(O\left(V_{n+1}\right)\varepsilon F\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O\left(V_{n}\right)\varepsilon F & \longrightarrow & O\left(V_{n}\right)\varepsilon E & \longrightarrow & \left(O\left(V_{n}\right)\varepsilon E\right) \mid \left(O\left(V_{n}\right)\varepsilon F\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The mapping $O\left(V_{n+1}, F\right) \longrightarrow O\left(V_n, F\right)$ is approximatively surjective and the spaces E and F are Fréchet b-spaces. Therefore at the projective limit, it follows from Corollary 2.2, that

$$\varprojlim_n O\left(V_n, E\right) \mid \varprojlim O\left(V_n, F\right) \simeq \varprojlim \left(O(V_n, E \mid F)\right)$$

We see then that $O\left(U, E\right) \mid O\left(U, F\right) \simeq O_1(U, E \mid F)$. This shows the Theorem.

Authors working in complex variables, for example Gunning and Rossi [4], consider holomorphic functions near to a compact set X i.e they define the following space

$$O\left([X]\right) \simeq \varinjlim_U O\left(U\right)$$

where U is a neigbourhood of X. Since the category $\mathbf{q}$ is stable under the inductive limits and the projective limits, we consider the quotient bornological space $O_1([X], E \mid F)$. It is an inductive limit of the inductive system of quotient bornological spaces $(O_1(U, E \mid F))_U$ i.e. $O_1([X], E \mid F) \simeq \varinjlim_U O_1(U, E \mid F)$ where U is a neighbourhood of X.

Theorem 2.4. *Let $E \mid F$ be a quotient bornological space and X a compact space. The quotient bornological spaces $O_1([X], E \mid F)$ and $O\left([X]\right)\varepsilon(E \mid F)$ are isomorphic.*

Proof. Consider Y compact, with $Y \subset U$ and $X \subset \overset{o}{Y}$ where $\overset{o}{Y}$ is the interior of Y. We have the following morphisms

$$O_1(U, E \mid F) \longrightarrow O\,([Y])\,\varepsilon(E \mid F) \longrightarrow O\,([X])\,\varepsilon(E \mid F).$$

At the inductive limit, we obtain a morphism

$$O_1([X]\,, E \mid F) \longrightarrow O\,([X])\,\varepsilon(E \mid F).$$

On the other hand, we have a morphism

$$O_1([Y]\,, E \mid F) \longrightarrow O\,([Y])\,\varepsilon(E \mid F).$$

But right exact functors resist to inductive limits in $\mathbf{q}$, and the ε-product $N\varepsilon.$ is a right exact functor whenever N is a nuclear b-space. Therefore we have a morphism

$$O\,([X])\,\varepsilon(E \mid F) \longrightarrow O_1([X]\,, E \mid F).$$

The two morphisms so defined are clearly inverse one of the other. This proves the Theorem.

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