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AN EXAMPLE OF MULTIPLICATIVELY SPECTRUM-PRESERVING MAPS BETWEEN NON-ISOMORPHIC SEMI-SIMPLE COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. In this paper we give an example of a multiplicatively spectrum-preserving map between two non-unital commutative semisimiple Banach algebras which are not *algebraically* isomorphic to each other.

1. INTRODUCTION

Molnár [6] initiated the study of multiplicatively spectrum-preserving maps on Banach algebras and proved among other theorems that a map T from a Banach algebra $C(\mathcal{X})$ of all complex-valued continuous functions on a first countable compact Hausdorff space $\mathcal{X}$ onto itself is an almost isomorphism in the sense that T is an algebra isomorphism times a weight with the values in $\{-1,1\}$ if T is multiplicatively spectrum preserving in the sense that the spectrum of the product of any two functions f and $g \in C(\mathcal{X})$ equals to the spectrum of the product of Tf and Tg. Rao and Roy [7] generalized the result for an arbitrary uniform algebra onto itself. Hatori, Miura and Takagi [3] studied the maps between arbitrary two uniform algebras which are multiplicatively range-preserving and showed that the maps are almost isomorphisms. In particular, they shoed the two uniform algebras are algebraically isomorphic to eacy other. Hatori, Miura and Takagi [4] consider the case where underlying algebras are unital semisimple commutative Banach algebras. They showed that the surjective maps between those commutative Banach algebras are almost isomophic if they are multiplicatively spectrum-preserving and the two Banach algebras are algebraically isomorphic to each other. Luttman and Tonev [5] considered multiplicatively preserving property for much more smaller set; peripheral ranges. They proved that if a surjective map between uniform algebra satisfies that the peripheral range of the product of any two functions equals to the peripheral range of the product of the images of those two functions, then the map is an almost isomorphism and these uniform algebras are isometrically isomorphic to each other as Banach algebras. For the case where the underlying Banach algebras

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need not be unital, Rao and Roy [8] consider maps from uniformly closed algebras of continuous functions which vanish at infinity onto itself.

In any case of the previous results the domain algebra and the image algebra of the given map are *algebraically* isomorphic. In this paper we give an example of a multiplicatively spectrum-preserving map between two non-unital commutative semisimiple Banach algebras which are not *algebraically* isomorphic to each other.

2. A Main Result

Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $D_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$, $\bar{D}_0 = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$ and $L = \{z \in \mathbb{R} : 1 \leq z \leq 2\}$. Put $X_0 = \bar{D}_0 \cup L$, $\overline{X_0} = X_0 \cup \{0\}$ and $X = X_0 \times \{1, 2\}$. For $i = 1$ and 2 define maps π_i from X_0 into X such that $\pi_i(z) = (z, i)$ respectively. Then the map π_i is a homeomorphism from X_0 onto $X_0 \times \{i\}$ for $i = 1,\ 2$. Put complex-valued functions f_A and f_B on X by $f_A(z, i) = z$ and $f_B(z, i) = (-1)^{i+1} z$. The algebra of all complex-valued continuous functions on $\overline{X_0}$ which is analytic on D is denoted by $P(\overline{X_0})$. We denote $P(X_0)$ by the restriction of $P(\overline{X_0})$ on X_0 and $P_{00}(X_0) = \{z^2 f(z) : f \in P(X_0)\}$. The algebra of all complex-valued continuous functions which vanish at infinity is denoted by $C_0(X)$. We denote $A_0 = \{f \in C_0(X) : f \circ \pi_1,\ f \circ \pi_2 \in P_{00}(X_0)\}$. Put $A = A_0 + \mathbb{C} f_A$ and $B = A_0 + \mathbb{C} f_B$. It is easy to see that A and B are closed subalgebras of $C_0(X)$ which strongly separate the points of X.

Theorem 2.1. *Let A and B be the Banach algebras which are defined above. Let $\eta : X \to \{-1, 1\}$ be $\eta(x, i) = (-1)^i$. Then the map $T : A \to B$ defined by $T(f) = \eta f$ is surjective and satisfies the equality*

$$\sigma(T(f)T(g)) = \sigma(fg), \quad f, g \in A.$$

The Banach algebras A and B are not algebraically isomorphic to each other.

Note that A is isometrically isomorphic to B as a Banach space.

Lemma 2.2. *For every $f \in A$ and $i = 1$ and 2, the function $f \circ \pi_i$ is analytically extended at the origin 0 and the derivative at the origin of both functions coincides, that is, $(f \circ \pi_1)'(0) = (f \circ \pi_2)'(0)$.*

Proof. It is simple that the extended function with the value 0 at the origin for $f \circ \pi_i$ is analytic on D. For the rest of the proof the extended function of $f \in A$ is also denoted by f. Let $f \in A$. Then f is docmposed by $f = f_0 + \lambda f_A$ for an $f_0 \in A_0$ and a complex number λ. Then we have

$$(f \circ \pi_i)'(0) = (f_0 \circ \pi_i)'(0) + \lambda (f_A \circ \pi_i)'(0)$$

for $i = 1,\ 2$. Since $(f_0 \circ \pi_i)'(0) = 0$ and $(f_A \circ \pi_i)'(0) = 1$, we see that $(f \circ \pi_1)'(0) = \lambda = (f \circ \pi_2)'(0)$. □

In the same way we see the following.

Lemma 2.3. *For every $f \in B$ and $i = 1$ and 2, the function $f \circ \pi_i$ is analytically extended at the origin* 0 *and* $(f \circ \pi_1)'(0) = -(f \circ \pi_2)'(0)$.

The maximal ideal space for A (resp. B) is denoted by M_A (resp. M_B). The space X is embedded in M_A (resp. M_B) by the natural embedding. We see the following.

Lemma 2.4. *The maximal ideal space M_A of A coincides with X itself.*

Proof. First we show that the maximal ideal space $M_{P_{00}(X_0)}$ is equal to X_0. It is trivial that $M_{P_{00}(X_0)} \supset X_0$, we show that the opposite inclusion. Supoose that $\phi \in M_{P_{00}(X_0)}$. By the definition the function z^2 is in $P_{00}(X_0)$ and we see that $\phi(z^2) \neq 0$. (Suppose not. Put an arbitrary $f \in P_{00}(X_0)$. Then there is a $g \in P(X_0)$ with $f = z^2 g$. We see that $z^2 g^2 \in P_{00}(X_0)$ and

$$\phi(f)^2 = \phi(f^2) = \phi(z^2)\phi(z^2 g^2) = 0.$$

It follows that ϕ vanishes since f is an arbitrary element in $P_{00}(X_0)$, which is a contradiction.) Put a function $\bar{\phi} : P(X_0) \to \mathbb{C}$ by

$$\bar{\phi}(f) = \frac{\phi(z^2 f)}{\phi(z^2)}$$

for each $f \in P(X_0)$. Then by a simple calculation we see that $\bar{\phi}$ is a non-zero complex homomorphism on $P(X_0)$, that is, $\bar{\phi}$ is in the maximal ideal space $M_{P(X_0)}$ of $P(X_0)$. Since $P(X_0)$ is algebraically isomorphic to $P(\overline{X_0})$, we see that $M_{P(X_0)} = \overline{X_0}$ (cf. [2, Corollary II 1.10]); there exists an $x \in \overline{X_0}$ such that $\bar{\phi}(f) = f(x)$ holds for every $f \in P(X_0)$. We see that $x \neq 0$. (Suppose not. Then we have

$$0 = \bar{\phi}(z^2) = \frac{\phi(z^4)}{\phi(z^2)} = \phi(z^2) \neq 0,$$

which is a contradiction.) We also see by a similar calculation that $\bar{\phi}|P_{00}(X_0) = \phi$. It follows that the equalities

$$\phi(f) = \bar{\phi}(f) = f(x)$$

hold for every $f \in P_{00}(X_0)$. Thus we see that $M_{P_{00}(X_0)} = X_0$.

Next we show that the maximal ideal space M_{A_0} of A_0 coincides with X. By the definition of the algebra A_0, A_0 is algebraically isomorphic to the direct product of two copies of $P_{00}(X_0)$, so the maximal ideal space of A_0 is homeomorphic to the topological sum of those of $P_{00}(X_0)$, which is homeomorphic to X.

Finally we show that $M_A = X$. We only need to show that $M_A \subset X$. Let $\phi \in M_A$. Since $(f_A)^2 \in A_0$ we see that the restriction $\phi|A_0$ of ϕ to A_0 is non-zero. So there exists an $x \in X$ such that the equality

$$\phi(f) = f(x)$$

holds for every $f \in A_0$. Put a function Z^2 on X by $Z^2(z,i) = z^2$ for every $(z,i) \in X$. Then $Z^2 f_A$ is in A_0 and $\phi(Z^2 f_A) = Z(x)^2 f_A(x)$. Since ϕ is multiplicative on A, we have that $\phi(Z^2 f_A) = \phi(Z^2)\phi(f_A)$. It follows that $\phi(f_A) = f_A(x)$ since $\phi(Z^2) = Z^2(x) \neq 0$. It follows that the equality

$$\phi(f) = f(x)$$

holds for every $f \in A$. We see that $M_A \subset X$. □

In a way similar to the above we see the following.

Lemma 2.5. *The maximal ideal space M_B of B is X itself.*

Lemma 2.6. *A is not algebraically isomorphic to B.*

Proof. Suppose that A is algebraically isomorphic to B. Let T be an algebra isomorphism from A onto B. Then there exists a homeomophism Φ from M_B onto M_A with

$$\widehat{T(f)} = \hat{f} \circ \Phi, \quad f \in A.$$

Since $M_A = X = M_B$ and $X = X_0 \times \{1\} \cup X_0 \times \{2\}$, we have that (1) $\Phi(X_0 \times \{1\}) = X_0 \times \{1\}$ and $\Phi(X_0 \times \{2\}) = X_0 \times \{2\}$; or (2) $\Phi(X_0 \times \{1\}) = X_0 \times \{2\}$ and $\Phi(X_0 \times \{2\}) = X_0 \times \{1\}$. We consider the case (1). (For the case (2) we can prove in a similar way and the proof is omitted.) In this case there exist self-homeomorphisms Φ_1 and Φ_2 on X_0 with $\Phi(x,1) = (\Phi_1(x),1)$ and $\Phi(x,2) = (\Phi_2(x),2)$ for every $x \in X_0$. Since $T(f_A) \circ \pi_1 = \Phi_1$ on X_0, $T(f_A) \circ \pi_1$ is an analytic automorphism on D_0 and $T(f_A) \circ \pi_1(1) = 1$. It follows that $T(f_A) \circ \pi_1$ is extended to the analytic automorphism on D which fixes the origin and 1; $T(f_A) \circ \pi_1(z) = z$ for every $z \in D$. So we have that $(T(f_A) \circ \pi_1)'(0) = 1$. In a way similar to the above, we see that $(T(f_A) \circ \pi_2)'(0) = 1$, which contradicts to Lemma 2.3. Thus we see that A is not algebraically isomorphic to B. □

Lemma 2.7. *Let $\eta : X \to \{-1,1\}$ be $\eta(x,i) = (-1)^i$. Then the map $T : A \to B$ defined by $T(f) = \eta f$ is surjective and satisfies the equality*

$$\sigma(T(f)T(g)) = \sigma(fg), \quad f,g \in A.$$

A proof is clear and is omitted.

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