Nihonkai Math. J.
Vol.18(2007), 1-9

RICCI-PSEUDO-SYMMETRIC REAL HYPERSURFACES IN COMPLEX SPACE FORMS

IN-BAE KIM, HYE JEONG PARK AND HYUNJUNG SONG

ABSTRACT. We characterize a Ricci-pseudo-symmetric real hypersurface M with associated function f in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. We show that f is a constant on M, and M is locally congruent to a real hypersurface of type A_2 if $c > 0$, and that of type A_0 if $c < 0$.

1. Introduction

The nonexistence of semi-parallel and semi-symmetric real hypersurfaces in a complex space form $M_n(c)$ has been established for $n \geq 3$ (see [1], [2], [3], [4] and [6]). Thus it is natural to find a weaker condition than the semi-parallelism or semi-symmetric one that allows to be classified the real hypersurfaces. Recently, G. A. Lobos and M. Ortega ([3]) studied the existence of pseudo-parallel real hypersurfaces in $M_n(c)$, $c \neq 0$.

Let M be real hypersurface in a complex space form, and let R and S be the curvature tensor and the Ricci operator of M. Given tangent vector fields X and Y on M, let $X \wedge Y$ denote the operator of the tangent bundle of M given by $Z \mapsto < Y, Z > X - < X, Z > Y$, where $<,>$ is the inner product. It can be extended to act as a derivation on S as follows:

$$(X \wedge Y \cdot S)Z = (X \wedge Y)SZ - S(X \wedge Y)Z.$$

2000 AMS Subject Classification: Primary 53C40; Secondary 53C15.

This research was supported by the research fund of Hankuk University of Foreign Studies.

The third author was supported by Korea Research Foundation Grant.(KRF-2003-037-C00009)

Keywords and phrases : pseudo-symmetric real hypersurface, Ricci operator, model spaces of type A.

The curvature operator $R(X,Y)$ can operate in the same way, that is,

$$(R(X,Y)\cdot S)Z = R(X,Y)SZ - SR(X,Y)Z.$$

A real hypersurface M in a complex space form $M_n(c)$ is called *Ricci-pseudo-symmetric with associated function* f if there is a real valued smooth function f on M such that

$$R(X,Y)\cdot S = f\ X\wedge Y\cdot S \tag{1.1}$$

for any tangent vector fields X and Y on M. The condition (1.1) is weaker than the semi-symmetric one, which is defined by $f=0$, and stronger than the cyclic Ryan one, which is defined by $\mathfrak{S}(R(X,Y)\cdot S)Z=0$ (see [1] and [5]). Under the cyclic Ryan condition, U.-H. Ki, H. Nakagawa and Y. J. Suh ([1]) proved that the structure vector field ξ of a cyclic Ryan real hypersurface M in $M_n(c)$, $c\neq 0$, $n\geq 3$, is principal, and M is locally congruent to one of the model spaces of type A and B.

The purpose of this paper is to investigate the existence of Ricci-pseudo-symmetric real hypersurfaces. Namely, we shall prove the following.

Theorem. *Let M be a connected Ricci-pseudo- symmetric real hypersurface with associated function f in a complex space form $M_n(c)$, $c\neq 0$, $n\geq 3$, of constant holomorphic sectional curvature c. Then $f=\frac{|c|}{4}$, and M is locally congruent to one of the followings:*

(1) If $c>0$, (A_2) tubes over totally geodesic complex projective spaces $P_k(\mathbb{C})$ $(1\leq k\leq n-2)$ with principal curvatures 0 of multiplicity 1, $\frac{\sqrt{c}}{2}$ of $n-1$ and $-\frac{\sqrt{c}}{2}$ of $n-1$.

(2) If $c<0$, (A_0) horospheres with principal curvatures $\sqrt{-c}$ of multiplicity 1 and $\frac{\sqrt{-c}}{2}$ of $2n-2$.

2. Preliminaries

Let M be a real hypersurface in an $n(\geq 3)$-dimensional complex space form $(M_n(c), <,>, J)$ of constant holomorphic sectional curvature c, and let N be a unit normal vector field on an open neighborhood in M. For a local tangent vector field X on the neighborhood, the images of X and N under the almost complex structure J of $M_n(c)$ can be expressed by

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where ϕ defines a linear transformation on the tangent space $T_p(M)$ of M at any point $p \in M$, and η and ξ denote a 1-form and a unit tangent vector field on the neighborhood respectively. Then, denoting the Riemannian metric on M induced from the metric on $M_n(c)$ by the same symbol $<,>$, it is easy to see that

$$<\phi X, Y> + <\phi Y, X> = 0, \qquad <\xi, X> = \eta(X)$$

for any tangent vector field X and Y on M. The collection $(\phi,\ <,>,\ \xi,\ \eta)$ is called an *almost contact metric structure* on M, and satisfies

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \\ &<\phi X, \phi Y> = <X, Y> - \eta(X)\eta(Y). \end{aligned} \tag{2.1}$$

Let ∇ be the Riemannian connection with respect to the metric $<,>$ on M, and A be the shape operator in the direction of N on M. Then we have

$$\nabla_X \xi = \phi A X, \qquad (\nabla_X \phi) Y = \eta(Y) A X - <AX, Y> \xi. \tag{2.2}$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are given by

$$\begin{aligned} R(X,Y)Z = &\frac{c}{4}\{<Y,Z>X - <X,Z>Y + <\phi Y, Z>\phi X \\ &- <\phi X, Z>\phi Y - 2<\phi X, Y>\phi Z\} \\ &+ <AY, Z>AX - <AX, Z>AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2<\phi X, Y>\xi\} \tag{2.4}$$

for any tangent vector fields X, Y and Z on M, where R is the Riemannian curvature tensor of M. If we denote the Ricci operator of M by S, then it follows from (2.3) that

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2 X, \tag{2.5}$$

where $m = trA = traceA$ is the mean curvature of M.

3. Ricci-pseudo-symmetric real hypersurfaces

Let M be a Ricci-pseudo-symmetric real hypersurface in $M_n(c)$, $c \neq 0$, $n \geq 3$, with the associated function f. Then it follows from (1.1) and (2.3) that

$$
\begin{aligned}
&(\frac{c}{4} - f)\{< Y, SZ > X - < X, SZ > Y - < Y, Z > SX \\
&+ < X, Z > SY\} + \frac{c}{4}\{< \phi Y, SZ > \phi X - < \phi Y, Z > S\phi X \\
&- < \phi X, SZ > \phi Y + < \phi X, Z > S\phi Y \\
&-2 < \phi X, Y > (\phi S - S\phi)Z\} \\
&+ < AY, SZ > AX - < AX, SZ > AY \\
&- < AY, Z > SAX + < AX, Z > SAY = 0
\end{aligned}
\tag{3.1}
$$

for any tangent vector fields X, Y and Z on M. Since the cyclic sum of $(X \wedge Y \cdot S)Z$ for the vectors X, Y and Z vanishes identically, M is a cyclic Ryan-space. Thus we see from the result of [1] that the structure vector field ξ of M is principal, that is,

$$A\xi = \alpha\xi. \tag{3.2}$$

From (2.5) and (3.2), we have

$$S\xi = k\xi, \qquad k = \frac{n-1}{2}c + m\alpha - \alpha^2. \tag{3.3}$$

Putting $X = Z = \xi$ into (3.1) and using (1.1), (3.2) and (3.3), we obtain

$$(\frac{c}{4} - f)(SY - kY) + \alpha(SAY - kAY) = 0 \tag{3.4}$$

for any tangent vector field Y on M.

Now we take a local orthonormal frame field $\{E_1,\ E_2,\ \ldots,\ E_{2n-1}\}$ on M. If we put $Y = Z = E_i$ into (3.1) and take summation over $i = 1,\ \ldots,\ 2n-1$, then we have

$$
\begin{aligned}
&\frac{3}{4}c\phi S\phi X + [(\frac{c}{4} - f)(2n-1) + \frac{3}{4}c]SX + ASAX - SA^2X \\
&+ mSAX - (trSA)AX - (\frac{c}{4} - f)(trS)X - \frac{3}{4}ck\eta(X)\xi \\
&= 0
\end{aligned}
\tag{3.5}
$$

for any tangent vector field X on M.

Let X_λ be a unit tangent vector field on M orthogonal to ξ such that $AX_\lambda = \lambda X_\lambda$. Then, from (2.5), we have $SX_\lambda = k_\lambda X_\lambda$, where $k_\lambda = \frac{2n+1}{4}c + m\lambda - \lambda^2$. Putting $X = X_\lambda$ into (3.5) and using (2.1), we obtain

$$S\phi X_\lambda = \ell_\lambda \phi X_\lambda, \tag{3.6}$$

where we have put

$$\ell_\lambda = \frac{4}{3c}\{[(2n-1)(\frac{c}{4} - f) + m\lambda + \frac{3}{4}c]k_\lambda - (\frac{c}{4} - f)trS - \lambda trSA\}.$$

Since $AS = SA$ on M, by putting $Y = \phi X_\lambda$ into (3.4) and using (3.6), we get

$$(\ell_\lambda - k)[\alpha A\phi X_\lambda + (\frac{c}{4} - f)\phi X_\lambda] = 0. \tag{3.7}$$

By putting $X = X_\lambda$ into (3.1) and using (3.6) yields

$$\begin{aligned}(3.8)\qquad &(\frac{c}{4} - f)[< (S - k_\lambda I)Y, Z > X_\lambda + < X_\lambda, Z > (S - k_\lambda I)Y] \\ &+\frac{c}{4}[< (S - \ell_\lambda I)\phi Y, Z > \phi X_\lambda + < \phi X_\lambda, Z > (S - \ell_\lambda I)\phi Y] \\ &-\frac{c}{2} < \phi X_\lambda, Y > (\phi S - S\phi)Z + \lambda < (S - k_\lambda I)AY, Z > X_\lambda \\ &+\lambda < X_\lambda, Z > (S - k_\lambda I)AY = 0\end{aligned}$$

for any tangent vector fields Y and Z on M. Putting $Y = \phi X_\lambda$ and $Z = X$ into (3.8) and making use of (3.6), we get

$$\begin{aligned}(3.9)\qquad &(\frac{c}{2} - f)(\ell_\lambda - k_\lambda)(< \phi X_\lambda, X > X_\lambda + < X_\lambda, X > \phi X_\lambda) \\ &-\frac{c}{2}(\phi S - S\phi)X + \lambda < (S - k_\lambda I)A\phi X_\lambda, X > X_\lambda \\ &+\lambda < X_\lambda, X > (S - k_\lambda I)A\phi X_\lambda = 0\end{aligned}$$

for any tangent vector field X on M. By taking inner product of the both sides of (3.9) with X_λ, we obtain

$$\lambda(S - k_\lambda I)A\phi X_\lambda = (c - f)(k_\lambda - \ell_\lambda)\phi X_\lambda.$$

Substituting this equation into (3.9), we have

$$(\phi S - S\phi)X = (k_\lambda - \ell_\lambda)[< \phi X_\lambda, X > X_\lambda + < X_\lambda, X > \phi X_\lambda] \tag{3.10}$$

for any tangent vector field X on M.

Since $n \geq 3$, we can choose a unit tangent vector field X_μ on M such that $AX_\mu = \mu X_\mu$, X_μ is orthogonal to both ξ and X_λ and is linearly independent to ϕX_λ. For this vector field X_μ, we have

$$SX_\mu = k_\mu X_\mu, \qquad S\phi X_\mu = \ell_\mu \phi X_\mu,$$

where we have put

$$k_\mu = \frac{2n-1}{4}c + m\mu - \mu^2$$

and

$$\ell_\mu = \frac{4}{3c}\{[(2n-1)(\frac{c}{4} - f) + m\mu + \frac{3}{4}c]k_\mu - (\frac{c}{4} - f)trS - \mu trSA\}.$$

By a similar argument as in (3.10), we also have

$$(\phi S - S\phi)X = (k_\mu - \ell_\mu)(< \phi X_\mu, X > X_\mu + < X_\mu, X > \phi X_\mu)$$

for any tangent vector field X on M. If we compare this equation with (3.10), then we see that

$$k_\lambda = \ell_\lambda, \qquad k_\mu = \ell_\mu, \tag{3.11}$$

since $\{\phi X_\lambda, X_\mu\}$ is linearly independent. From (3.10) and (3.11), we have

$$\phi S = S\phi \quad \text{on } M. \tag{3.12}$$

Putting $Y = X_\mu$ and $Z = \phi X_\lambda$ into (3.8) and making use of (3.5) and (3.12), we obtain

$$(k_\mu - k_\lambda)[(\frac{c}{4} - f + \lambda\mu) < X_\mu, \phi X_\lambda > X_\lambda + \frac{c}{4}\phi X_\mu] = 0.$$

Since $\{X_\lambda, \phi X_\mu\}$ is linearly independent, we get

$$k_\lambda = k_\mu. \tag{3.13}$$

Therefore it is easy to see from (3.11) and (3.13) that

$$SX = k_\lambda X \tag{3.14}$$

for any tangent vector field X orthogonal to ξ. It is well-known ([5]) that a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$, does not admit an Einstein real hypersurface. Thus we also see that

$$k_\lambda \neq k. \tag{3.15}$$

Let X be a principal direction orthogonal to ξ associated to λ, that is, $AX = \lambda X$. Then, putting $Y = \phi X$ into (3.4) and using (3.14) and (3.15), we have

$$\alpha A\phi X + (\frac{c}{4} - f)\phi X = 0. \tag{3.16}$$

From (3.4), (3.14) and (3.15), we also obtain

$$\frac{c}{4} - f + \alpha\lambda = 0. \tag{3.17}$$

4. Proof of Theorem

At first, we shall prove the following.

Lemma 4.1. *Let M be a Ricci-pseudo-symmetric real hypersurface with the associated function f in $M_n(c)$, $c \neq 0$, $n \geq 3$. Then*

(1) *for any non-zero tangent vector X orthogonal to ξ such that $AX = \lambda X$, we have $\lambda \neq 0$ and $A\phi X = \frac{f}{\lambda}\phi X$,*

(2) *M has at most three distinct principal curvatures,*

(3) *if M has three distinct principal curvatures, then the principal curvature α vanishes identically,*

(4) *the multiplicity of α is equal to 1.*

Proof. (1) Since ξ is principal, it is well-known ([5]) that M satisfies

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0.$$

Applying X to this equation, we have

$$2(2\lambda - \alpha)A\phi X = (2\alpha\lambda + c)\phi X. \tag{4.1}$$

If we compare (3.16) with (4.1) and make use of (3.17), then we obtain $\lambda A\phi X = f\phi X$, and this equation together with (3.17) gives rise to $\lambda \neq 0$.

(2) Assume that M has $r(\geq 3)$ distinct principal curvatures $\lambda_1, \ldots, \lambda_r$, where $\lambda_i \neq \alpha$ for $i = 1, \ldots, r$. Since $k_\lambda = \frac{2n+1}{4}c + m\lambda_i - \lambda_i^2$ by (3.14), we get $m = \lambda_i + \lambda_j$ for $1 \leq i \neq j \leq r$. Thus we obtain $\lambda_2 = \lambda_3 = \ldots = \lambda_r$ and it contradicts.

(3) Let $\lambda(\neq \alpha)$ and $\mu(\neq \alpha)$ be the distinct principal curvatures. Then it follows from (3.17) that $\frac{c}{4} - f + \alpha\lambda = \frac{c}{4} - f + \alpha\mu$ and hence $\alpha = 0$.

(4) Assume that the multiplicity of α is greater than 2. Then there is a non-zero tangent vector X orthogonal to ξ such that $AX = \alpha X$. By (1), we have $\alpha \neq 0$ and $A\phi X = \frac{f}{\alpha}\phi X$. comparing this equation with (3.16), we obtain $c = 0$ and hence a contradiction. □

Proof of Theorem. Since it is known ([5]) that there is no umbilical real hypersurfaces in $M_n(c)$, we can only consider two cases where M has two and three distinct principal curvatures because of (2) of Lemma 4.1.

(Case I) M has two distinct principal curvatures α and λ.

Since the multiplicity of α is equal to 1 by (4) of Lemma 4.1, we have $AX = \lambda X$ for any non-zero tangent vector X orthogonal to ξ, and, by (1), $\lambda \neq 0$ and $A\phi X = \frac{f}{\lambda}\phi X$. Since we see that $\lambda = \frac{f}{\lambda}$, that is, $\lambda^2 = f > 0$, it follows from (3.17) that λ is a solution of

$$\lambda^2 - \alpha\lambda - \frac{c}{4} = 0.$$

By the discriminant of the above quadratic equation, we see that $c = -\alpha^2 < 0$ and $\lambda = \frac{\alpha}{2}$, and hence we have $f = -\frac{c}{4}(c < 0)$, $\alpha = \sqrt{-c}$ and $\lambda = \frac{\sqrt{-c}}{2}$.

(Case II) M has three distinct principal curvatures α, λ and μ.

Since $\alpha = 0$ by (3) of Lemma 4.1, we see from (3.17) that $f = \frac{c}{4}$. For any non-zero tangent vectors X and Y orthogonal to ξ such that $AX = \lambda X$ and $AY = \mu Y$, we have $\lambda\mu \neq 0$, $A\phi X = \frac{f}{\lambda}\phi X$ and $A\phi Y = \frac{f}{\mu}\phi Y$ by (1) of Lemma 4.1. It is easily seen that $\lambda = \frac{f}{\lambda}$ if and only if $\mu = \frac{f}{\mu}$.

We consider the case where $\lambda \neq \frac{f}{\lambda}$, that is, $\mu = \frac{f}{\lambda}$. Then we see that $\lambda\mu = f = \frac{c}{4}$ and the multiplicity of λ (resp. μ) is equal to $n-1$. Since $SX = k_\lambda X$ for any tangent vector X orthogonal to ξ by (3.14), it follows from (2.5) that $m = \lambda + \mu$. Therefore we get $m = (n-1)(\lambda + \mu)$ and hence $\lambda + \mu = 0$ because of $n \geq 3$. Since $f = -\lambda^2 = \frac{c}{4} < 0$, M must be locally congruent to a real hypersurface of type A_2 or B in a complex hyperbolic space $H_n(\mathbb{C})$, if it exists. It is known that the principal curvature α of real hypersurfaces of type A_2 and B in $H_n(\mathbb{C})$ is not equal to zero, and hence the case where $\lambda \neq \frac{f}{\lambda}$ does not occur.

Finally we see that $\lambda = \frac{f}{\lambda}$, that is, $\lambda^2 = \mu^2 = f = \frac{c}{4}$. Since we have $\lambda = -\mu$, it follows from (3.14) that $m = 0$ and hence M is minimal. It is easy to see that the multiplicity of λ is equal to $2p$ and that of μ is $2q$, where $p,\ q \geq 1$ and $p+q = n-1$. Since $\alpha = 0$ and $m = 2p\lambda + 2q\mu = 2(p-q)\lambda$, we get $p = q = \frac{n-1}{2}$. Therefore M has the principal curvatures 0 of multiplicity 1, $\frac{\sqrt{c}}{2}$ of $n-1$ and $-\frac{\sqrt{c}}{2}$ of $n-1$ in a complex projective space $P_n(\mathbb{C})$. $\square$

References

1. U.-H. Ki, H. Nakagawa and Y. J. Suh, *Real hypersurfaces with harmonic Weyl tensor of a complex space form*, Hiroshima Math. J., 20(1990), 93-102
2. M. Kimura and S. Maeda, *On real hypersurfaces of a complex projective space*, Math. Z., 202(1989), 299-311
3. G. A. Lobos and M. Ortega, *Pseudo-parallel real hypersurfaces in complex space forms*, Bull. Korean Math. Soc., 41(2004), 609-618
4. S. Maeda, *Real hypersurfaces of complex projective spaces*, Math. Ann., 263(1983), 473-478
5. R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*, Tight and taut submanifolds, Math. Sci. Res. Publ., 32(1997), Cambridge Univ. Press, Cambridge
6. R. Niebergall and P. J. Ryan, *Semi-parallel and semi-symmetric real hypersurfaces in complex space forms*, Kyungpook Math. J., 38(1998), 227-234

Department of Mathematics
Hankuk University of Foreign Studies
Seoul 130-791
Korea

E-mail address:

In-Bae Kim, ibkim@hufs.ac.kr
Hye Jeong Park, chang@kangnung.ac.kr
Hyunjung Song, hsong@hufs.ac.kr

Received January 26, 2007
Revised May 7, 2007