Nihonkai Math. J.
Vol.17(2006), 89-101

A Note on Boundary Controllability of Neutral Integrodifferential Systems in Banach Spaces

K.Balachandran and A.Leelamani

Abstract: In this note we establish a set of sufficient conditions for boundary controllability of neutral integrodifferential systems in Banach spaces. The results are obtained by using the strongly continuous semigroup theory and the Sadovskii fixed point theorem.

Key Words: Boundary controllability, neutral integrodifferential systems, semigroup theory, fixed point theorem.

2000 AMS Subject Classification: 93 B 05

1. Introduction

The problem of controllability of nonlinear systems and integrodifferential systems including delay systems has been studied by many researchers [3]. Controllability of neutral functional integrodifferential systems in abstract spaces was studied by Balachandran et.al [1]. Fu [6,7] studied the controllability of neutral functional differential systems in abstract phase space with the help of the Sadovskii fixed point theorem. Motivation for neutral functional differential equations can be found in [9-11]. Several authors [4,5,16] have developed many abstract settings to describe the boundary control systems in which the control must be taken in sufficiently smooth functions for the existence of regular solutions to state space system. Lasiecka [13] established the regularity of optimal boundary control for parabolic equations. Recently Balachandran and Anandhi [2] discussed the boundary controllability of neutral integrodifferential systems in Banach spaces by means of Schaefer's fixed point theorem. The purpose of this note is to derive a set of sufficient conditions for the boundary controllability of neutral integrodifferential systems in phase space by using the Sadovskii fixed point theorem. These conditions are weaker than those conditions obtained in [2].

2. Preliminaries

Let E and U be a pair of real Banach spaces with the norms $\|\cdot\|$ and $\|\cdot\|_U$ respectively. Let σ be a linear, closed and densely defined operator with domain $D(\sigma) \subseteq E$ and range $R(\sigma) \subseteq E$ and let τ be a linear operator with domain $D(\tau) \subseteq E$ and range $R(\tau) \subseteq X$, a Banach space with norm $\|\cdot\|_X$.

Consider the boundary control neutral integrodifferential system of the form

$$\begin{aligned} \frac{d}{dt}[x(t) - h(t, x_t)] &= \sigma x(t) + f\left(t, x_t, \int_0^t g(t, s, x_s) ds\right), \\ \tau x(t) &= B_1 u(t), \quad t \in J = [0, b], \\ x_0 &= \phi \in \mathcal{B}, \end{aligned} \tag{1}$$

where the control function $u(\cdot)$ is in $L^2(J, U)$, a Banach space of admissible control functions, $B_1 : U \to X$ is a linear continuous operator, the nonlinear operators $f : J \times \mathcal{B} \times E \to E$, $h : J \times \mathcal{B} \to E$ and $g : \Delta \times \mathcal{B} \to E$ are continuous functions and $\Delta = \{(t, s) : 0 \le s \le t \le b\}$, $\mathcal{B}$ is the phase space to be specified later.

Let $A : E \to E$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ with domain

$$D(A) = \{x \in D(\sigma); \tau x = 0\}, Ax = \sigma x, \quad \text{for} \quad x \in D(A).$$

We shall make the following hypotheses:

(H_1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to the graph norm of $D(\sigma)$.

(H_2) There exists a linear continuous operator $B : U \to E$ such that

$$\sigma B \in L(U, E), \tau(Bu) = B_1 u,$$

for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \le C\|B_1 u\|_X$, for all $u \in U$, where C is a constant.

(H_3) For all $t \in (0, b]$ and $u \in U, T(t)Bu \in D(A)$. Moreover, there exists a positive function $\gamma \in L^1(0, b)$ such that $\|AT(t)B\|_{L(U,E)} \le \gamma(t)$ almost everywhere for $t \in (0, b)$.

Let $A : D(A) \to E$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ defined on a Banach space E. Let $0 \in \rho(A)$, the resolvent set of A, then we define the fractional power $(-A)^\alpha$, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D((-A)^\alpha)$ which is dense in E. Further $D((-A)^\alpha)$ is a Banach space under the norm

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \text{ for } x \in D((-A)^\alpha)$$

and is denoted by E_α. The imbedding $E_\alpha \hookrightarrow E_\beta$ for $0 < \alpha < \beta \leq 1$ is compact whenever the resolvent operator of A is compact. For more results on fractional powers of operators, one can refer to [14].

We assume that the delay $x_t : (-\infty, 0] \to E$ defined by $x_t(\theta) = x(t+\theta)$ belong to some abstract phase space $\mathcal{B}$, which will be a linear space of functionals mapping $(-\infty, 0]$ into E endowed with the seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfies the following axioms :[8,12]

(A1) If $x : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then, for every $t \in [0, b)$, the following conditions hold:

  (i) x_t is in $\mathcal{B}$,
  (ii)$\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$.

  Here $H \geq 0$ is a constant, $K, M : [0, \infty) \to [0, \infty), K$ is continuous and M is locally bounded and H, K, M are independent of $x(t)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A3) The space $\mathcal{B}$ is complete.

Let $x(t)$ be the solution of (1). Then we can define a function $z(t) = x(t) - Bu(t)$ and from our assumption, we see that $z(t) \in D(A)$. Hence (1) can be written in terms of A and B as

$$\begin{aligned} \frac{d}{dt}[x(t) - h(t, x_t)] &= Az(t) + \sigma Bu(t) + f\left(t, x_t, \int_0^t g(t, s, x_s)ds\right), \quad t \in J, \quad (2) \\ x(t) &= z(t) + Bu(t), \\ x_0 &= \phi. \end{aligned}$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$\begin{aligned} \dot{z}(t) &= Az(t) + \frac{d}{dt}h(t, x_t) + \sigma Bu(t) - B\dot{u}(t) + f\left(t, x_t, \int_0^t g(t, s, x_s)ds\right), \\ z(0) &= \phi(0) - Bu(0) \end{aligned}$$

and the solution of (1) is given by

$$\begin{aligned} x(t) &= T(t)[\phi(0) - Bu(0)] + Bu(t) + \int_0^t T(t-s)\frac{d}{ds}h(s, x_s)ds \\ &\quad + \int_0^t T(t-s)\Big[\sigma Bu(s) - B\dot{u}(s) + f\left(s, x_s, \int_0^s g(s, \tau, x_\tau)d\tau\right)\Big]ds. \quad (3) \end{aligned}$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of the solution for the general inputs

$u \in L^1(J, U)$. Integrating (3) by parts, we get

$$\begin{aligned} x(t) &= T(t)[\phi(0) - h(0,\phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s, x_s)ds \\ &\quad + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds \\ &\quad + \int_0^t T(t-s) f\left(s, x_s, \int_0^s g(s,\tau,x_\tau)d\tau\right) ds. \end{aligned} \tag{4}$$

Thus (4) is well defined and it is called a mild solution of the system (1).

We say the system (1) is said to be *null controllable* on the interval J if for every $\phi \in \mathcal{B}$, there exists a control $u \in L^2(J, U)$ such that $x(\cdot)$ of (1) satisfies $x(b) = 0$.

In order to establish our results we need the following fixed point theorem due to Sadovskii [15].

Sadovskii's Theorem: Let P be a condensing operator on a Banach space X, that is, P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) < \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(H) \subset H$ for a convex, closed and bounded set H of X, then P has a fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).

Further we make the following assumptions:

(H_4) A is the infinitesimal generator of an analytic semigroup of compact linear operators $T(t)$ on E such that $\|T(t)\| \leq K_1$ for some $K_1 > 0$ and for any $\alpha \geq 0$, there exists a positive constant $K_2(\alpha) > 0$ such that

$$\|(-A)^\alpha T(t)\| \leq K_2 t^{-\alpha}. \tag{5}$$

(H_5) $h : J \times \mathcal{B} \to E$ is a continuous function and there exist constants $\beta \in (0,1)$ and $L_1, L_2 > 0$ such that the Lipschitz condition

$$\|(-A)^\beta h(s_1, \phi_1) - (-A)^\beta h(s_2, \phi_2)\| \leq L_1 [|s_1 - s_2| + \|\phi_1 - \phi_2\|_{\mathcal{B}}]$$

is satisfied for $0 \leq s_1, s_2 \leq b, \phi_1, \phi_2 \in \mathcal{B}$, and that the inequality

$$\|(-A)^\beta h(t, \phi)\| \leq L_2[\|\phi\|_{\mathcal{B}} + 1] \tag{6}$$

holds for $t \in J, \phi \in \mathcal{B}$.

(H_6) The function $g : \Delta \times \mathcal{B} \to E$ is a continuous function such that there is a positive function $a \in L^1([0,t])$ such that

$$\|g(t, s, x_s)\| \leq a(t,s)\|x\|_{\mathcal{B}} \text{ and } \delta = \sup_{t \in J} \int_0^t a(t,s)ds.$$

(H_7) The function $f : J \times \mathcal{B} \times E \to E$ satisfies the following conditions:

(i) For each $t \in J$, the function $f(t, \cdot, \cdot) : \mathcal{B} \times E \to E$ is continuous and for each $(\phi, x) \in \mathcal{B} \times E$ the function $f(\cdot, \phi, x) : J \to E$ is strongly measurable.

(ii) For each positive integer k, there exists $\mu_k \in L^1(0, b)$ such that

$$\sup_{\|x\|_{\mathcal{B}}, \|y\| \le k} \|f(t, x, y)\| \le \mu_k(t) \text{ and } \liminf_{k \to \infty} \frac{1}{k} \int_0^b \mu_k(s) ds = \gamma < \infty.$$

(H_8) There exist constants $M_1, M_2 > 0$ such that $\|\sigma B\|_{L(U,E)} \le M_1$ and $\int_0^b \gamma(t) dt \le M_2$.

(H_9) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \int_0^b [T(b-s)\sigma - AT(b-s)] Bu(s) ds$$

induces a bounded invertible operator $\tilde{W}$ defined on $L^2(J, U)/kerW$ and there exists a positive constant $M_3 > 0$ such that $\|\tilde{W}^{-1}\| \le M_3$.

Let $y(\cdot) : (-\infty, b) \to E$ be the function defined by

$$y(t) = \begin{cases} T(t)\phi(0), & 0 \le t \le b, \\ \phi(t), & -\infty < t < 0, \end{cases}$$

then $y_0 = \phi$ and the map $t \to y_t$ is continuous. Take $N = \sup\{\|y_t\|_{\mathcal{B}} : 0 \le t \le b\}$.

For simplicity, let us take $M_0 = \|(-A)^{-\beta}\|, K_b = \sup\{K(t) : 0 \le t \le b\}$,

$$\begin{aligned} L^* &= L_1 K_b \Big(M_0 + \frac{K_2 b^\beta}{\beta} \Big) \text{ and} \\ M^* &= [1 + bK_1 M_1 M_3 + M_2 M_3] \Big(M_0 L_2 (N+1) + (N+1) K_2 L_2 \frac{b^\beta}{\beta} \Big) \\ &\quad + (K_1 \|\phi(0) - h(0, \phi)\|)(bK_1 M_1 M_3 + M_2 M_3) + K_1 \|h(0, \phi)\|. \end{aligned}$$

(H_{10}) Further assume that

$$L^* < 1 \tag{7}$$

and

$$(1 + bK_1 M_1 M_3 + M_2 M_3) \Big[L_2 K_b \Big(M_0 + \frac{K_2 b^\beta}{\beta} \Big) + K_1 K_b (1 + \delta) \gamma \Big] < 1. \tag{8}$$

3.Main Results

Theorem 3.1: If the assumptions $(H_1) - (H_{10})$ are satisfied and $\phi \in \mathcal{B}$, then the bounday control neutral integrodifferential system (1) is null controllable on the interval J.

Proof: Using the assumption (H_9), for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned} u(t) &= -\tilde{W}^{-1}\Big\{T(b)[\phi(0) - h(0,\phi)] + h(b,x_b) + \int_0^b AT(b-s)h(s,x_s)ds \\ &\quad + \int_0^b T(b-s)f\left(s,x_s,\int_0^s g(s,\tau,x_\tau)d\tau\right)ds\Big\}(t). \end{aligned}$$

It shall be shown that when using this control the operator S defined by

$$\begin{aligned} (Sx)(t) &= T(t)[\phi(0) - h(0,\phi)] + h(t,x_t) + \int_0^t AT(t-s)h(s,x_s)ds \\ &\quad + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds \\ &\quad + \int_0^t T(t-s)f\left(s,x_s,\int_0^s g(s,\tau,x_\tau)d\tau\right)ds, \quad 0 \le t \le b, \end{aligned}$$

has a fixed point $x(\cdot)$. Then $x(\cdot)$ is a mild solution of system (1), and it is easy to verify that

$$x(b) = (Sx)(b) = 0,$$

which implies that the system is null controllable.

Next we will prove that the operator S has a fixed point.

For each $z \in C(J;E)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & 0 \le t \le b, \\ 0, & -\infty < t < 0. \end{cases}$$

If $x(\cdot)$ satisfies (4), we can decompose it as $x(t) = z(t) + y(t), 0 \le t \le b$, which implies that $x_t = \bar{z}_t + y_t$, for every $0 \le t \le b$, and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= -T(t)h(0,\phi) + h(t,\bar{z}_t + y_t) + \int_0^t AT(t-s)h(s,\bar{z}_s + y_s)ds \\ &\quad + \int_0^t T(t-s)\sigma Bu(s)ds - \int_0^t AT(t-s)Bu(s)ds \\ &\quad + \int_0^t T(t-s)f\left(s,\bar{z}_s + y_s,\int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right)ds. \end{aligned}$$

Let P be the operator on $C(J;E)$ defined by

$$(Pz)(t) = -T(t)h(0,\phi) + h(t,\bar{z}_t + y_t) + \int_0^t AT(t-s)h(s,\bar{z}_s + y_s)ds$$

$$+\int_0^t T(t-s)\sigma Bu(s)ds - \int_0^t AT(t-s)Bu(s)ds$$
$$+\int_0^t T(t-s)f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right) ds.$$

Obviously the operator S has a fixed point if and only if P has a fixed point, so we have to prove that P has a fixed point.

For each positive integer k, let

$$B_k = \{z \in C(J;E) : z(0) = 0, \|z(t)\| \le k, 0 \le t \le b\},$$

then B_k, for each k, is a bounded closed convex set in $C(J;E)$. Since by (5) and (6) the following relation holds

$$\begin{aligned}\|AT(t-s)h(s,\bar{z}_s + y_s)\| &\le \|(-A)^{1-\beta}T(t-s)(-A)^{\beta}h(s,\bar{z}_s+y_s)\|\\ &\le K_2L_2(t-s)^{\beta-1}[kK_b + N + 1],\end{aligned}$$

then it follows that $AT(t-s)h(s,\bar{z}_s + y_s)$ is integrable on $[0,t]$, so P is well defined on B_k. We claim that there exists a positive integer k such that $PB_k \subseteq B_k$. If it is not true, then for each positive integer k, there is a function $z_k \in B_k$, but $Pz_k \notin B_k$, that is, $\|Pz_k(t)\| > k$ for some $t \in [0,b]$. However, on the other hand, we have

$$\begin{aligned}k &\le \|(Pz_k)(t)\|\\
&\le \|-T(t)h(0,\phi) + h(t,\bar{z}_{k,t} + y_t) + \int_0^t AT(t-s)h(s,\bar{z}_{k,s}+y_s)ds\\
&\quad + \int_0^t \big[T(t-s)\sigma B - AT(t-s)B\big]\tilde{W}^{-1}\big\{-T(b)[\phi(0) - h(0,\phi)]\\
&\quad -h(b,\bar{z}_{k,b} + y_b) - \int_0^b AT(b-s)h(s,\bar{z}_{k,s}+y_s)ds\\
&\quad -\int_0^b T(b-s)f\left(s,\bar{z}_{k,s}+y_s,\int_0^s g(s,\tau,\bar{z}_{k,\tau}+y_\tau)d\tau\right)ds\big\}(s)ds\\
&\quad +\int_0^t T(t-s)f\left(s,\bar{z}_{k,s}+y_s,\int_0^s g(s,\tau,\bar{z}_{k,\tau}+y_\tau)d\tau\right)ds\|\\
&\le K_1\|h(0,\phi)\| + \|h(t,\bar{z}_{k,t}+y_t)\| + \int_0^t \|AT(t-s)h(s,\bar{z}_{k,s}+y_s)\|ds\\
&\quad + \int_0^t [K_1M_1 + \gamma(t-s)]M_3\big\{K_1\|\phi(0) - h(0,\phi)\| + \|h(b,\bar{z}_{k,b}+y_b)\|\\
&\quad + \int_0^b \|AT(b-s)h(s,\bar{z}_{k,s}+y_s)\|ds\\
&\quad + K_1\int_0^b \|f\left(s,\bar{z}_{k,s}+y_s,\int_0^s g(s,\tau,\bar{z}_{k,\tau}+y_\tau)d\tau\right)\|ds\big\}ds\\
&\quad + K_1\int_0^t \|f\left(s,\bar{z}_{k,s}+y_s,\int_0^s g(s,\tau,\bar{z}_{k,\tau}+y_\tau)d\tau\right)\|ds.\end{aligned}$$

Since

$$\|\int_0^t AT(t-s)h(s,\bar{z}_{k,s}+y_s)ds\| \le \|\int_0^t (-A)^{1-\beta}T(t-s)(-A)^{\beta}h(s,\bar{z}_{k,s}+y_s)ds\|$$

$$
\begin{aligned}
&\leq \int_0^t \frac{K_2}{(t-s)^{1-\beta}} L_2(\|\bar{z}_{k,s} + y_s\|_{\mathcal{B}} + 1)ds \\
&\leq \frac{1}{\beta} K_2 L_2 b^\beta (kK_b + N + 1)
\end{aligned}
$$

and

$$
\int_0^t \|f\left(s, \bar{z}_{k,s} + y_s, \int_0^s g(s,\tau,\bar{z}_{k,\tau} + y_\tau)d\tau\right)\|ds \leq \int_0^b \mu_{k^*}(s)ds
$$

where $k^* = (1+\delta)(kK_b + N)$, there holds

$$
\begin{aligned}
k &\leq K_1\|h(0,\phi)\| + M_0 L_2(kK_b + N + 1) + \frac{1}{\beta} b^\beta K_2 L_2(kK_b + N + 1) \\
&\quad + (bK_1M_1 + M_2)M_3\Big\{K_1\|\phi(0) - h(0,\phi)\| + M_0L_2(kK_b + N + 1) \\
&\quad + \frac{1}{\beta} b^\beta K_2 L_2(kK_b + N + 1) + K_1 \int_0^b \mu_{k^*}(\tau)d\tau\Big\} + K_1 \int_0^b \mu_{k^*}(s)ds \\
&\leq M^* + M_0 L_2 kK_b[1 + bK_1M_1M_3 + M_2M_3] \\
&\quad + K_2L_2kK_b\frac{b^\beta}{\beta}[1 + bK_1M_1M_3 + M_2M_3] \\
&\quad + K_1 \int_0^b \mu_{k^*}(s)ds[1 + bK_1M_1M_3 + M_2M_3] \\
&\leq M^* + (1 + bK_1M_1M_3 + M_2M_3)\Big(M_0L_2kK_b + K_2L_2kK_b\frac{b^\beta}{\beta} + K_1\int_0^b \mu_{k^*}(s)ds\Big).
\end{aligned}
$$

Dividing on both sides by k and taking the lower limit, we get

$$
(1 + bK_1M_1M_3 + M_2M_3)\left[L_2K_b\Big(M_0 + \frac{K_2 b^\beta}{\beta}\Big) + K_1K_b(1+\delta)\gamma\right] \geq 1.
$$

This contradicts (8). Hence $PB_k \subseteq B_k$, for some positive number k.

Now define the operators P_1, P_2 on B_k by

$$
\begin{aligned}
(P_1 z)(t) &= -T(t)h(0,\phi) + h(t, \bar{z}_t + y_t) + \int_0^t AT(t-s)h(s,\bar{z}_s + y_s)ds \\
&\text{and} \\
(P_2 z)(t) &= \int_0^t \big(T(t-s)\sigma - AT(t-s)\big)Bu(s)ds \\
&\quad + \int_0^t T(t-s)f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right)ds,
\end{aligned}
$$

for $0 \leq t \leq b$, respectively. We will show that P_1 is a contraction mapping and P_2 is a compact operator.

To prove that P_1 is a contraction, we take $z_1, z_2 \in B_k$, then for each $t \in [0, b]$ and by (A1 (iii)) and (H_5), we have

$$\begin{aligned}
\|(P_1z_1)(t) - (P_1z_2)(t)\| &\leq \|h(t, \bar{z}_{1,t} + y_t) - h(t, \bar{z}_{2,t} + y_t)\| \\
&\quad + \|\int_0^t AT(t-s)\Big[h(s, \bar{z}_{1,s} + y_s) - h(s, \bar{z}_{2,s} + y_s)\Big]ds\| \\
&\leq M_0L_1\|\bar{z}_{1,t} - \bar{z}_{2,t}\|_{\mathcal{B}} + \int_0^t K_2(t-s)^{\beta-1}L_1\|\bar{z}_{1,s} - \bar{z}_{2,s}\|_{\mathcal{B}}\, ds \\
&\leq \Big(M_0L_1K_b + K_2L_1K_b\frac{b^\beta}{\beta}\Big) \sup_{0\leq s\leq b} \|z_1(s) - z_2(s)\| \\
&\leq L_1K_b\Big(M_0 + K_2\frac{b^\beta}{\beta}\Big) \sup_{0\leq s\leq b} \|z_1(s) - z_2(s)\| \\
&\leq L^* \sup_{0\leq s\leq b} \|z_1(s) - z_2(s)\|.
\end{aligned}$$

Thus P_1 satisfies contraction condition with $L^* < 1$ by (7).

To prove that P_2 is compact, first we prove that P_2 is continuous on B_k. Let $\{z_n\} \subseteq B_k$ with $z_n \to z$ in B_k, then for each $s \in [0, b]$, $\bar{z}_{n,s} \to \bar{z}_s$ and by (H_7) (i), we have

$$\begin{aligned}
&f\left(s, \bar{z}_{n,s} + y_s, \int_0^s g(s, \tau, \bar{z}_{n,\tau} + y_\tau)d\tau\right) \\
&\to f\left(s, \bar{z}_s + y_s, \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau\right) \quad as \ \ n \to \infty.
\end{aligned}$$

Since

$$\begin{aligned}
\Big\|f\left(s, \bar{z}_{n,s} + y_s, \int_0^s g(s, \tau, \bar{z}_{n,\tau} + y_\tau)d\tau\right) - f\left(s, \bar{z}_s + y_s, \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau\right)\Big\| \\
\leq 2\mu_{k^*}(s),
\end{aligned}$$

then by the continuity of $(-A)^\beta g$ and the dominated convergence theorem we have

$$\begin{aligned}
\|P_2z_n - P_2z\| &= \sup_{0\leq t\leq b} \|\int_0^t \Big(T(t-s)\sigma - AT(t-s)\Big)B[u_n(s) - u(s)]ds \\
&\quad + \int_0^t T(t-s)\Big[f\left(s, \bar{z}_{n,s} + y_s, \int_0^s g(s, \tau, \bar{z}_{n,\tau} + y_\tau)d\tau\right) \\
&\quad - f\left(s, \bar{z}_s + y_s, \int_0^s g(s, \tau, \bar{z}_\tau + y_\tau)d\tau\right)\Big]ds\| \\
&\to 0, \ \ as \ \ n \to \infty,
\end{aligned}$$

that is, P_2 is continuous.

Next we show that the family $\{P_2z : z \in B_k\}$ is an equicontinuous family of functions. Let $z \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq b$,

$$\|(P_2z)(t_1) - (P_2z)(t_2)\|$$

$$\begin{aligned}
&= \Big\| \int_0^{t_1} T(t_1 - s)\sigma Bu(s)ds - \int_0^{t_1} AT(t_1 - s)Bu(s)ds \\
&\quad + \int_0^{t_1} T(t_1 - s) f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right) ds \\
&\quad - \int_0^{t_2} T(t_2 - s)\sigma Bu(s)ds + \int_0^{t_2} AT(t_2 - s)Bu(s)ds \\
&\quad - \int_0^{t_2} T(t_2 - s) f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right) ds \Big\| \\
&\le \int_0^{t_1} \Big\| \big(T(t_1 - s) - T(t_2 - s)\big)\sigma Bu(s)\Big\| ds + \int_{t_1}^{t_2} \Big\| T(t_2 - s)\sigma Bu(s)\Big\| ds \\
&\quad + \int_0^{t_1} \Big\| A\big(T(t_1 - s) - T(t_2 - s)\big) Bu(s)\Big\| ds + \int_{t_1}^{t_2} \Big\| AT(t_2 - s)Bu(s)\Big\| ds \\
&\quad + \int_0^{t_1} \Big\| \big(T(t_1 - s) - T(t_2 - s)\big) f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right)\Big\| ds \\
&\quad + \int_{t_1}^{t_2} \Big\| T(t_2 - s) f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right)\Big\| ds.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|u(s)\| &\le \|\tilde{W}^{-1}\| \Big\{ K_1 \|\phi(0) - h(0,\phi)\| + \|(-A)^{-\beta}\| \|(-A)^{\beta} h(b, \bar{z}_b + y_b)\| \\
&\quad + \int_0^b \|(-A)^{1-\beta} T(b-s)\| \|(-A)^{\beta} h(s, \bar{z}_s + y_s)\| ds \\
&\quad + \int_0^b \|T(b-s)\| \|f\left(s, \bar{z}_s + y_s, \int_0^s g(s,\tau,\bar{z}_\tau + y_\tau)d\tau\right)\| ds \Big\} \\
&\le M_3 \Big\{ K_1 \|\phi(0) - h(0,\phi)\| + M_0 L_2 (kK_b + N + 1) \\
&\quad + K_2 L_2 (kK_b + N + 1)\frac{b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s) ds \Big\}
\end{aligned}$$

and $\mu_{k^*}(s) \in L^1$. We see that

$$\begin{aligned}
&\|(P_2 z)(t_1) - (P_2 z)(t_2)\| \\
&\le \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| M_1 M_3 \Big\{ K_1 \|\phi(0) - h(0,\phi)\| + M_0 L_2 (kK_b + N + 1) \\
&\qquad + K_2 L_2 (kK_b + N + 1)\frac{b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s) ds \Big\} ds \\
&\quad + \int_{t_1}^{t_2} \|T(t_2 - s)\| M_1 M_3 \Big\{ K_1 \|\phi(0) - h(0,\phi)\| + M_0 L_2 (kK_b + N + 1) \\
&\qquad + K_2 L_2 (kK_b + N + 1)\frac{b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s) ds \Big\} ds \\
&\quad + \int_0^{t_1} \|A(T(t_1 - s) - T(t_2 - s))B\| M_3 \Big\{ K_1 \|\phi(0) - h(0,\phi)\| + M_0 L_2 (kK_b + N + 1) \\
&\qquad + K_2 L_2 (kK_b + N + 1)\frac{b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s) ds \Big\} ds
\end{aligned}$$

$$+\int_{t_1}^{t_2} \|AT(t_2-s)B\|M_3\Big\{K_1\|\phi(0)-h(0,\phi)\|+M_0L_2(kK_b+N+1)$$
$$+K_2L_2(kK_b+N+1)\frac{b^\beta}{\beta}+K_1\int_0^b \mu_{k^*}(s)ds\Big\}ds$$
$$+\int_0^{t_1}\|T(t_1-s)-T(t_2-s)\|\mu_{k^*}(s)ds+\int_{t_1}^{t_2}\|T(t_2-s)\|\mu_{k^*}(s)ds,$$

tends to zero independent of $z \in B_k$ as $t_1 \to t_2$, since the compactness of $T(t)$ implies the continuity of $T(t)$ in t in the uniform operator topology (see page 202-203, [10]). Hence P_2 maps B_k into an equicontinuous family of functions.

It remains to prove that $V(t) = \{(P_2z)(t) : z \in B_k\}$ is relatively compact in E. Let $0 < t \leq b$ be fixed, $0 < \epsilon < t$, for $z \in B_k$, we define

$$\begin{aligned}(P_{2,\epsilon}z)(t) &= \int_0^{t-\epsilon} T(t-s)\sigma Bu(s)ds - \int_0^{t-\epsilon} AT(t-s)Bu(s)ds \\ &\quad + \int_0^{t-\epsilon} T(t-s)f\left(s, \bar{z}_s+y_s, \int_0^s g(s,\tau,\bar{z}_\tau+y_\tau)d\tau\right) ds \\ &= T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)\sigma Bu(s)ds - T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)Bu(s)ds \\ &\quad + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f\left(s,\bar{z}_s+y_s,\int_0^s g(s,\tau,\bar{z}_\tau+y_\tau)d\tau\right) ds.\end{aligned}$$

Using the estimation of $\|u(s)\|$ and by the compactness of $T(t)$ we prove $V_\epsilon(t) = \{(P_{2,\epsilon}z)(t) : z \in B_k\}$ is relatively compact in E for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $z \in B_k$, we have

$$\begin{aligned}&\|(P_2z)(t)-(P_{2,\epsilon}z)(t)\| \\ &\leq \int_{t-\epsilon}^t \|T(t-s)\sigma Bu(s)\|ds + \int_{t-\epsilon}^t \|AT(t-s)Bu(s)\|ds \\ &\quad + \int_{t-\epsilon}^t \left\|T(t-s)f\left(s,\bar{z}_s+y_s,\int_0^s g(s,\tau,\bar{z}_\tau+y_\tau)d\tau\right)\right\|ds \\ &\leq \int_{t-\epsilon}^t \|T(t-s)\|M_1M_3\Big[K_1\|\phi(0)-h(0,\phi)\|+\|h(b,\bar{z}_b+y_b)\| \\ &\quad + \int_0^b \|AT(b-s)h(s,\bar{z}_s+y_s)\|ds + \int_0^b \|T(b-s)\|\mu_{k^*}(s)ds\Big]ds \\ &\quad + \int_{t-\epsilon}^t \|AT(t-s)B\|M_3\Big[K_1\|\phi(0)-h(0,\phi)\|+\|h(b,\bar{z}_b+y_b)\| \\ &\quad + \int_0^b \|AT(b-s)h(s,\bar{z}_s+y_s)\|ds + \int_0^b \|T(b-s)\|\mu_{k^*}(s)ds\Big]ds \\ &\quad + \int_{t-\epsilon}^t \|T(t-s)\|\mu_{k^*}(s)ds.\end{aligned}$$

Therefore there are relatively compact sets arbitrarily close to the set $V(t) = \{(P_2z)(t) : z \in B_k\}$, and hence the set $V(t)$ is also relatively compact in E.

Thus by the Arzela-Ascoli theorem P_2 is a compact operator. These arguments show that $P = P_1 + P_2$ is a condensing mapping on B_k, and by the Sadovskii fixed point theorem [15] there exists a fixed point $z(\cdot)$ for P on B_k. If we define $x(t) = z(t) + y(t), -\infty < t \leq b$, then it is easy to see that $x(\cdot)$ is a mild solution of (1) satisfying $x_0 = \phi, x(b) = 0$. Hence the system (1) is null controllable on J.

Acknowledgement: The authors are thankful to the referee for the improvement of the paper.

References

[1] K.Balachandran and E.R.Anandhi, Neutral functional integrodifferential control systems in Banach spaces, *Kybernetika*, **39** (2003), 359-367.

[2] K.Balachandran and E.R.Anandhi, Boundary Controllability of neutral integrodifferential systems in Banach spaces, *Nihonkai Mathematical Journal*, **15** (2004), 1-13.

[3] K.Balachandran and J.P.Dauer, Controllability of nonlinear systems in Banach spaces; A survey, *Journal of Optimization Theory and Applications*, **115** (2002), 7-28.

[4] A.V.Balakrishnan, Applied Functional Analysis, Springer-Verlag, New York, 1976.

[5] R.F.Curtain and H.J.Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Springer Verlag, New York,1995.

[6] X.L.Fu, Controllability of neutral functional differential systems in abstract space, *Applied Mathematics and Computation*, **141** (2003), 281-296.

[7] X.L.Fu, Controllability of abstract neutral functional differential systems with unbounded delay, *Applied Mathematics and Computation*, **151** (2004), 299-314.

[8] J.Hale and J.Kato, Phase space for retarted equations with infinite delay, *Funkcialaj Ekvacioj*, **21** (1978), 11-41.

[9] H.R.Henriquez, Periodic solutions of quasilinear partial functional differential equations with unbounded delay, *Funkcialaj Ekvacioj*, **37** (1994), 329-343.

[10] E.Hernandez, Existence results for partial neutral functional integrodifferential equations with unbounded delay, *Journal of Mathematical Analysis and Applications*, **292** (2004), 194-210.

[11] E.Hernandez and H.R.Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, *Journal of Mathematical Analysis and Applications*, **221** (1998), 452-475.

[12] Y.Hino, S.Murakami and T.Naito, Functional Differential Equations with Infinite Delay, *Lecture Notes in Mathematics*, 1473, Springer-Verlag, Berlin, 1991.

[13] I.Lasiecka, Boundary control of parabolic systems; regularity of solutions, *Applied Mathematics and Optimization*,**4** (1978), 301-327.

[14] A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[15] B.N.Sadovskii, On a fixed point principle, *Functional Analysis and Applications*, **1** (1967), 74-76.

[16] D.Washburn, A bound on the boundary input map for parabolic equations with application to time optimal control, *SIAM Journal on Control and Optimization*, **17** (1979), 652-671.

Department of Mathematics
Bharathiar University
Coimbatore 641 046
India
e-mail: balachandran_k@lycos.com

Received 13 August, 2005   Revised 5 June, 2006