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A NOTE ON THE RADON-NIKODYM TYPE THEOREM FOR OPERATORS ON SELF-DUAL CONES

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ABSTRACT. Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. We consider an order for operators preserving a self-dual cone $\mathcal{H}^+$. Let A, B be positive semi-definite operators on $\mathcal{H}$ such that A preserves $\mathcal{H}^+$ and B belongs to a strong closure of the positive part of an order automorphism group on $\mathcal{H}^+$. We prove that if A is majorized by B, then there exists a positive semi-definite operator c in the center $Z(Q\mathcal{M}|_{Q\mathcal{H}})$ with $\| c \| \leq 1$ such that $QA|_{Q\mathcal{H}} = cB|_{Q\mathcal{H}}$ where Q is a support projection of B.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $(\cdot, \cdot)$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be self-dual if $\mathcal{H}^+ = \{\xi \in \mathcal{H} | (\xi, \eta) \geq 0 \ \forall \eta \in \mathcal{H}^+\}$. We denote the isometric involution with respect to $\mathcal{H}^+$ by J. Put $\mathcal{H}^J = \mathcal{H}^+ - \mathcal{H}^+$. Then $\mathcal{H}^J = \{\xi \in \mathcal{H} | J\xi = \xi\}$. Every element $\xi \in \mathcal{H}$ is written as $\xi = \xi_1 + i\xi_2$ for $\xi_1, \xi_2 \in \mathcal{H}^J$. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $L(\mathcal{H})$. We denote the set of all operators in $L(\mathcal{H})$ preserving $\mathcal{H}^J$ by $L(\mathcal{H})^J$. For a fixed self-dual cone $\mathcal{H}^+$, we shall denote for $A, B \in L(\mathcal{H})^J$ by

$$A \trianglelefteq B$$

if $(B - A)(\mathcal{H}^+) \subset \mathcal{H}^+$. Then the relation '$\trianglelefteq$ ' defines an ordered vector space on $L(\mathcal{H})^J$. The group of all order automorphisms on $\mathcal{H}^+$ is denoted by $GL(\mathcal{H}^+)$. We shall write '$\leq$' as the usual order defined on the set of all Hermitian operators on $\mathcal{H}$.

Recall a self-dual cone associated with a standard von Neumann algebra in the sense of Haagerup [2], which appears in the form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ where $\mathcal{M}$ is a von

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Neumann algebra on $\mathcal{H}$. For example, put for $A \in \mathcal{M}$

$$\hat{A} : \xi \mapsto AJAJ\xi \quad \text{for all} \quad \xi \in \mathcal{H}.$$

Then $\hat{A} \trianglerighteq O$ from the standard form.

In A. Connes [1] and B. Iochum [3], they characterized an element of $GL(\mathcal{H}^+)$ for an orientable or a facially homogeneous cone $\mathcal{H}^+$. In this note we shall investigate the strong closure of the positive part of $GL(\mathcal{H}^+)$ from the point of view of the inequality concerned with the order for operators preserving a self-dual cone associated with a standard form.

2. Main Results

We need some lemmas to prove the main theorem.

Lemma 2.1. *For a standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$,*

$$\| XJXJ \| = \| X \|^2$$

holds for all $X \in \mathcal{M}$.

Proof. We first assume $X = X^*$. Then for each spectral projection P of X, the projection $PJPJ$ becomes a spectral projection of $XJXJ$. Here we remark that $PJPJ \neq O$. Indeed, using the fact that $JZJ = Z^*$ for each element Z in the center of $\mathcal{M}$, the central support of P is equal to that of JPJ. Hence we have $PJPJ \neq O$. Take the spectral projection P such that the difference $XP - \| X \| P$ is small. Then the difference $(XJXJ)PJPJ - \| X \|^2 PJPJ$ is also small and we obtain the desired equality.

In the general case, for X, we obtain that

$$\begin{aligned} \| XJXJ \|^2 &= \| (XJXJ)^*(XJXJ) \| = \| X^*XJX^*XJ \| \\ &= \| X^*X \|^2 = \| X \|^4 . \end{aligned}$$

This completes the proof. □

Lemma 2.2. *For a standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$, if $A, B \in \mathcal{M}$ and $A \geq O, B \geq O$, then the following conditions are equivalent:*

(i) $O \leq A \leq B$.

(ii) $O \leq AJAJ \leq BJBJ$.

Proof. The implication (i) ⇒ (ii) is immediate from the commutativity of A and JAJ. The implication (ii) ⇒ (i) is shown as follows:

We may assume A and B to be invertible. Let $B^{-\frac{1}{2}}JB^{-\frac{1}{2}}J$ operate on (ii) by multiplication from the right and left. Then

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}}JB^{-\frac{1}{2}}AB^{-\frac{1}{2}}J \leq I.$$

It follows from Lemma 2.1 that

$$\| B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \|^2 = \| B^{-\frac{1}{2}}AB^{-\frac{1}{2}}JB^{-\frac{1}{2}}AB^{-\frac{1}{2}}J \| \leq 1.$$

Hence $\| B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \| \leq 1$, so $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \leq I$. Consequently $A \leq B$. $\square$

Lemma 2.3. *For a standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$, suppose that $A \in L(\mathcal{H})$, and $B \in \mathcal{M}$ is an injective operator with a dense range. Then, $O \trianglelefteq A \trianglelefteq BJBJ$ if and only if there exists an element $C \in Z(\mathcal{M})$ with $O \leq C \leq I$ such that $A = CBJBJ$.*

Proof. Consider the polar decomposition $B = U|B|$ of B. By the assumption on B, we obtain that U is a unitary element of $\mathcal{M}$, and so $\hat{U} = UJUJ \trianglerighteq O$. Hence $\hat{U}^* \trianglerighteq O$ since $(\hat{U}^*\xi, \eta) = (\xi, \hat{U}\eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. Then we may assume that B is positive semi-definite. Let $B = \displaystyle\int_0^{\|B\|} \lambda dE_\lambda$ be a spectral decomposition of B. Put $P_n = \displaystyle\int_{\frac{1}{n}}^{\|B\|} dE_\lambda$ for $n \in \mathbb{N}$. Since $P_n \in \mathcal{M}$ implies $\hat{P}_n = P_nJP_nJ \trianglerighteq O$, it follows that

$$O \trianglelefteq \hat{P}_n A \hat{P}_n \trianglelefteq \hat{P}_n \hat{B} \hat{P}_n.$$

Since $P_nBP_n (= BP_n)$ is invertible on $P_n\mathcal{H}$, where the inverse shall be denoted by $(P_nBP_n)^{-1}$ if there is no possibility of confusion, it follows that

$$(\hat{P}_n\hat{B}\hat{P}_n)^{-1} = \widehat{P_nBP_n}^{\,-1} = ((P_nBP_n)^{-1})\hat{}\,.$$

This means that $\hat{P}_n\hat{B}\hat{P}_n$ is an order isomorphism of $\hat{P}_n\mathcal{H}$. This yields

$$O \trianglelefteq \hat{P}_nA\hat{P}_n(\hat{P}_n\hat{B}\hat{P}_n)^{-1} \trianglelefteq \hat{P}_n.$$

Then, under the reduced standard form $(\hat{P}_n\mathcal{M}|_{\hat{P}_n\mathcal{H}}, \hat{P}_n\mathcal{H}, \hat{P}_nJ|_{\hat{P}_n\mathcal{H}}, \hat{P}_n\mathcal{H}^+)$, there exists an element c_n in an order ideal $Z_{\hat{P}_n\mathcal{H}^+}$ of $\hat{P}_n\mathcal{H}$ with $\| c_n \| \leq 1$ such that

$$\hat{P}_nA\hat{P}_n(\hat{P}_n\hat{B}\hat{P}_n)^{-1}\xi = c_n\xi$$

for all $\xi \in \hat{P}_n\mathcal{H}$. Here, in a standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$, the orrder ideal of $\mathcal{H}$ is defined as

$$Z_{\mathcal{H}^+} = \{T \in L(\mathcal{H}) | \exists\alpha > 0, -\alpha I \trianglelefteq T \trianglelefteq \alpha I\}.$$

By [3, Theorem VI.1.2 (iii)] we obtain that $c_n \in Z(\hat{P}_n\mathcal{M}|_{\hat{P}_n\mathcal{H}})$. We note that $Z(\hat{P}_n\mathcal{M}|_{\hat{P}_n\mathcal{H}}) = Z(\mathcal{M})|_{\hat{P}_n\mathcal{H}}$. Since $\hat{P}_n \leq \hat{P}_{n+1}$ and $\hat{P}_n$ commutes with $\hat{B}$ and c_m for $m \geq n$, it follows for $\xi \in \hat{P}_n\mathcal{H}$ that

$$\begin{aligned} c_{n+1}\xi &= \hat{P}_nc_{n+1}\hat{P}_n\xi = \hat{P}_n\left(\hat{P}_{n+1}A\hat{P}_{n+1}(\hat{P}_{n+1}\hat{B}\hat{P}_{n+1})^{-1}\right)\hat{P}_n\xi \\ &= \hat{P}_nA\hat{P}_n(\hat{P}_n\hat{B}\hat{P}_n)^{-1}\xi = c_n\xi. \end{aligned}$$

Put $\mathcal{S} = \bigcup_{n=1}^{\infty} \hat{P}_n\mathcal{H}$, which is a dense set in $\mathcal{H}$. Then we define the operator

$$C\xi = \lim_{n\to\infty} c_n\hat{P}_n\xi \text{ for all } \xi \in \mathcal{S}.$$

From the boundedness of $\{c_n\hat{P}_n\}$ the operator C has a continuous extension on $\mathcal{H}$, which shall be denoted by the same notation. Thus $0 \le C \le I$. Furthermore, when $m \ge n$, $c_m\hat{P}_m$ commutes with both $\hat{P}_nX\hat{P}_n$ and $\hat{P}_nJXJ\hat{P}_n$ with all $X \in \mathcal{M}$. This yields that $C\hat{P}_nX\hat{P}_n = \hat{P}_nX\hat{P}_nC$ and $C\hat{P}_nJXJ\hat{P}_n = \hat{P}_nJXJ\hat{P}_nC$. In view of $\hat{P}_n \to I$ as $n \to \infty$, we have $CX = XC$ and $CJXJ = JXJC$, and so $C \in Z(\mathcal{M})$. Consequently,

$$\begin{aligned} A &= \text{s-}\lim_{n\to\infty} \hat{P}_nA\hat{P}_n \\ &= \text{s-}\lim_{n\to\infty} c_n\hat{P}_n\hat{B}\hat{P}_n \\ &= C\hat{B}. \end{aligned}$$

The converse implication holds from the following fact. If $C \in Z(\mathcal{M})$ with $O \le C \le I$, then $I - C \ge O$, and so $I - C = (I-C)^{\frac{1}{2}}J(I-C)^{\frac{1}{2}}J \trianglerighteq O$. Hence $\hat{B} - C\hat{B} = (I-C)\hat{B} \trianglerighteq O$. This completes the proof. □

Theorem 2.4. *Suppose that $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ is a standard form. Let $A, B \in L(\mathcal{H})$ and $A \ge O, B \ge O$. Suppose that B is a strong limit of a monotone net (in the sense of '$\le$') of the positive semi-definite operators from $GL(\mathcal{H}^+)$.*

- (i) *There exists a positive semi-definite operator K from $\mathcal{M}$ such that $B = KJKJ$.*
- (ii) *If $O \trianglelefteq A \trianglelefteq B$, then*
  - (1) *there exits a positive semi-definite operator c from the center $Z(\hat{P}\mathcal{M}|_{\hat{P}\mathcal{H}})$ with $\| c \| \le 1$ such that $\hat{P}A|_{\hat{P}\mathcal{H}} = cB|_{\hat{P}\mathcal{H}}$;*
  - (2) *$O \trianglelefteq (\hat{P}A|_{\hat{P}\mathcal{H}})^\lambda \trianglelefteq (B|_{\hat{P}\mathcal{H}})^\lambda$ for all $\lambda \in [0, \infty)$.*

Here $\hat{P} = PJPJ$ for the support projection P of K.

Proof. (i): A positive semi-definite operator from $GL(\mathcal{H}^+)$ is written in the form K_0JK_0J for some invertible positive semi-definite operator $K_0 \in \mathcal{M}$ by [1, Theorem 3.3]. In the case that B is a strong limit of a decreasing net $\{K_iJK_iJ\}$ of such invertible positive semi-definite operators K_i, it follows from Lemma 2.2 that $\{K_i\}$ is also decreasing. Since $\{K_i\}$ is bounded, there exists a positive semi-definite operator $K \in \mathcal{M}$ which is a strong limit of $\{K_i\}$. Thus $B = KJKJ$. In the case that B is a strong limit of an increasing net $\{K_iJK_iJ\}$ of invertible positive semi-definite operators K_i, since a strong limit of a bounded increasing net of invertible positive semi-definite operators is invertible, B belongs to $GL(\mathcal{H}^+)$.

(ii): We first claim that if P is a support projection of K then $PJPJ$ is a support projection of $KJKJ$. This follows from the fact that for any two positive semi-definite operators A, B such that $AB = BA$, we obtain $s(AB) = s(A)s(B)$. Here $s(\cdot)$ means a support projection. Indeed consider the abelian von Neumann algebra generated by A and B. Then

$$s(AB) = \text{s-}\lim_{n\to\infty}(AB)^{\frac{1}{n}} = \text{s-}\lim_{n\to\infty}A^{\frac{1}{n}}B^{\frac{1}{n}} = s(A)s(B).$$

Next, since $\hat{P} = PJPJ \trianglerighteq O$, the assumption $O \trianglelefteq A \trianglelefteq B$ implies $O \trianglelefteq \hat{P}A\hat{P} \trianglelefteq \hat{P}B\hat{P}$. We obtain that the restriction operator $K|_{\hat{P}\mathcal{H}}$ is an injective positive semi-definite operator in $\hat{P}\mathcal{M}|_{\hat{P}\mathcal{H}}$ with a dense range. By Lemma 2.3 we can then choose a positive semi-definite operator c from the center of $\hat{P}\mathcal{M}|_{\hat{P}\mathcal{H}}$ with $\| c \| \leq 1$ satisfying $\hat{P}A|_{\hat{P}\mathcal{H}} = cB|_{\hat{P}\mathcal{H}}$. Thus (1) holds. Furthermore, since for every $\lambda \in [0, \infty)$,

$$(B|_{\hat{P}\mathcal{H}})^{\lambda} - (\hat{P}A|_{\hat{P}\mathcal{H}})^{\lambda} = (I - c^{\lambda})B^{\lambda}JB^{\lambda}J|_{\hat{P}\mathcal{H}} \trianglerighteq O,$$

we obtain (2). This completes the proof. □

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