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SCALAR CONSERVATION LAWS WITH BOUNDARY CONDITIONS AND ROUGH DATA MEASURE SOLUTIONS *

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Abstract. Uniqueness and existence of L^∞ solutions to initial boundary value problems for scalar conservation laws, with continuous flux functions, is derived by L^1 contraction of Young measure solutions. The classical Kruzkov entropies, extended in Bardos, LeRoux and Nedelec's sense to boundary value problems, are sufficient for the contraction. The uniqueness proof uses the essence of Kruzkov's idea with his symmetric entropy and entropy flux functions, but the usual doubling of variables technique is replaced by the simpler fact that mollified measure solutions are in fact smooth solutions. The mollified measures turn out to have not only weak but also strong boundary entropy flux traces. Another advantage with the Young measure analysis is that the usual assumption of Lipschitz continuous flux functions can be relaxed to continuous fluxes, with little additional work.

1. Background to Scalar Conservation Laws with Boundary Conditions. DiPerna [11] showed that measure valued solutions are useful to prove convergence of approximations to scalar conservation laws: convergence follows by verifying that the approximations are uniformly bounded in L^∞, weakly consistent with all entropy inequalities and consistent with the initital data, cf. also [3], [4], [10], [15] and [19]. The work [18] extended DiPerna's result to include boundary conditions based on Bardos, LeRoux and Nedelec's boundary conditions for the Kruzkov entropies, derived in [2] to establish uniqueness and existence of solutions with bounded variation. Here we derive a uniqueness result for the initial boundary value problem of scalar conservation laws with continuous flux functions and initial-boundary data in L^∞. The analysis is a combination of the existence and uniqueness result for the initial value problems of scalar conservation laws in [20], based on measure valued solutions in L^p, and the initial boundary value conditions for Young measures in [18], using weak entropy flux traces. The existence and uniqueness for L^∞ solutions by Otto [16], with Lipschitz continuous fluxes, uses boundary entropy flux pairs related to Bardos, LeRoux and Nedelec's boundary entropy inequalities for all convex entropies. The present work, with continuous flux functions shows that the Kruzkov entropies, in Bardos, LeRoux and Nedelec's sense, are sufficient for L^1 contraction of Young measure solutions, which in turn implies uniqueness of L^∞ solutions. The uniqueness proof uses the essence of Kruzkov's idea with his symmetric entropy and entropy flux functions, but the usual doubling of variables technique is replaced by the simpler fact that mollified measure solutions are in fact smooth solutions. The mollified measures turn out to also have strong boundary entropy flux traces. Existence and uniqueness for the pure initial value problem, with continuous flux functions, was established by semi group methods in [9] and by measure solutions in [20].

At the hart of the matter of initial boundary value problems to scalar conservation laws is the trace of entropy fluxes, which define the boundary condition. The first study [2] used solutions with bounded variation and hence their trace exist directly. The work [18] used the equation in the interior domain to show that the entropy flux,

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for measure solutions, has a trace on the boundary. Otto [16] derived traces of entropy fluxes for L^∞ solutions. Vasseur showed in [22] that also the solution itself has a trace, provided the flux is smooth and satisfies a certain non-degeneracy condition required by the Averging Lemma technique. The corresponding conditions of the trace for the initial data was derived in [11] and further studied in [8], [22]. Young measures have been used also to study the behavior of solutions to initial boundary value problems of some hyperbolic systems of conservation laws, see [7], [13].

The plan of the paper is: Section 2 gives an introduction to Young measure solutions, the statements of the contraction of measure solutions and the uniqueness and existence of L^∞ solutions and its relation to the work by Otto [16]. Section 3 proves the uniqueness result and Section 4 derives the existence part.

2. Measure Valued Solutions and the Results. Let Ω be a bounded open set in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and outward unit normal vector n. Consider for $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ the scalar nonlinear conservation law

$$\partial_t u + \mathrm{div}_x f(u) = 0, \quad \text{on } \Omega \times \mathbb{R}_+, \tag{1}$$

with the Bardos, LeRoux and Nedelec [2], [12] boundary condition on $\partial\Omega \times \mathbb{R}_+$, for all $k \in \mathbb{R}$,

$$\mathrm{sgn}((u(\hat{x},t) - k) - \mathrm{sgn}(a(\hat{x},t) - k))(f(u(\hat{x},t)) - f(k)) \cdot n(\hat{x}) \geq 0, \tag{2}$$

and the initial condition

$$u(\cdot, 0) = u_0, \quad \text{on } \Omega, \tag{3}$$

where $f = (f_1, ..., f_d) : \mathbb{R} \to \mathbb{R}^d$, $\mathrm{div}_x f(u(x,t)) = \sum_{i=1}^{d} \partial f_i(u(x,t)/\partial x_i$ and $\mathbb{R}_+ \equiv (0, \infty)$.

Young measure solutions can be constructed from the weak limit of approximate solutions to (1), cf. [3], [4], [10], [15] and [19]. Consider for instance vanishing viscosity solutions u_h satisfying

$$\begin{aligned} \partial_t u_h + \mathrm{div} f(u_h) &= h\Delta u_h, \text{ in } \Omega \times \mathbb{R}_+, \\ u_h &= \quad a, \quad \text{on } \partial\Omega \times \mathbb{R}_+, \\ u_h &= \quad u_0, \quad \text{on } \Omega \times \{0\}, \end{aligned}$$

Provided the data has bounded variation, [2] shows that u_h converge a.e. to a function satisfying (1-3) with the uniform bound

$$\|u_h(\cdot, t)\|_{L^\infty(\Omega \times \mathbb{R}_+)} \leq K.$$

Following [21] and [11], one can therefore extract a subsequence $\{u_{h_j}\}$ with an associated Young measure valued mapping $\nu_{(\cdot)} : \Omega \times \mathbb{R}_+ \to \mathrm{Prob}([-K, K])$, such that for any $g \in \mathcal{C}(\mathbb{R})$ and all $\phi \in L^1(\Omega \times \mathbb{R}_+)$

$$\begin{aligned} \lim_{j \to \infty} \int_{\Omega \times \mathbb{R}_+} g(u_{h_j}(y))\phi(y)dy &= \int_{\Omega \times \mathbb{R}_+} \int_{\mathbb{R}} g(\lambda) d\nu_y(\lambda) \; \phi(y)dy \\ &\equiv \int_{\Omega \times \mathbb{R}_+} \langle \nu_y, g(\lambda) \rangle \phi(y)dy. \end{aligned}$$

The work [18] formulated conditions for Young measure solutions to the initial boundary value problem. The definition is based on transversally averaged traces, on $\partial\Omega \times I\!R_+$, of Young measures. To define this trace introduce first, in a neighborhood of $\partial\Omega$, the change of coordinates

$$\Omega \ni x \to (\hat{x}, x^{\perp}) \in \partial\Omega \times (0, \kappa)$$

$$\hat{x} = x - x^{\perp} n(\hat{x}) \tag{4}$$

for some $\kappa > 0$, then we have from [18]

LEMMA 2.1. *Suppose $\nu : \Omega \times I\!R_+ \to Prob([-K, K])$ is a Young measure associated to a uniformly L^∞ bounded sequence of functions,*

$$\|u_n\|_{L^\infty(\Omega \times I\!R_+)} \le K, \quad n \in I\!N,$$

then there is a sequence of positive real numbers, $x_j^{\perp} \to 0$, and a Young measure $\gamma\nu : \partial\Omega \times I\!R_+ \to Prob([-K, K])$ such that, for every $g \in \mathcal{C}([-K, K])$, the $L^\infty(\partial\Omega \times I\!R_+)$ weak star limit

$$\begin{aligned} \lim_{j\to\infty} \int_{\partial\Omega\times I\!R_+} &\langle \nu_{(x(\hat{x}, x_j^{\perp}), t)}, g(\lambda)\rangle \phi(\hat{x}, t) d\hat{x} dt \\ &= \int_{\partial\Omega \times I\!R_+} \langle \gamma\nu_{(\hat{x},t)}, g(\lambda)\rangle \phi(\hat{x}, t) d\hat{x} dt, \quad \forall \phi \in L^1(\partial\Omega \times I\!R_+) \end{aligned} \tag{5}$$

holds, where $d\hat{x}$ is the Lebesgue measure on $\partial\Omega$.

Based on this trace and the measure valued solutions for initial value problems introduced by DiPerna [11], the work [18] defines

DEFINITION 2.2. *A Young measure, ν, with its trace, $\gamma\nu$, satisfying the assumptions in Lemma 2.1 is a measure solution to problem (1-3) if for all non negative test functions $\varphi \in \mathcal{C}_c^1(\overline{\Omega} \times I\!R_+)$ and for all $k \in I\!R$*

$$\begin{aligned} \int_{\Omega\times I\!R_+} \Big(&\langle \nu_{x,t}(\lambda), |\lambda - k|\rangle \partial_t \varphi(x, t) \\ &+ \langle \nu_{x,t}(\lambda), sgn(\lambda - k)(f(\lambda) - f(k))\rangle \cdot \nabla_x \varphi(x, t)\Big) dx dt \\ - \int_{\partial\Omega\times I\!R_+} &\langle \gamma\nu_{x,t}(\lambda), (f(\lambda) - f(k))\rangle \cdot n(x) sgn(a - k)\varphi(x, t) d\hat{x} dt \ge 0, \end{aligned} \tag{6}$$

and

$$\lim_{t\to 0+} \int_{\Omega} \langle \nu_{(x,t)}(\lambda), |\lambda - u_0(x)|\rangle dx = 0. \tag{7}$$

The strong convergence (7) can often be verified by a standard combination of weak convergence and convexity, see [11] and (55).

REMARK 2.3. *The Young trace measure $\gamma\nu$ satisfying the limit (5) is not uniquely determined by ν. However, the equation (6) implies that the trace of the fluxes for measure solutions*

$$\langle \gamma\nu_{x,t}(\lambda), sgn(\lambda - k)(f(\lambda) - f(k))\rangle \text{ and } \langle \gamma\nu_{x,t}(\lambda), f(\lambda) - f(k)\rangle \quad \forall k \in I\!R$$

are in fact uniquely defined on $L^1(\partial\Omega \times I\!R_+)$, see [18] and Section 3.

The main results of this paper are

THEOREM 2.4. *Assume that $u_0 \in L^\infty(\Omega)$, $a \in L^\infty(\partial\Omega \times I\!R_+)$, $f \in [\mathcal{C}(I\!R)]^d$ and that ν and σ are Young measure solutions to (1-3), in the sense of Definition 2.2, then the contraction*

$$\partial_t \int_\Omega \langle \nu_{x,t}(\lambda) \otimes \sigma_{x,t}(\mu), |\lambda - \mu| \rangle dx \leq 0,$$

holds in the distribution sense on $I\!R_+$. If in addition ν and σ satisfy the same initial condition (7), then there exists a unique solution $u \in L^\infty(\Omega \times I\!R_+)$ such that

$$\nu_y = \sigma_y = \delta_{u(y)}, \quad \text{for a.e. } y \in \Omega \times I\!R_+.$$

THEOREM 2.5. *Suppose that the data u_0, a and f satisfy the assumptions in Theorem 2.4. Then there exist an L^∞ solution $u \in L^\infty(\Omega \times I\!R_+)$, with $\nu_\cdot = \delta_{u(\cdot)}$ satisfying (6-7).*

The measure tensor product $\nu_y \otimes \sigma_y$ is defined for all $g \in \mathcal{C}(I\!R^2)$ by

$$\langle \nu_y \otimes \sigma_y, g(\lambda, \mu) \rangle \equiv \int_{I\!R} \int_{I\!R} g(\lambda, \mu) d\nu_y(\lambda) d\sigma_y(\mu).$$

We will often omit the integration variables λ and μ and write ν_y and σ_y instead of $\nu_y(\lambda)$ and $\sigma_y(\mu)$. Let us for $k \in I\!R$ denote the Kruzkov entropy pairs by

$$\big(|\lambda - k|, q(\lambda, k)\big) \equiv \big(|\lambda - k|, \operatorname{sgn}(\lambda - k)(f(\lambda) - f(k))\big).$$

REMARK 2.6. *Definition 2.2 is equivalent, cf. (39), to the distribution formulations for all $k \in I\!R$*

$$\partial_t \langle \nu_{x,t}, |\lambda - k| \rangle + \operatorname{div}_x \langle \nu_{x,t}, q(\lambda, k) \rangle \leq 0, \quad \text{in } \mathcal{D}'(\Omega \times I\!R_+), \tag{8}$$

$$\langle \gamma\nu_{x,t}, \big(sgn(\lambda - k) - (sgn(a - k)\big)\big(f(\lambda) - f(k)\big) \rangle \cdot n \geq 0, \text{ in } \mathcal{D}'(\partial\Omega \times I\!R_+). \tag{9}$$

The two inequalities (8-9) are the Young measure form of the Bardos, LeRoux and Nedelec [2] entropy condition and boundary entropy flux condition for uniqueness and existence of BV solutions. These conditions, based on the Kruzkov entropies in the interior and on the boundary, are a subset of the conditions Otto uses for uniqueness and existence of L^∞ solutions. Therefore the unique solution $u \in L^\infty(\Omega \times I\!R_+)$ in Theorem 2.4 is the unique solution constructed by Otto in [16].

There is a related formulation of (6) for measure solutions introduced in [6],[5] to study convergence of the SPH (Smoothed Particle Hydrodynamics) method. This formulation turns out to be well suited for the convergence of approximate solutions of (1) and requires somewhat less than Definition 2.2 for the boundary integral term.

3. The Proof of the Uniqueness Theorem 2.4. An attractive surprise of measure valued solutions is that a standard regularization of a non smooth measure valued solution remains a measure valued solution in the interior domain, also for a nonlinear problem, as first shown in [11]: let, for $\varepsilon > 0$, the function ω_ε be a standard mollifier on $\mathbb{R}^{d+1}$

$$\omega_\varepsilon(y) = \varepsilon^{-(d+1)}\omega(y/\varepsilon) \quad \forall y \in \mathbb{R}^{d+1}, \tag{10}$$

satisfying $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$, supp $\omega \subset \{y \in \mathbb{R}^{d+1} : |y| \le 1\}$ and

$$\omega \ge 0, \quad \int_{\mathbb{R}^{d+1}} \omega(y)dy = 1. \tag{11}$$

Then the positive measure ν^ε defined by

$$\langle \nu_y^\varepsilon, g\rangle = \int_{\Omega\times\mathbb{R}_+} \langle \nu_z, g\rangle \omega_\varepsilon(z-y)dz \quad \forall g \in \mathcal{C}(\mathbb{R}),\ \forall y \in \Omega \times \mathbb{R}_+$$

is a *Young* measure in the interior domain

$$\Omega_+^\delta \equiv \{y \in \Omega \times \mathbb{R}_+ : \text{distance}\,(y, \partial(\Omega\times\mathbb{R}_+)) > \delta\},$$

for $\varepsilon \le \delta$, and ν_y^ε depends smoothly on y.

The choise $(x,t) \in \Omega_+^\varepsilon$ implies $0 \le \omega_\varepsilon(\cdot - (x,t)) \in \mathcal{C}_c^\infty(\Omega\times\mathbb{R}_+)$ and $\partial_{y_i}\omega_\varepsilon(z-y) = -\partial_{z_i}\omega_\varepsilon(z-y)$. Consequently (6) with test function $\omega_\varepsilon(\cdot - (x,t))$ establishes that ν^ε is also a measure valued solution in the interior domain Ω_+^ε, i.e.

$$\begin{aligned}
&\partial_t\langle \nu_{(x,t)}^\varepsilon, |\lambda - k|\rangle + \operatorname{div}_x\langle \nu_{(x,t)}^\varepsilon, q(\lambda,k)\rangle \\
&= \int_{\Omega\times\mathbb{R}_+} \Big(\langle \nu_{(x',t')}, |\lambda-k|\rangle \partial_t\omega_\varepsilon((x',t')-(x,t)) \\
&\qquad + \langle \nu_{(x',t')}, q(\lambda,k)\rangle \cdot \nabla_x\omega_\varepsilon((x',t')-(x,t))\Big)dx'dt' \\
&= -\int_{\Omega\times\mathbb{R}_+} \Big(\langle \nu_{(x',t')}, |\lambda-k|\rangle \partial_{t'}\omega_\varepsilon((x',t')-(x,t)) \\
&\qquad + \langle \nu_{(x',t')}, q(\lambda,k)\rangle \cdot \nabla_{x'}\omega_\varepsilon((x',t')-(x,t))\Big)dx'dt' \\
&\le 0, \quad \text{for} \quad (x,t) \in \Omega_+^\varepsilon.
\end{aligned} \tag{12}$$

Note that ν^ε is defined as a positive measure on $\Omega\times\mathbb{R}_+$, however as Young measure solution of (1) it is well defined only on Ω_+^ε and therefore its behavoir in the boundary layer is crucial, which is the focus of this paper. To analyze the behavior near the boundary we shall in Step 2 below slightly modify the mollifier.

The proof of the theorem has four steps, based on six claims proved below:

Step 1 (the interior domain). The regularized measures $\nu_{(x,t)}^\varepsilon$ and $\sigma_{(x,t)}^\varepsilon$, which by (12-12) are smooth Young measure solutions away from the boundary, satisfy **Claim 1a:**

$$\partial_t\langle \nu_{(x,t)}^\varepsilon \otimes \sigma_{(x,t)}^\varepsilon, |\lambda-\mu|\rangle + \operatorname{div}_x\langle \nu_{(x,t)}^\varepsilon \otimes \sigma_{(x,t)}^\varepsilon, q(\lambda,\mu)\rangle \le 0,\ (x,t) \in \Omega_+^\varepsilon. \tag{13}$$

Step 2 (the boundary contribution). Our analysis near the boundary uses a mollifier which depends on two parameters providing different mollification in a surface related to $\partial\Omega \times I\!\!R_+$ and its normal direction. As a sub step for a general smooth curved boundary we consider first the simpler special case when the boundary of Ω is the plane $x_1 = 0$, so that $x^\perp \equiv x_1$ and $\hat{x} \equiv (x_2, \ldots, x_d)$; **Claim 2b** then treats the general case with a smoothly curved boundary by local transformations to coordinates with planar boundary. In the planar boundary case, $x_1 = 0$, let the mollifiers $\omega_{\varepsilon^\perp} \in \mathcal{C}_c^\infty(I\!\!R)$ and $\omega_{\hat{\varepsilon}} \in \mathcal{C}_c^\infty(I\!\!R^d)$ satisfy (10-11) with $d+1$ replaced by 1 and d, respectively. Define for $(x,t) = \Big((x_1, \hat{x}), t\Big) \in \Omega \times I\!\!R_+$ and $\varepsilon^\perp > 0,\ \hat{\varepsilon} > 0$ the mollifier more precisely by

$$\omega_\varepsilon(x,t) \equiv \omega_{\varepsilon^\perp}(x_1)\omega_{\hat{\varepsilon}}(\hat{x},t). \tag{14}$$

This new mollifier with the two dimensional mollifier parameter $\varepsilon = (\varepsilon^\perp, \hat{\varepsilon})$ defines by (12) again measure solutions ν^ε and σ^ε in the interior domain and we use the notation

$$\begin{aligned} &\nu^{\varepsilon^\perp, \hat{\varepsilon}} \equiv \nu^\varepsilon, \sigma^{\varepsilon^\perp, \hat{\varepsilon}} \equiv \sigma^\varepsilon, \\ &\nu^{\hat{\varepsilon}} \equiv \nu^{0,\hat{\varepsilon}},\ \sigma^{\hat{\varepsilon}} \equiv \sigma^{0,\hat{\varepsilon}}. \end{aligned} \tag{15}$$

Note that the new ν^ε and σ^ε satisfy (13) in Ω_+^δ, for $\delta \geq \varepsilon^\perp$, since with a two dimensional mollification parameter $\varepsilon = (\varepsilon^\perp, \hat{\varepsilon},)$ and a planar boundary only the $x^\perp$-convolution part, $\omega_{\varepsilon^\perp}$, of the mollification interfere with the boundary. Take the limit $\varepsilon^\perp \to 0+$ in this version of (13) to obtain **Claim 1b:**

$$\partial_t \langle \nu^{\hat{\varepsilon}}_{(x,t)} \otimes \sigma^{\hat{\varepsilon}}_{(x,t)}, |\lambda - \mu| \rangle + \ \mathrm{div}_x \langle \nu^{\hat{\varepsilon}}_{(x,t)} \otimes \sigma^{\hat{\varepsilon}}_{(x,t)}, q(\lambda,\mu) \rangle \leq 0, \quad \text{in } \mathcal{D}'(\Omega \times I\!\!R_+). \tag{16}$$

Define, for any non negative $\theta \in \mathcal{C}_c^1(I\!\!R_+)$, the $L^\infty(0,\kappa)$ function

$$A(x^\perp) \equiv \int_{\partial\Omega \times I\!\!R_+} \langle \nu^{\hat{\varepsilon}}_{(x(\hat{x},x^\perp),t)} \otimes \sigma^{\hat{\varepsilon}}_{(x(\hat{x},x^\perp),t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) \theta(t) d\hat{x} dt,$$

then equation (16) and (4) imply $A' \geq B$ in $\mathcal{D}'(0,\kappa)$ for some $\kappa > 0$, where $\|B\|_{L^\infty(0,\kappa)}$ is bounded. Consequently, the function A has bounded variation and therefore the limit

$$\lim_{x^\perp \to 0+} A(x^\perp) \ \text{ exists.} \tag{17}$$

Combine this limit and (16), with test functions approaching $\theta(t)1_\Omega(x)$, to obtain

$$\begin{aligned} &\partial_t \int_\Omega \langle \nu^{\hat{\varepsilon}}_{(x,t)} \otimes \sigma^{\hat{\varepsilon}}_{(x,t)}, |\lambda - \mu| \rangle dx \\ &\quad \leq - \lim_{x^\perp \to 0+} \int_{\partial\Omega} \langle \nu^{\hat{\varepsilon}}_{(x(\hat{x},x^\perp),t)} \otimes \sigma^{\hat{\varepsilon}}_{(x(\hat{x},x^\perp),t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) d\hat{x} \\ &\quad \underbrace{=}_{\text{by (17)}} - \lim_{j \to \infty} \int_{\partial\Omega} \langle \nu^{\hat{\varepsilon}}_{(x(\hat{x},x_j^\perp),t)} \otimes \sigma^{\hat{\varepsilon}}_{(x(\hat{x},x_j^\perp),t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) d\hat{x} \\ &\quad \underbrace{=}_{\textbf{Claim 2a}} - \int_{\partial\Omega} \langle \gamma\nu^{\hat{\varepsilon}}_{(x(\hat{x},0),t)} \otimes \gamma\sigma^{\hat{\varepsilon}}_{(x(\hat{x},0),t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) d\hat{x}, \quad \text{in } \mathcal{D}'(I\!\!R_+). \end{aligned} \tag{18}$$

The proof of the last equality, Claim 2a, is based on polynomial approximation of q and the weak convergence in Lemma 2.1. Let $\hat{\varepsilon} \to 0+$ in (18) as in Claim 1b to conclude

$$\begin{aligned}
&\partial_t \int_\Omega \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \rangle dx \\
&\le - \int_{\partial\Omega} \langle \gamma\nu_{(x,t)} \otimes \gamma\sigma_{(x,t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) d\hat{x}, \quad \text{in } \mathcal{D}'(I\!R_+).
\end{aligned} \tag{19}$$

Step 3 (the boundary term provides a contraction). The boundary term in (19) has the right sign for a contraction:

Claim 3:

$$\langle \gamma\nu_{(\hat{x},t)} \otimes \gamma\sigma_{(\hat{x},t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) \ge 0, \quad \text{a.e. on } \partial\Omega \times I\!R_+.$$

Step 4 (reduction to a point mass). Steps 1-3 yield the contraction

$$\partial_t \int_\Omega \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \rangle dx \le 0. \tag{20}$$

The initial conditions imply

$$\begin{aligned}
&\limsup_{t\to 0} \int_\Omega \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - \mu| \rangle dx \\
&\le \limsup_{t\to 0} \int_\Omega \langle \nu_{(x,t)} \otimes \sigma_{(x,t)}, |\lambda - u_0| + |u_0 - \mu| \rangle dx \\
&= \limsup_{t\to 0} \int_\Omega \langle \nu_{(x,t)}, |\lambda - u_0| \rangle dx \\
&\quad + \limsup_{t\to 0} \int_\Omega \langle \sigma_{(x,t)}, |u_0 - \mu| \rangle dx = 0,
\end{aligned}$$

and we conclude by (20) that for a.e. $y \in \Omega \times I\!R_+$

$$\langle \nu_y \otimes \sigma_y, |\lambda - \mu| \rangle \ = 0. \tag{21}$$

Therefore the support of $\nu_y \otimes \sigma_y$ is on the line $\lambda = \mu$ and, since the measure is a tensor product, the support of ν_y and σ_y must be a common single point, say $u(y)$, i.e. **Claim 4:** $\nu_y = \sigma_y = \delta_{u(y)}$. Since ν_y and σ_y have support in $[-K, K]$ and are measurable in y, the function u belongs to $L^\infty(\Omega \times I\!R_+)$.

Proof of Claim 1a. Since the measures ν_y^ε and σ_y^ε depend smoothly on y, we can directly compute the derivatives by the chain rule

$$\begin{aligned}
&\partial_t \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, |\lambda - \mu| \rangle + \ \mathrm{div}_x \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, q(\lambda,\mu) \rangle \\
= \ &\Big(\langle \partial_t \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, |\lambda - \mu| \rangle + \langle \nabla_x \ \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, q(\lambda,\mu) \rangle \Big) \\
+ \ &\Big(\langle \nu^\varepsilon_{(x,t)} \otimes \partial_t \sigma^\varepsilon_{(x,t)}, |\lambda - \mu| \rangle + \langle \nu^\varepsilon_{(x,t)} \otimes \nabla_x \ \sigma^\varepsilon_{(x,t)}, q(\lambda,\mu) \rangle \Big) \\
\equiv \ &I + II.
\end{aligned}$$

The fact that ν^ε is a Young measure solution in the sense of (12) implies $I \le 0$ on Ω_+^ε. Similarly, the measure σ^ε is also a solution and the symmetry $q(\lambda,\mu) = q(\mu,\lambda)$ imply $II \le 0$, on Ω_+^ε, which proves (13).

Proof of Claim 1b. Let

$$P_M(\lambda,\mu) \equiv \sum_{n=0}^{M}\sum_{m=0}^{M} p_{nm}\lambda^n\mu^m$$

be polynomials approximating the Kruzkov flux $q(\lambda,\mu)$ (or entropy $|\lambda-\mu|$) such that

$$\|q - P_M\|_{[\mathcal{C}([-K,K]^2)]^d} \to 0, \quad \text{as } M \to \infty. \tag{22}$$

The approximation to the identity property of $\omega_{\varepsilon^\perp}$, cf. [17], yields for any $\phi \in \mathcal{C}_c(\Omega \times I\!R_+)$

$$\begin{aligned}
&\lim_{\varepsilon^\perp \to 0+} \int_{\Omega \times I\!R_+} \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, q(\lambda,\mu)\rangle \phi(x,t)dxdt \\
&= \sum_{n,m\le M} \lim_{\varepsilon^\perp \to 0+} \int_{\Omega \times I\!R_+} p_{nm}\langle \nu^\varepsilon_{(x,t)}, \lambda^n\rangle\langle \sigma^\varepsilon_{(x,t)}, \mu^m\rangle \phi(x,t)dxdt \\
&\quad + \lim_{\varepsilon^\perp \to 0+} \int_{\Omega \times I\!R_+} \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, q - P_M\rangle \phi(x,t)dxdt \\
&= \int_{\Omega \times I\!R_+} \langle \nu^{\hat\varepsilon}_{(x,t)} \otimes \sigma^{\hat\varepsilon}_{(x,t)}, P_M(\lambda,\mu)\rangle \phi(x,t)dxdt \\
&\quad + \lim_{\varepsilon^\perp \to 0+} \int_{\Omega \times I\!R_+} \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, q - P_M\rangle \phi(x,t)dxdt
\end{aligned}$$

The norm of the Young measures are uniformly bounded by 1. Therefore the convergence

$$\Big|\int_{\Omega \times I\!R_+} \langle \nu^\varepsilon_{(x,t)} \otimes \sigma^\varepsilon_{(x,t)}, P_M - q\rangle \phi(x,t)dxdt\Big| \le \|P_M - q\|_{[\mathcal{C}_c]^d}\|\phi\|_{L^1(\Omega\times I\!R_+)} \to 0,$$

as $M \to \infty$, is uniform in ε, which together with analogous estimates for the entropy $|\lambda-\mu|$ proves Claim 1b.

Proof of Claim 2a. Let as in Claim 1b

$$P_M(\lambda,\mu) \equiv \sum_{n=0}^{M}\sum_{m=0}^{M} p_{nm}\lambda^n\mu^m$$

be polynomials approximating the Kruzkov flux $q(\lambda,\mu)$ with

$$\|q - P_M\|_{[\mathcal{C}([-K,K]^2)]^d} \to 0, \quad \text{as } M \to \infty. \tag{23}$$

Then the function q can be replaced by the polynomial P_M with neglicable errors in the integrals of (18) for sufficiently large M, since norm of the measures ν^ε_y, σ^ε_y, $\gamma\nu^\varepsilon_y$ and $\gamma\sigma^\varepsilon_y$ are uniformly bounded in ε and y. A combination of Fubinis theorem (a), dominated convergence (b) and the weak limits (c) in Lemma 2.1 with test function

$\omega_{\hat\varepsilon}$ verifies Claim 2a:

$$
\begin{aligned}
&\lim_{j\to\infty}\int_{\partial\Omega}\langle \nu^{\hat{\hat\varepsilon}}_{(x(\hat x,x_j^\perp),t)}\otimes\sigma^{\hat{\hat\varepsilon}}_{(x(\hat x,x_j^\perp),t)},P_M(\lambda,\mu)\rangle\cdot n(\hat x)d\hat x\\
&\underbrace{=}_{(a)}\sum_{n,m\le M}\lim_{j\to\infty}\int_{\partial\Omega}\langle\nu^{\hat{\hat\varepsilon}}_{(x(\hat x,x_j^\perp),t)},\lambda^n\rangle\langle\sigma^{\hat{\hat\varepsilon}}_{(x(\hat x,x_j^\perp),t)},\mu^m\rangle p_{nm}\cdot n(\hat x)d\hat x\\
&=\sum_{n,m\le M}\lim_{j\to\infty}\int_{\partial\Omega}\Big(\int_{\partial\Omega\times I\!R_+}\langle\nu_{(x_j^\perp,y)},\lambda^n\rangle\omega_{\hat\varepsilon}(y-(\hat x,t))dy\\
&\qquad\times\int_{\partial\Omega\times I\!R_+}\langle\sigma_{(x_j^\perp,y)},\mu^m\rangle\omega_{\hat\varepsilon}(y-(\hat x,t))dy\Big)p_{nm}\cdot n(\hat x)d\hat x\\
&\underbrace{=}_{(b,c)}\sum_{n,m\le M}\int_{\partial\Omega}\Big(\int_{\partial\Omega\times I\!R_+}\langle\gamma\nu_y,\lambda^n\rangle\omega_{\hat\varepsilon}(y-(\hat x,t))dy\\
&\qquad\times\int_{\partial\Omega\times I\!R_+}\langle\gamma\sigma_y,\mu^m\rangle\omega_{\hat\varepsilon}(y-(\hat x,t))dy\Big)p_{nm}\cdot n(\hat x)d\hat x\\
&=\int_{\partial\Omega}\langle\gamma\nu^{\hat{\hat\varepsilon}}_{(x(\hat x,0),t)}\otimes\gamma\sigma^{\hat{\hat\varepsilon}}_{(x(\hat x,0),t)},P_M(\lambda,\mu)\rangle\cdot n(\hat x)d\hat x.
\end{aligned}
$$

Proof of Claim 2b. In the case of a smoothly curved boundary $\partial\Omega$ we will change to local coordinates where the boundary is a plane, to mollify tangent to the plane in a neighborhood of the boundary. In an interior domain we use Claim 1a with the standard mollifier (10-11).

Partition $\partial\Omega$ into a finite set of overlapping patches $\mathcal{P}_n$ of open sub sets of $\partial\Omega$ of sufficiently small diameter. For any patch introduce a coordinate system tangent to some point $x_{\mathcal{P}_n}$ in $\mathcal{P}_n$ and let $\hat x'$ be the orthogonal projection of $\hat x\in\mathcal{P}_n$ onto the tangent plane. Extend the flattening coordinates $\hat x'$ to Ω by

$$x'=(x^\perp,\hat x')\quad\text{ where } x=x^\perp n(\hat x)+\hat x,\ 0\le x^\perp\le\kappa,\ \hat x\in\mathcal{P}_n.$$

Introduce also the notation $x'=(x_1',\hat{x'})$ with $x_1'=x^\perp$ and $\hat{x'}=(\hat x_2',\dots,\hat x_d')$ and write formally $\partial\Omega'\equiv\cup_n\hat x'(\mathcal{P}_n)$ and $\Omega'\equiv\cup_n x'((0,\kappa)\times\mathcal{P}_n)$. Let $\Omega^\delta\equiv\{y\in\Omega:\text{distance}\,(y,\partial\Omega)>\delta\}$. We will use a partition of unity subordinate to the patches $\mathcal{P}_n$

$$1_{\partial\Omega}=\sum_n\chi_n,\quad 0\le\chi_n\in\mathcal{C}^\infty(\partial\Omega)\text{ and supp}\,\chi_n\subset\mathcal{P}_n.$$

Let the mollifiers $\omega_\varepsilon^\perp\in\mathcal{C}_c^\infty(I\!R)$ and $\hat\omega_\varepsilon\in\mathcal{C}_c^\infty(I\!R^d)$ satisfy (10-11) with $d+1$ replaced by 1 and d, respectively, where in addition $\text{supp}\,\omega_\varepsilon^\perp\subset[0,\varepsilon]$. Define for $\varepsilon>0$ and $(x^\perp,\hat x,t)\in[0,\kappa]\times\mathcal{P}_n\times I\!R_+$ a mollifier tangential to $\partial\Omega$ by

$$\omega_\varepsilon^n(x'(x),t)\equiv\omega_\varepsilon^\perp(x_1'(x))\hat\omega_\varepsilon(\hat x'(x),t).\tag{24}$$

Write the gradient and the volume measure in the local coordinates as

$$
\begin{aligned}
\partial_{x_i}&=\sum_j\frac{\partial x_j'}{\partial x_i}\partial_{x_j'}\equiv\sum_j X_j^i(x')\partial_{x_j'},\\
dx'&=|\det(\frac{\partial x'}{\partial x})|dx\equiv J(x)dx.
\end{aligned}
$$

Use Riesz representation theorem to define the Young measure ν^ϵ, for all $g \in \mathcal{C}(I\!R)$ and for $x^\perp(y') \le \kappa - \varepsilon$, by

$$\begin{aligned}
&\langle \nu^\varepsilon_{(x(y'),t)}, g\rangle = \\
&= \sum_n \int_{([0,\kappa]\times\hat{x}'(\mathcal{P}_n))\times I\!R_+} \langle \nu_{(x(z'),t')}, g\rangle \omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\chi_n(x(y'))dz'dt' \\
&\equiv \int_{\Omega'\times I\!R_+} \langle \nu_{(x(z'),t')}, g\rangle \omega_\varepsilon\Big((z',t')-(y',t);\, y'\Big)dz'dt'. \qquad (25)
\end{aligned}$$

The last identity is used only as a notation to avoid writing the sum over all χ_n when the partition is not crucial. The partition is important when ν^ε is differentiated below. To prepare for this differentiation define also $m^\nu_\varepsilon \in L^\infty(\Omega \times I\!R_+)$ by

$$\begin{aligned}
m^\nu_\varepsilon(x(y'),t) = &\sum_{ijn} \int_{\Omega'\times I\!R_+} \langle \nu_{(x(z'),t')}, q_i\rangle \\
&\Big(X^i_j(y') - X^i_j(z')\Big)\partial_{y'_j}\omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\chi_n(x(y'))dz'dt' \\
&+\sum_{ijn} \int_{\Omega\times I\!R_+} \langle \nu_{(x(z'),t')}, q_i\rangle \qquad (26)\\
&X^i_j(z')\partial_{y'_j}\omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\chi_n(x(y'))\Big(J(x(z')) - J(x(y'))\Big)dx(z')dt' \\
&+\sum_{ijn} \int_{\Omega'\times I\!R_+} \langle \nu_{(x(z'),t')}, q_i\rangle \omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\nabla_x\chi_n(x(y'))dz'dt'.
\end{aligned}$$

Following (12), the facts

$$\begin{aligned}
&\omega^n_\varepsilon(z'(\cdot)-(y',t)) \in \mathcal{C}^1_c(\Omega\times I\!R_+) \quad \text{for } x(y') \in (\Omega - \Omega^{\kappa-\varepsilon}) \times I\!R_+, \\
&\partial_{y'_j}\omega^n_\varepsilon(z'-y') = -\partial_{z'_j}\omega^n_\varepsilon(z'-y'),
\end{aligned}$$

imply together with (6), for the test function

$$\sum_n \omega^n_\varepsilon((z'(\cdot),t')-(y',t))\chi_n(x(y'))J(x(y')),$$

that

$$\begin{aligned}
&\int_{\Omega\times I\!R_+} \langle \nu_{(x(z'),t')}, |\lambda-\mu|\rangle \qquad (27)\\
&\quad \sum_n \underbrace{\partial_t}_{-\partial_{t'}} \omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\chi_n(x(y'))J(x(y'))dx(z')dt' \\
&+\sum_i \int_{\Omega\times I\!R_+} \langle \nu_{(x(z'),t')}, q_i\rangle \\
&\quad \sum_n \underbrace{\sum_j X^i_j(z')\partial_{y'_j}}_{-\partial_{x(z')_i}} \omega^n_\varepsilon\Big((z',t')-(y',t)\Big)\chi_n(x(y'))J(x(y'))dx(z')dt' \le 0.
\end{aligned}$$

The next step is to combine (25-27) to verify that ν^ε is approximately a measure solution, for $(x^\perp(y'),\hat{x}(y'),t) \in (0,\kappa-\varepsilon]\times\partial\Omega\times I\!R_+$,

$$\partial_t\langle \nu^\varepsilon_{(x(y'),t)}, |\lambda-k|\rangle + \mathrm{div}_x\langle \nu^\varepsilon_{(x(y'),t)}, q(\lambda,k)\rangle \le m^\nu_\varepsilon(x(y'),t), \qquad (28)$$

as follows. In this left hand side, write the gradient $\partial_{x_i(y')} = \sum_j X^i_j(y')\partial_{y'_j}$ in the local coordinates and the volume measure $dz' = J(x(z'))dx(z')$ in the global coordinates. The remainder m'_ε is constructed from the splitting

$$\begin{aligned} J(x(z'))\sum_j X^i_j(y')\partial_{y'_j} = J(x(y'))&\sum_j X^i_j(z')\partial_{y'_j} \\ &+J(x(z'))\sum_j \Big(X^i_j(y') - X^i_j(z')\Big)\partial_{y'_j} \\ &+\Big(J(x(z')) - J(x(y'))\Big)\sum_j X^i_j(z')\partial_{y'_j}, \end{aligned}$$

where the first term in this right hand side yields, by (27), a contribution to (28) with the right sign. The other terms in the splitting gives the remainder m'_ε.

For two Young measure solutions ν^{ε_1} and σ^{ε_2} we then obtain as in Claim 1a for $(x,t) \in (\Omega - \Omega^{\kappa-\max(\varepsilon_1,\varepsilon_2)}) \times I\!R_+$

$$\begin{aligned} &\partial_t\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda - \mu|\rangle + \mathrm{div}_x\langle \nu^{\varepsilon_1}_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu)\rangle \\ &\quad \le \langle \sigma^{\varepsilon_2}_{(x,t)}, m^\nu_{\varepsilon_1}\rangle + \langle \nu^{\varepsilon_1}_{(x,t)}, m^\sigma_{\varepsilon_2}\rangle \equiv m'_\varepsilon(x,t), \end{aligned} \tag{29}$$

where the function m'_ε satisfies

LEMMA 3.1. *There is a constant C, depending only on Ω, K, f, and the partition $\{\mathcal{P}_n\}$, such that*

$$|m'_\varepsilon(x,t)| \le C.$$

Proof of the lemma. Let $g(x,y) \equiv \langle \nu_{(x,t)} \otimes \sigma_{(y,t)}, q\rangle \in [L^\infty(\Omega\times\Omega)]^d$. We have

$$\begin{aligned} m'_\varepsilon(x(z'),t) = &\sum_{jn_1n_2}\int_{\Omega'^2} g(x(x'),x(y')) \\ &\cdot(X_j(x') - X_j(z'))\partial_{x'_j}\omega^{n_1}_{\varepsilon_1}(x'-z')\omega^{n_2}_{\varepsilon_2}(y'-z')\chi_{n_1}(x(z'))\chi_{n_2}(x(z'))dx'dy' \\ &+\sum_{jn_1n_2}\int_{\Omega'^2} g(x(x'),x(y')) \\ &\cdot(X_j(y') - X_j(z'))\partial_{y'_j}\omega^{n_2}_{\varepsilon_2}(y'-z')\omega^{n_1}_{\varepsilon_1}(x'-z')\chi_{n_1}(x(z'))\chi_{n_2}(x(z'))dx'dy' \\ &+\sum_{jn_1n_2}\int_{\Omega'^2} g(x(x'),x(y'))\cdot X_j(x')\partial_{x'_j}\omega^{n_1}_{\varepsilon_1}(x'-z')\omega^{n_2}_{\varepsilon_2}(y'-z') \\ &\chi_{n_1}(x(z'))\chi_{n_2}(x(z'))\Big(J(x(x')) - J(x(z'))\Big)dx'dy' \\ &+\sum_{jn_1n_2}\int_{\Omega'^2} g(x(x'),x(y'))\cdot X_j(y')\partial_{y'_j}\omega^{n_2}_{\varepsilon_2}(y'-z')\omega^{n_1}_{\varepsilon_1}(x'-z') \\ &\chi_{n_1}(x(z'))\chi_{n_2}(x(z'))\Big(J(x(y')) - J(x(z'))\Big)dx'dy' \\ &+\sum_{jn_1n_2}\int_{\Omega'^2} g(x(x'),x(y')) \\ &\cdot\omega^{n_2}_{\varepsilon_2}(y'-z')\omega^{n_1}_{\varepsilon_1}(x'-z')\nabla_x\Big(\chi_{n_1}(x(z'))\chi_{n_2}(x(z'))\Big)dx'dy'. \end{aligned}$$

Use that $\|g\|_{L^\infty}$ is bounded, $X \in \mathcal{C}^1$ and $J \in \mathcal{C}^1$, and the ω_ε approximation to the identity to obtain the uniform bound

$$|m'_\varepsilon(x,t)| \le C \text{ for } (x,t) \in \Omega \times I\!R_+. \qquad \square \tag{30}$$

To finish the proof of Claim 2b, let now $\varepsilon_1 \to 0+$ in (29) and use polynomial approximation of q and $|\lambda - \mu|$ as in Claim 1b to conclude that for any non negative $\phi \in \mathcal{C}^1_c((\Omega - \Omega)^{\kappa - \varepsilon_2} \times I\!R_+)$

$$\begin{aligned}\int_{\Omega \times I\!R_+} \Big(&\langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda - \mu| \rangle \partial_t \phi + m'_\varepsilon \phi \\ &+ \langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \rangle \cdot \nabla_x \phi \Big) dx dt \ge 0.\end{aligned} \tag{31}$$

This equation implies as in (17) that

$$\lim_{x^\perp \to 0+} \int_{\partial\Omega \times I\!R_+} \langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x})\, d\hat{x} dt \quad \text{exists.} \tag{32}$$

Define the non negative cut off function $\phi_\delta \in \mathcal{C}^1(I\!R)$ with $0 \le \phi_\delta(x^\perp) \le 1$ and $\phi_\delta(x^\perp) = 0$ for $x^\perp \ge \delta$ and $\phi_\delta(x^\perp) = 1$ for $x^\perp \le \delta/2$. Test functions approaching $1_\Omega(x)\phi_\delta(x^\perp)\theta(t)$ in (31) yield as in (18) by Fubinis theorem, (32), dominated convergence, Lemma 2.1 and the factorization (24) of ω_ε

$$\begin{aligned}&\partial_t \int_\Omega \langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, |\lambda - \mu| \rangle \phi_\delta(x^\perp) dx \\ &\quad - \int_\Omega m'_\varepsilon(x,t) \phi_\delta(x^\perp) dx - \int_\Omega \langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \rangle \cdot \nabla_x \phi_\delta(x^\perp) dx \\ &\le - \lim_{x^\perp \to 0+} \int_{\partial\Omega} \langle \nu_{(x,t)} \otimes \sigma^{\varepsilon_2}_{(x,t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) d\hat{x} \\ &= - \lim_{j \to \infty} \int_{\Omega'} \Big(\int_{\partial\Omega} \langle \nu_{(x(x_j^\perp, \hat{x}),t)} \otimes \sigma_{(x(y_1', \hat{y}'),t)}, q(\lambda,\mu) \rangle \cdot n(\hat{x}) \\ &\qquad\qquad \omega_{\varepsilon_2}(y' - (x_j^\perp, \hat{x}'); \hat{x}) d\hat{x} \Big) dy' \\ &= - \int_{\partial\Omega} \langle \gamma\nu_{(x(0,\hat{x}),t)} \otimes \sigma^{\varepsilon_2}_{(x(0,\hat{x}),t)}, q(\lambda,\mu) \rangle d\hat{x} \ \text{ in } \mathcal{D}'(I\!R_+).\end{aligned} \tag{33}$$

Similarly the four results (a) Fubinis theorem, (b) dominated convergence, (c) the uniform convergence

$$\int_{\partial\Omega'} \hat{\omega}_{\varepsilon_0}(\hat{z}' - \hat{x}') \hat{\omega}_{\varepsilon_2}(\hat{y}' - \hat{x}') d\hat{x}' \to \hat{\omega}_{\varepsilon_0}(\hat{z}' - \hat{y}') \quad \text{as } \varepsilon_2 \to 0+$$

and (d) the function

$$a(\hat{z}', \cdot) \equiv \int_{\partial\Omega'} \langle \gamma\nu_{(x(0,\hat{z}'),t)} \otimes \sigma_{(x(\cdot,\hat{y}'),t)}, q(\lambda,\mu) \cdot n(\hat{y}') \rangle \hat{\omega}_{\varepsilon_0}(\hat{z}' - \hat{y}') d\hat{y}'$$

having bounded variation as a function of y_1', for fixed $\hat{z}'$, imply in $\mathcal{D}'(I\!R_+)$

$$\begin{aligned}
&\lim_{\varepsilon_2\to 0+}\int_{\partial\Omega}\langle\gamma\nu^{\varepsilon_0}_{(x(0,\hat{x}),t)}\otimes\sigma^{\varepsilon_2}_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}\\
&\underbrace{=}_{\text{(a,b)}}\int_{\partial\Omega'}\lim_{\varepsilon_2\to 0+}\int_{I\!R_+}\int_{\partial\Omega'}\langle\gamma\nu_{(x(0,\hat{z}'),t)}\otimes\sigma_{(x(y_1',\hat{y}'),t)},q(\lambda,\mu)\rangle\cdot\\
&\qquad\cdot\Big(\int_{\partial\Omega}n(\hat{x})\hat{\omega}_{\varepsilon_0}(\hat{z}'-\hat{x}';\hat{x}')\hat{\omega}_{\varepsilon_2}(\hat{y}'-\hat{x}';\hat{x}')d\hat{x}\Big)d\hat{y}'\omega^{\perp}_{\varepsilon_2}(y_1')dy_1'd\hat{z}'\\
&\underbrace{=}_{\text{(a,c,d), Lemma 2.1}}\int_{\partial\Omega}\langle\gamma\nu^{\varepsilon_0}_{(x(0,\hat{x}),t)}\otimes\gamma\sigma_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}.
\end{aligned}\tag{34}$$

Approximation of q by polynomials as in Claim 1b shows the uniform, in ε_2, convergence

$$\lim_{\varepsilon_0\to 0+}\int_{\partial\Omega}\langle(\gamma\nu^{\varepsilon_0}-\gamma\nu)_{(x(0,\hat{x}),t)}\otimes\sigma^{\varepsilon_2}_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}=0,\tag{35}$$

$$\lim_{\varepsilon_0\to 0+}\int_{\partial\Omega}\langle(\gamma\nu^{\varepsilon_0}-\gamma\nu)_{(x(0,\hat{x}),t)}\otimes\gamma\sigma_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}=0.\tag{36}$$

Finally, let $\varepsilon_2\to 0+$ in (33) and use (34-36) to conclude

$$\begin{aligned}
&\partial_t\int_\Omega\langle\nu_{(x,t)}\otimes\sigma_{(x,t)},|\lambda-\mu|\rangle\phi_\delta(x^\perp)dx-\limsup_{\varepsilon\to 0+}\int_\Omega m_\varepsilon'(x,t)\phi_\delta(x^\perp)dx\\
&\quad-\int_\Omega\langle\nu_{(x,t)}\otimes\sigma_{(x,t)},q(\lambda,\mu)\rangle\cdot\nabla\phi_\delta(x^\perp)dx\\
&\le-\int_{\partial\Omega}\langle\gamma\nu_{(x(0,\hat{x}),t)}\otimes\gamma\sigma_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}.
\end{aligned}\tag{37}$$

We have $0\le\phi_\delta\le 1$. Consequently $0\le 1-\phi_\delta\in\mathcal{C}^\infty_c(\Omega)$ and the limit $\varepsilon\to 0+$ of (13), obtained as in Claim 1b, therefore implies

$$\begin{aligned}
&\partial_t\int_\Omega\langle\nu_{(x,t)}\otimes\sigma_{(x,t)},|\lambda-\mu|\rangle(1-\phi_\delta(x^\perp))dx\\
&\quad-\int_\Omega\langle\nu_{(x,t)}\otimes\sigma_{(x,t)},q(\lambda,\mu)\rangle\cdot\nabla(1-\phi_\delta(x^\perp))dx\le 0.
\end{aligned}\tag{38}$$

Use Lemma 3.1 to obtain the uniform bound

$$\int_\Omega m_\varepsilon'(x,t)\phi_\delta(x^\perp)dx=\mathcal{O}(\delta),$$

and add (37) and (38) with $\delta\to 0+$ to prove the claim

$$\begin{aligned}
&\partial_t\int_\Omega\langle\nu_{(x,t)}\otimes\sigma_{(x,t)},|\lambda-\mu|\rangle dx\\
&\quad\le-\int_{\partial\Omega}\langle\gamma\nu_{(x(0,\hat{x}),t)}\otimes\gamma\sigma_{(x(0,\hat{x}),t)},q(\lambda,\mu)\rangle\cdot n(\hat{x})d\hat{x}.
\end{aligned}$$

Proof of Claim 3. Take the test function $\varphi(x,t) = \phi(\hat{x},t)\chi_\kappa(x^\perp)$ in (6), with

$$\chi_\kappa(x^\perp) = \begin{cases} (1 - \frac{x^\perp}{\kappa})^2, & 0 \le x^\perp < \kappa \\ 0 & x^\perp \ge \kappa, \end{cases}$$

and let $\kappa \to 0$. Lemma 2.1 and the fact that the $L^\infty(0,\kappa)$ function

$$\int_{\partial\Omega\times I\!R_+} \langle \nu_{(x(\hat{x},\cdot),t)}, q(\lambda,k)\rangle \cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt$$

has bounded variation on $(0,\kappa)$, cf. (17), imply that for all $k \in I\!R$

$$\begin{aligned} \int_{\partial\Omega\times I\!R_+} &\langle \gamma\nu_{(\hat{x},t)}, (\operatorname{sgn}(\lambda-k) - \operatorname{sgn}(a-k))(f(\lambda)-f(k))\rangle \\ &\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt \ge 0. \end{aligned} \tag{39}$$

This inequality shows that a.e. on $\partial\Omega \times I\!R_+$ for all $k \in I\!R$

$$\langle \gamma\nu_{(\hat{x},t)}, (\operatorname{sgn}(\lambda-k) - \operatorname{sgn}(a-k))(f(\lambda)-f(k))\rangle \cdot n(\hat{x}) \ge 0. \tag{40}$$

We similarly obtain a.e. on $\partial\Omega \times I\!R_+$ for all $k \in I\!R$

$$\langle \gamma\sigma_{(\hat{x},t)}, (\operatorname{sgn}(\mu-k) - \operatorname{sgn}(a-k))(f(\mu)-f(k))\rangle \cdot n(\hat{x}) \ge 0. \tag{41}$$

To verify Claim 3 we will divide the integration

$$\int_{I\!R^2} q(\lambda,\mu)d(\gamma\nu(\lambda))d(\gamma\sigma(\mu)) \tag{42}$$

into six disjoint domains and their boundaries, described by the figure below,

$$\begin{aligned} I\!R^2 = &\{\mu < a, \lambda < \mu\} \cup \{\mu > a, \lambda > \mu\} \cup \{\mu = a\} \\ &\cup\{\lambda < a, \mu < \lambda\} \cup \{\lambda > a, \mu > \lambda\} \cup \{\lambda = a\} \\ &\cup\{\mu < a, \lambda > a\} \cup \{\mu > a, \lambda < a\} \cup \{\mu = \lambda\} \end{aligned} \tag{43}$$

and check the sign of the integrals $\int qd(\gamma\nu)d(\gamma\sigma)$ over each domain.

The boundaries: Take first $k = a(\hat{x},t)$ in (40) and (41) to get

$$\begin{aligned} \langle \gamma\nu, q(\lambda,a)\rangle \cdot n \ge 0, \\ \langle \gamma\sigma, q(\mu,a)\rangle \cdot n \ge 0, \end{aligned}$$

which together with $q(\mu,\mu) = 0$ imply

$$\begin{aligned} &\int_{\mu=a} qd(\gamma\nu)d(\gamma\sigma) \ge 0, \\ &\int_{\lambda=a} qd(\gamma\nu)d(\gamma\sigma) \ge 0, \\ &\int_{\mu=\lambda} qd(\gamma\nu)d(\gamma\sigma) = 0. \end{aligned} \tag{44}$$

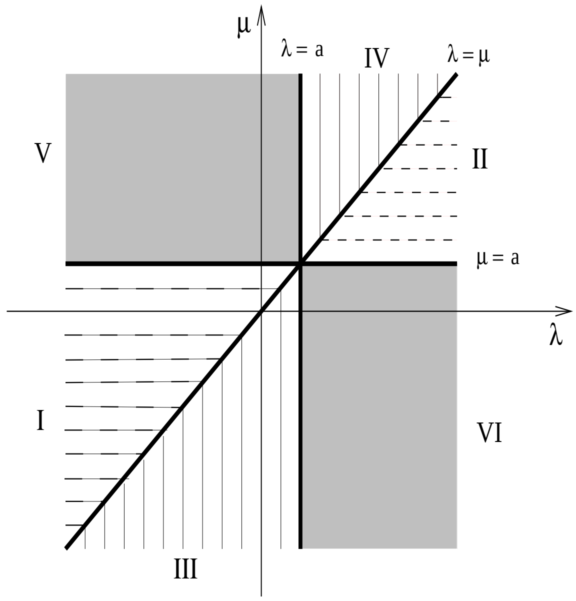

FIG. 1. *Spliting of the domain*

The domains I&II: Take $k = \mu$ in (40) to get

$$\int_{I\!R} (\mathrm{sgn}(\lambda - \mu) - \mathrm{sgn}(a - \mu))\,(f(\lambda) - f(\mu)) \cdot n \; d\gamma\nu(\lambda) \geq 0.$$

Use the representation

$$\mathrm{sgn}(\lambda - \mu) - \mathrm{sgn}(a - \mu) = \begin{cases} 0 & if \quad \lambda > \mu \;\; and \;\; \mu < a \\ 2 \;\; \mathrm{sgn}(\lambda - \mu) & if \quad \lambda < \mu \;\; and \;\; \mu < a, \\ 0 & if \quad \lambda < \mu \;\; and \;\; \mu > a \\ 2 \;\; \mathrm{sgn}(\mu - \lambda) & if \quad \lambda > \mu \;\; and \;\; \mu > a, \end{cases}$$

to obtain

$$\int_{\lambda<\mu<a} q(\mu, \lambda) \cdot n \; d\gamma\nu(\lambda) \geq 0, \quad \text{and} \quad \int_{\lambda>\mu>a} q(\mu, \lambda) \cdot n \; d\gamma\nu(\lambda) \geq 0, \tag{45}$$

which after integration with respect to $d\gamma\sigma(\mu)$ on the two sets $(-\infty, a)$ and (a, ∞), respectivly, imply

$$\int_{\mu<a} \int_{\lambda<\mu} q(\mu, \lambda) \cdot n \; d\gamma\nu(\lambda) d\gamma\sigma(\mu) \geq 0, \tag{46}$$
$$\int_{\mu>a} \int_{\lambda>\mu} q(\mu, \lambda) \cdot n \; d\gamma\nu(\lambda) d\gamma\sigma(\mu) \geq 0.$$

The domains III&IV: Let us similarly take $k = \lambda$ in (41) and integrate it with respect to $d\gamma\nu_{(\hat{x},t)}(\lambda)$ over the sets $(-\infty, a)$ and (a, ∞), respectivly, to obtain

$$\int_{\lambda<a}\int_{\mu<\lambda} q(\mu,\lambda)\cdot n\ d\gamma\sigma d\gamma\nu \geq 0, \quad \text{and} \quad \int_{\lambda>a}\int_{\mu>\lambda} q(\mu,\lambda)\cdot n\ d\gamma\sigma d\gamma\nu \geq 0. \tag{47}$$

The domains V&VI: The limit $\mu \to a+$ in (45) gives

$$\int_{\lambda>a} q(\lambda,a)\cdot n\ d\gamma\nu \geq 0,$$

and similarly

$$\int_{\mu<a} q(\mu,a)\cdot n\ d\gamma\sigma \geq 0,$$

which by the splitting $f(\lambda) - f(\mu) = f(\lambda) - f(a) + f(a) - f(\mu)$ imply

$$\begin{aligned} &\int_{\mu<a}\int_{\lambda>a} q(\lambda,\mu)\cdot n\ d\gamma\nu d\gamma\sigma \\ &= \gamma\sigma(\mu<a)\int_{\lambda>a} q(\lambda,a)\cdot n\ d\gamma\nu + \gamma\nu(\lambda>a)\int_{\mu<a} q(a,\mu)\cdot n\ d\gamma\sigma \geq 0, \end{aligned} \tag{48}$$

and analogously

$$\begin{aligned} &\int_{\mu>a}\int_{\lambda<a} q(\lambda,\mu)\cdot n\ d\gamma\nu d\gamma\sigma \\ &= \gamma\sigma(\mu>a)\int_{\lambda<a} q(\lambda,a)\cdot n\ d\gamma\nu + \gamma\nu(\lambda<a)\int_{\mu>a} q(a,\mu)\cdot n\ d\gamma\sigma \geq 0. \end{aligned} \tag{49}$$

The combinination of (42-44), (46-49) proves the claim.

Proof of Claim 4 . Suppose the contrary, that $\lambda_1 \neq \lambda_2$ and $\lambda_1 \in supp\ \nu_y, \lambda_2 \in supp\ \sigma_y$. Then there are bounded continuous non negative functions Ψ_i on $I\!R$ with $\lambda_i \in supp\ \Psi_i,\ i = 1, 2,\ supp\ \Psi_1 \cap supp\ \Psi_2 = \emptyset$ and $\langle\nu_y, \Psi_1\rangle\ > 0$ and $\langle\sigma_y, \Psi_2\rangle\ > 0$. Thus by Fubinis theorem and (21)

$$\begin{aligned} 0 &< \int_{I\!R\times I\!R} \Psi_1(\lambda)\Psi_2(\mu) d\nu_y(\lambda) d\sigma_y(\mu) \\ &\leq \Big\|\frac{\Psi_1(\lambda)\Psi_2(\mu)}{|\lambda-\mu|}\Big\|_{L^\infty} \int_{I\!R\times I\!R} |\lambda-\mu| d\nu_y(\lambda) d\sigma_y(\mu) = 0, \end{aligned}$$

which is a contradiction. Therefore $\nu_y = \sigma_y = \delta_{u(y)}$ for a.e. y.

4. Proof of the Existence Theorem 2.5. Approximate in $\mathcal{C}(I\!R)$ the flux f, in (6), by $f^\delta \in \mathcal{C}^1(I\!R)$ and use Otto's existence result [16] for the problems with smooth fluxes f^δ (or alternatively, approximate also the initial and the boundary data uniformly by functions with bounded varitation and use the existence in [2]). This approximation shows that the corresponding solutions $u^\delta \in L^\infty(\Omega \times I\!R_+)$ satisfy the uniform bound

$$\|u^\delta\|_{L^\infty} \leq K.$$

Therefore there is a Young measure ν associated to a sub sequence, $\{u^\delta\}$. We shall verify that ν is a measure solutions with L^∞ initial and boundary data, so that by the Uniqueness Theorem 2.4 the Young measure is in fact an L^∞ solution.

Solution in the interior. Let

$$q^\delta(\lambda,k) \equiv \operatorname{sgn}(\lambda-k)(f^\delta(\lambda)-f^\delta(k)).$$

In the interior domain the solution u^δ satisfies

$$\partial_t|u^\delta-k| + \operatorname{div}_x q^\delta(u^\delta,k) \le 0, \text{ in } \mathcal{D}'(\Omega\times I\!R_+) \tag{50}$$

and we obtain directly the distribution limit

$$\begin{aligned}0 &\ge \text{d-lim}_{\delta\to 0}\left(\partial_t|u^\delta-k| + \operatorname{div}_x q^\delta(u^\delta,k)\right)\\ &= \partial_t\langle\nu,|\lambda-k|\rangle + \operatorname{div}_x\langle\nu,q(\lambda,k)\rangle\\ &\quad + \underbrace{\text{d-lim}_{\delta\to 0}\operatorname{div}_x(q^\delta\left(u^\delta,k\right) - q(u^\delta,k))}_{=\,0}\end{aligned}$$

so that (8) holds, i.e. ν is a measure solution in the interior domain. It remains to verify that ν also satisfies the boundary conditions (9) and the initial condition (7).

The boundary condition. Define for any $\phi\in\mathcal{C}^1(\partial\Omega\times I\!R_+)$

$$\begin{aligned}A_\delta(x^\perp) &\equiv \int_{\partial\Omega\times I\!R_+} q^\delta\left(u^\delta\left(x(\hat{x},x^\perp),t\right),k\right)\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt,\\ C_\delta(x^\perp) &\equiv \int_{\partial\Omega\times I\!R_+} f^\delta\left(u^\delta\left(x(\hat{x},x^\perp),t\right)\right)\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt.\end{aligned}$$

Then by equation (50) we have

$$A'_\delta \ge B_\delta \tag{51}$$

in the distribution sense on $(0,\kappa)$, for $0<\kappa$ independent of δ, where $B_\delta\in L^\infty(0,\kappa)$ are uniformly bounded, $\|B_\delta\|_{L^\infty}=\mathcal{O}(1)$. Similarly, the equation

$$\partial_t u^\delta + \operatorname{div}_x f^\delta(u^\delta) = 0, \text{ in } \mathcal{D}'(\Omega\times I\!R_+)$$

shows that $C'_\delta = D_\delta$ in $\mathcal{D}'(0,\kappa)$, where $D_\delta,\in L^\infty(0,\kappa)$ are uniformly bounded, $\|D_\delta\|_{L^\infty}=\mathcal{O}(1)$. Therefore A_δ and C_δ have bounded variation and the limits $\lim_{x^\perp\to 0+}A_\delta(x^\perp)$ and $\lim_{x^\perp\to 0+}C_\delta(x^\perp)$ exist. Consequently, using the notation $w(0+)\equiv\lim_{x^\perp\to 0+}w(x^\perp)$, we have

$$\begin{aligned}0 &= \lim_{\kappa\to 0+}\lim_{\delta\to 0+}\frac{1}{\kappa}\int_0^\kappa\int_0^{x^\perp} D_\delta(z)dzdx^\perp\\ &= \lim_{\kappa\to 0+}\lim_{\delta\to 0+}\frac{1}{\kappa}\int_0^\kappa\left(C_\delta(x^\perp)-C_\delta(0+)\right)dx^\perp\\ &= \lim_{\kappa\to 0+}\frac{1}{\kappa}\int_0^\kappa\int_{\partial\Omega\times I\!R_+}\langle\nu,f(\lambda)\rangle\cdot n\phi(\hat{x},t)d\hat{x}dtdx^\perp\\ &\quad - \lim_{\delta\to 0+}\int_{\partial\Omega\times I\!R_+} f^\delta(u^\delta\left(x(\hat{x},0+),t\right))\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt\\ &= \int_{\partial\Omega\times I\!R_+}\langle\gamma\nu,f(\lambda)\rangle\cdot n\phi(\hat{x},t)d\hat{x}dt\\ &\quad - \lim_{\delta\to 0+}\int_{\partial\Omega\times I\!R_+} f^\delta(u^\delta\left(x(\hat{x},0+),t\right))\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt.\end{aligned} \tag{52}$$

Since u^δ is a solution satisfying the boundary inequalities (6), we also know that

$$A_\delta(0+) \geq \int_{\partial\Omega\times\mathbb{R}_+} \operatorname{sgn}(a-k)\left(f^\delta\left(u^\delta\left(x(\hat{x},0+),t\right)\right) - f^\delta(k)\right)\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt$$

so that by the last equality in (52)

$$\liminf_{\delta\to 0+} A_\delta(0+) \geq \int_{\partial\Omega\times\mathbb{R}_+} \operatorname{sgn}(a-k)\langle\gamma\nu, f(\lambda)-f(k)\rangle\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt.$$

This and (51) show that

$$\begin{aligned}
0 &= \lim_{\kappa\to 0+}\lim_{\delta\to 0+}\frac{1}{\kappa}\int_0^\kappa\int_0^{x^\perp} B_\delta(z)dzdx^\perp && (53)\\
&\leq \lim_{\kappa\to 0+}\liminf_{\delta\to 0+}\frac{1}{\kappa}\int_0^\kappa (A_\delta(x^\perp)-A_\delta(0+))dx^\perp \\
&\leq \lim_{\kappa\to 0+}\frac{1}{\kappa}\int_0^\kappa\int_{\partial\Omega\times\mathbb{R}_+}\langle\nu, q(\lambda,k)\rangle\cdot n\phi(\hat{x},t)d\hat{x}dtdx^\perp \\
&\qquad -\int_{\partial\Omega\times\mathbb{R}_+}\operatorname{sgn}(a-k)\langle\gamma\nu, f(\lambda)-f(k)\rangle\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt \\
&= \int_{\partial\Omega\times\mathbb{R}_+}\langle\gamma\nu, q(\lambda,k)\rangle\cdot n\phi(\hat{x},t)d\hat{x}dt \\
&\qquad -\int_{\partial\Omega\times\mathbb{R}_+}\operatorname{sgn}(a-k)\langle\gamma\nu, f(\lambda)-f(k)\rangle\cdot n(\hat{x})\phi(\hat{x},t)d\hat{x}dt && (54)
\end{aligned}$$

which proves that the boundary condition (9) is satisfied, cf. Remark (2.6).

The initial condition. The initial condition (7) follows from a standard combination, cf. [11] or Claim 5 below, of the following weak convergence and convexity

$$\begin{aligned}
&(i) \quad \lim_{t\to 0+}\int_\Omega\langle\nu_{x,t}, \lambda-u_0(x)\rangle\phi(x)dx = 0, \quad \forall\phi\in\mathcal{C}^1(\Omega), \\
&(ii) \quad \int_\Omega\langle\nu_{x,t},\lambda^2\rangle dx \leq \int_\Omega u_0^2(x)dx.
\end{aligned} \qquad (55)$$

To prove the estimates (i) and (ii), use first that the equation for u^δ implies the uniform bounds

$$\begin{aligned}
&\int_\Omega\left(u^\delta(x,t)-u_0(x)\right)\phi(x)dx = \mathcal{O}(t), \\
&\int_\Omega(u^\delta(x,t))^2dx \leq \int_\Omega(u_0(x))^2dx,
\end{aligned}$$

and take their limits as $\delta\to 0+$ to obtain (55).

Claim 5. The estimates (i) and (ii) in (55) imply the initial condition.

Proof. Let $\phi_n\in\mathcal{C}^1(\Omega)$ approximate u_0 in $L^1(\Omega)$, so that $\|u_0-\phi_n\|_{L^1}\to 0$, then the weak limit (i) in (55) shows that

$$\lim_{t\to 0+}\int_\Omega\langle\nu_{x,t},\lambda\rangle u_0(x)dx = \int_\Omega(u_0(x))^2dx,$$

and consequently Jensen's inequality and the convexity (ii) in (55) imply

$$\begin{aligned}
&\frac{1}{\int_\Omega dx}(\lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, |\lambda - u_0(x)|\rangle dx)^2 \\
&\le \lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, (\lambda - u_0(x))^2\rangle dx \\
&= \lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, \lambda^2\rangle dx + \int_\Omega u_0^2 dx - 2\lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, \lambda\rangle u_0 dx \le 0,
\end{aligned}$$

so that

$$\lim_{t\to 0+}\int_\Omega \langle \nu_{x,t}, |\lambda - u_0|\rangle dx = 0.$$

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