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UNIQUENESS OF SOLUTIONS FOR AN ELLIPTIC EQUATION MODELING MEMS*

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Abstract. We show among other things, that for small voltage, the stable solution of the basic nonlinear eigenvalue problem modelling a simple electrostatic MEMS is actually the unique solution, provided the domain is star-shaped and the dimension is larger or equal than 3. In two dimensions, we need the domain to be either strictly convex or symmetric. The case of a power permittivity profile is also considered. Our results, which use an approach developed by Schaaf [13], extend and simplify recent results by Guo and Wei [7], [8].

Key words. MEMS, Stable solutions, Quenching branch.

AMS subject classifications. 35J60, 35B32, 35D10, 35J20

1. Introduction. We study the effect of the parameter λ, the dimension N, the profile f and the geometry of the domain $\Omega \subset \mathbb{R}^N$, on the question of uniqueness of the solutions to the following elliptic boundary value problem with a singular nonlinearity:

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (S)_{\lambda,f}$$

This equation has been proposed as a model for a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid ground plate located at height $z = 1$. See [10, 11]. A voltage – directly proportional to the parameter λ – is applied, and the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^*, the pull-in voltage.

In [9] a fine ODE analysis of the radially symmetric case with a constant profile $f \equiv 1$ on a ball B, yields the following bifurcation diagram that describes the L^∞-norm of the solutions u – which in this case necessarily coincides with $u(0)$ – in terms of the corresponding voltage λ.

The question whether the diagram above describes realistically the set of all solutions in more general domains and for non-constant profiles, and whether rigorous mathematical proofs can be given for such a description, has been the subject of many recent investigations. See [3, 4, 5, 7, 8].

We summarize in the following two theorems some of the established results concerning Figure 1. First, for every solution u of $(S)_{\lambda,f}$, we consider the linearized operator

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3}$$

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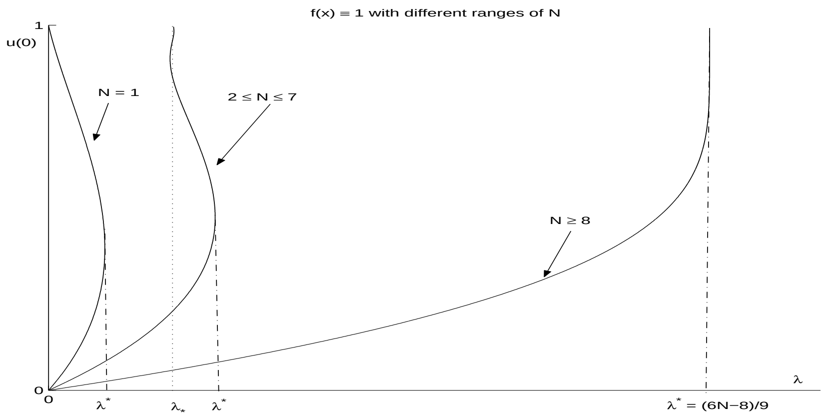

FIG. 1. Plots of $u(0)$ versus λ for profile $f(x) \equiv 1$ defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with different ranges of N. In the case $N \geq 8$, we have $\lambda^* = 2(3N-4)/9$.

and its eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, \dots\}$ (with the convention that eigenvalues are repeated according to their multiplicities). The Morse index $m(u, \lambda)$ of a solution u is the largest k for which $\mu_{k,\lambda}(u)$ is negative. A solution u of $(S)_{\lambda,f}$ is said to be *stable* (resp., *semi-stable*) if $\mu_{1,\lambda}(u) > 0$ (resp., $\mu_{1,\lambda}(u) \geq 0$).

A description of the first stable branch and of the higher unstable ones is given in the following.

THEOREM A [3, 4, 5]. *Suppose f is a smooth nonnegative function in Ω. Then, there exists a finite $\lambda^* > 0$ such that*

1. *If $0 \leq \lambda < \lambda^*$, there exists a (unique) minimal solution u_λ of $(S)_{\lambda,f}$ such that $\mu_{1,\lambda}(u_\lambda) > 0$. It is also unique in the class of all semi-stable solutions.*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_{\lambda,f}$.*
3. *If $1 \leq N \leq 7$, then $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a solution of $(S)_{\lambda^*,f}$ such that $\mu_{1,\lambda^*}(u^*) = 0$, and u^* – referred to as the extremal solution of problem $(S)_{\lambda,f}$ – is the unique solution.*
4. *If $1 \leq N \leq 7$, there exists λ_2^* with $0 < \lambda_2^* < \lambda^*$ such that for any $\lambda \in (\lambda_2^*, \lambda^*)$, problem $(S)_{\lambda,f}$ has a second solution U_λ with $\mu_{1,\lambda}(U_\lambda) < 0$ and $\mu_{2,\lambda}(U_\lambda) > 0$. Moreover, at $\lambda = \lambda_2^*$ there exists a second solution $U^* := \lim_{\lambda \downarrow \lambda_2^*} U_\lambda$ with*
$$\mu_{1,\lambda_2^*}(U^*) < 0 \quad \text{and} \quad \mu_{2,\lambda_2^*}(U^*) = 0.$$
5. *Given a more specific potential f in the form*
$$f(x) = \left(\prod_{i=1}^{k} |x - p_i|^{\alpha_i} \right) h(x)\,, \quad \inf_{\Omega} h > 0, \tag{1}$$
with points $p_i \in \Omega$, $\alpha_i \geq 0$, and given u_n a solution of $(S)_{\lambda_n,f}$, we have the equivalence
$$\|u_n\|_\infty \to 1 \quad \Longleftrightarrow \quad m(u_n, \lambda_n) \to +\infty$$
as $n \to +\infty$.

It was also shown in [4] that the permittivity profile f can dramatically change the bifurcation diagram, and totally alter the critical dimensions for compactness. Indeed, the following theorem summarizes the result related to the effect of power law profiles.

THEOREM B [4]. *Assume Ω is the unit ball B and f in the form*

$$f(x) = |x|^\alpha h(|x|)\,, \quad \inf_B h > 0.$$

Then we have

1. *If $N \geq 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, the extremal solution u^* is again a classical solution of $(S)_{\lambda^*,f}$ such that $\mu_{1,\lambda^*}(u^*) = 0$.*
2. *If $N \geq 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, the conclusion of Theorem A-(4) still holds true.*
3. *On the other hand, if either $2 \leq N \leq 7$ or $N \geq 8$, $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, for $f(x) = |x|^\alpha$ necessarily we have that*

$$u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}\,, \qquad \lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}.$$

The bifurcation diagram suggests the following conjectures:

1. For $2 \leq N \leq 7$ there exists a curve $(\lambda(t), u(t))_{t\geq 0}$ in the solution set

$$\mathcal{V} = \Big\{(\lambda, u) \in (0, +\infty) \times C^1(\bar{\Omega}) :\ u \text{ is a solution of } (S)_{\lambda,f}\Big\}, \qquad (2)$$

   starting from $(0,0)$ at $t = 0$ and going to “infinity": $\|u(t)\|_\infty \to 1$ as $t \to +\infty$, with infinitely many bifurcation or turning points in $\mathcal{V}$.
2. In dimension $N \geq 2$ and for any profile f, there exists a unique solution for small voltages λ.
3. For $2 \leq N \leq 7$ there exist exactly two solutions for λ in a small left neighborhhod of λ^*.

Conjectures 1 and 2 have been established for power law profiles in the radially symmetric case [7], and for the case where $f \equiv 1$ and Ω is a suitably symmetric domain in $\mathbb{R}^2$ [8]. Indeed, in these cases Guo and Wei first show that

$$\lambda_* = \inf\{\lambda > 0 :\ (S)_{\lambda,f} \text{ has a non-minimal solution}\} > 0,$$

and then apply the fine bifurcation theory developed by Buffoni, Dancer and Toland [1] to verify the validity of Conjecture 1 in that case. The fact that $\lambda_* > 0$ then allows them to carry out some limiting argument and to prove that the Morse index of $u(t)$ blows up as $t \to +\infty$, which is crucial for showing that infinitely many bifurcation or turning points occur along the curve. Thanks to Theorem A-(5), we shall be able in Section 2 to show the validity of Conjecture 1 in general domains Ω, by circumventing the need to prove that $\lambda_* > 0$. On the other hand, we shall prove in Section 3 that indeed $\lambda_* > 0$ for a large class of domains, and therefore we have uniqueness for small voltage. Our proofs simplify considerably those of Guo and Wei [7, 8], and extend them to general star-shaped domains Ω and power law profiles $f(x) = |x|^\alpha$, $\alpha \geq 0$.

Conjecture 3 has been shown in [3] in the class of solutions u with $m(u, \lambda) \leq k$, for every given $k \in \mathbb{N}$, and is still open in general.

2. A quenching branch of solutions. The first global result on the set of solutions in general domains was proved by the first author in [3]. By using a degree argument (repeated below), he showed the following result.

THEOREM 2.1. *Assume $2 \leq N \leq 7$ and f be as in (1). There exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and associated solution u_n of $(S)_{\lambda_n,f}$ so that*

$$m(u_n,\lambda_n) \to +\infty \qquad \text{as } n \to +\infty.$$

We now introduce some notation from Section 2.1 of [1]. Set

$$X = Y = \{u \in C^1(\bar{\Omega}) :\ u = 0 \text{ on } \partial\Omega\}, \quad U = (0,+\infty) \times \{u \in X : \|u\|_\infty < 1\},$$

and define the real analytic function $F : \mathbb{R} \times U \to Y$ as $F(\lambda,u) = u - \lambda K(u)$, where $K(u) = -\Delta^{-1}\left(f(x)(1-u)^{-2}\right)$ is a compact operator on every closed subset in $\{u \in X : \|u\|_\infty < 1\}$ and Δ^{-1} is the Laplacian resolvent with homogeneous Dirichlet boundary condition. The solution set $\mathcal{V}$ given in (2) rewrites as

$$\mathcal{V} = \{(\lambda,u) \in U :\ F(\lambda,u) = 0\},$$

and the projection of $\mathcal{V}$ onto X is defined as

$$\Pi_X \mathcal{V} = \{u \in X :\ \exists\, \lambda \text{ so that } (\lambda,u) \in \mathcal{V}\}.$$

Proof. In view of Theorem A-(5), we have the equivalence

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_{\Omega} u = 1 \qquad \Longleftrightarrow \qquad \sup_{(\lambda,u)\in\mathcal{V}} m(u,\lambda) = +\infty.$$

Arguing by contradiction, we can assume that

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_{\Omega} u \leq 1 - 2\delta, \qquad \sup_{(\lambda,u)\in\mathcal{V}} m(u,\lambda) < +\infty \tag{3}$$

for some $\delta \in (0,\frac{1}{2})$. By Theorem 1.3 in [3] one can find $\lambda_1, \lambda_2 \in (0,\lambda^*)$, $\lambda_1 < \lambda_2$, so that $(S)_{\lambda,f}$ possesses

- for λ_1, only the (non degenerate) minimal solution u_{λ_1} which satisfies $m(u_{\lambda_1},\lambda_1) = 0$;
- for λ_2, only the two (non degenerate) solutions u_{λ_2}, U_{λ_2} satisfying $m(u_{\lambda_2},\lambda_2) = 0$ and $m(U_{\lambda_2},\lambda_2) = 1$, respectively.

Consider a δ-neighborhood $\mathcal{V}_\delta$ of $\Pi_X\mathcal{V}$:

$$\mathcal{V}_\delta := \{u \in X :\ \mathrm{dist}_X(u,\Pi_X\mathcal{V}) \leq \delta\}.$$

Note that (3) gives that $\mathcal{V}$ is contained in a closed subset of $\{u \in X : \|u\|_\infty < 1\}$:

$$\mathcal{V}_\delta \subset \{u \in X :\ \|u\|_\infty \leq 1 - \delta\}.$$

We can now define the Leray-Schauder degree d_λ of $F(\lambda,\cdot)$ on $\mathcal{V}_\delta$ with respect to zero, since by definition of $\Pi_X\mathcal{V}$ (the set of all solutions) $\partial\mathcal{V}_\delta$ does not contain any solution of $(S)_{\lambda,f}$ for any value of λ. Since d_λ is well defined for any $\lambda \in [0,\lambda^*]$, by homotopy $d_{\lambda_1} = d_{\lambda_2}$. To get a contradiction, let us now compute d_{λ_1} and d_{λ_2}. Since the only zero of $F(\lambda_1, \cdot)$ in $\mathcal{V}_\delta$ is u_{λ_1} with Morse index zero, we have $d_{\lambda_1} = 1$. Since $F(\lambda_2, \cdot)$ has in $\mathcal{V}_\delta$ exactly two zeroes u_{λ_2} and U_{λ_2} with Morse index zero and one, respectively, we have $d_{\lambda_2} = 1 - 1 = 0$. This contradicts $d_{\lambda_1} = d_{\lambda_2}$, and the proof is complete. □

We can now combine Theorem A-(5) with the fine bifurcation theory in [1] to establish a more precise multiplicity result. See also [2].

Observe that $\mathcal{A}_0 := \{(\lambda, u_\lambda) : \lambda \in (0, \lambda^*)\}$ is a maximal arc-connected subset of

$$S := \{(\lambda, u) \in U : F(\lambda, u) = 0 \text{ and } \partial_u F(\lambda, u) : X \to Y \text{ is invertible}\}$$

with $\mathcal{A}_0 \subset S$. Assume that the extremal solution u^* is a classical solution so to have $u^* \in (\bar{S} \cap U) \setminus S$. Assumption (C1) of Section 2.1 in [1] does hold in our case. As far as condition (C2):

$$\{(\lambda, u) \in U : F(\lambda, u) = 0\} \text{ is open in } \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0\},$$

let us stress that it is a weaker statement than requiring U to be an open subset in $\mathbb{R} \times X$. In our case, the map $F(\lambda, u)$ is defined only in U (and not in the whole X), and then condition (C2) does not make sense. However, we can replace it with the new condition (C2):

$$U \text{ is an open set in } \mathbb{R} \times X,$$

which does hold in our context. Since (C2) is used only in Theorem 2.3-(iii) in [1] to show that S is open in $\bar{S}$, our new condtion (C2) does not cause any trouble in the arguments of [1].
Since $\partial_u F(\lambda, u)$ is a Fredholm operator of index 0, by a Lyapunov-Schmidt reduction we have that assumptions (C3)-(C5) do hold in our case (let us stress that these conditions are local and U is an open set in $\mathbb{R} \times X$).
Setting $\bar{\lambda} = 0$ and defining the map $\nu : U \to [0, +\infty)$ as $\nu(\lambda, u) = \frac{1}{1 - \|u\|_\infty}$, conditions (C6)-(C8) do hold in view of the property $\lambda \in [0, \lambda^*]$. Theorem 2.4 in [1] then applies and gives the following.

THEOREM 2.2. *Assume u^* a classical solution of $(S)_{\lambda^*, f}$. Then there exists an analytic curve $(\hat{\lambda}(t), \hat{u}(t))_{t \geq 0}$ in $\mathcal{V}$ starting from $(0, 0)$ and so that $\|\hat{u}(t)\|_\infty \to 1$ as $t \to +\infty$. Moreover, $\hat{u}(t)$ is a non-degenerate solution of $(S)_{\hat{\lambda}(t), f}$ except at isolated points.*

By the Implicit Function Theorem, the curve $(\hat{\lambda}(t), \hat{u}(t))$ can only have isolated intersections. If we now use the usual trick of finding a minimal continuum in $\{(\hat{\lambda}(t), \hat{u}(t)) : t \geq 0\}$ joining $(0, 0)$ to "infinity", we obtain a continuous curve $(\lambda(t), u(t))$ in $\mathcal{V}$ with no self-intersections which is only piecewise analytic. Clearly, $\partial_u F(\lambda, u) : X \to Y$ is still invertible along the curve except at isolated points.

Let now $2 \leq N \leq 7$ and f be as in (1). By the equivalence in Theorem A-(5) we get that $m(\lambda(t), u(t)) \to +\infty$ as $t \to +\infty$, and then $\mu_{k, \lambda(t)}(u(t)) < 0$ for t large, for every $k \geq 1$. Since $\mu_{k, \lambda(0)}(u(0)) = \mu_{k, 0}(0) > 0$ and $u(t)$ is a non-degenerate solution of $(S)_{\lambda(t), f}$ except at isolated points, we find $t_k > 0$ so that $\mu_{k, \lambda(t)}(u(t))$ changes from positive to negative sign across t_k. Since $\mu_{k+1, \lambda(t)}(u(t)) \geq \mu_{k, \lambda(t)}(u(t))$, we can choose t_k to be non-increasing in k and to have $t_k \to +\infty$ as $k \to +\infty$.
To study secondary bifurcations, we will use the gradient structure in the problem.

Setting $(\lambda_k, u_k) := (\lambda(t_k), u(t_k))$, we have that $(\lambda_k, u_k) \notin S$. Choose $\delta > 0$ small so that $\|u_k\|_\infty < 1 - \delta$, and replace the nonlinearity $(1-u)^{-2}$ with a regularized one:

$$f_\delta(u) = \begin{cases} (1-u)^{-2} & \text{if } u \leq 1-\delta\,, \\ \delta^{-2} & \text{if } u \geq 1-\delta, \end{cases}$$

and the map $F(\lambda, u)$ with the corresponding one $F_\delta(\lambda, u)$. We replace X and Y with $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. The map $F_\delta(\lambda, u)$ can be considered as a map from $\mathbb{R} \times X \to Y$ with a gradient structure:

$$\partial_u \mathcal{J}_\delta(\lambda, u)[\varphi] = \langle F_\delta(\lambda, u), \varphi\rangle_{L^2(\Omega)}$$

for every $\lambda \in \mathbb{R}$ and $u, \varphi \in X$, where $\mathcal{J}_\delta : \mathbb{R} \times X \to \mathbb{R}$ is the functional given by

$$\mathcal{J}_\delta(\lambda, u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega f(x) G_\delta(u)\, dx\,, \qquad G_\delta(u) = \int_0^u f_\delta(s) ds.$$

Assumptions (G1)-(G2) in Section 2.2 of [1] do hold. We have that $(\lambda(t), u(t)) \in S$ for t close to t_k and $m(\lambda(t), u(t))$ changes across t_k. If $\lambda(t)$ is injective, by Proposition 2.7 in [1] we have that $(\lambda(t_k), u(t_k))$ is a bifurcation point. Then we get the validity of Conjecture 1 as claimed below.

THEOREM 2.3. *Assume $2 \leq N \leq 7$ and f be as in (1). Then there exists a continuous, piecewise analytic curve $(\lambda(t), u(t))_{t\geq 0}$ in $\mathcal{V}$, starting from $(0,0)$ and so that $\|\hat{u}(t)\|_\infty \to 1$ as $t \to +\infty$, which has either infinitely many turning points, i.e. points where $(\lambda(t), u(t))$ changes direction (the branch locally "bends back"), or infinitely many bifurcation points.*

REMARK 2.1. In [7] the above analysis is performed in the radial setting to obtain a curve $(\lambda(t), u(t))_{t\geq 0}$, as given by Theorem 2.3, composed by radial solutions and so that $m_r(\lambda(t), u(t)) \to +\infty$ as $t \to +\infty$, $m_r(\lambda, u)$ being the radial Morse index of a solution (λ, u). In this way, it can be shown that bifurcation points can't occur and then $(\lambda(t), u(t))_{t\geq 0}$ exhibits infinitely many turning points. Moreover, they can also deal with the case where $N \geq 8$ and $\alpha > \alpha_N$.

3. Uniqueness of solutions for small voltage in star-shaped domains. We address the issue of uniqueness of solutions of the singular elliptic problem

$$\begin{cases} -\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

for $\lambda > 0$ small, where $\alpha \geq 0$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$. We shall make crucial use of the following extension of Pohozaev's identity due to Pucci and Serrin [12].

PROPOSITION 3.1. *Let v be a solution of the boundary value problem*

$$\begin{cases} -\Delta v = f(x, v) & \text{in } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then for any $a \in \mathbb{R}$ and any $h \in C^2(\Omega;\mathbb{R}^N) \cap C^1(\bar{\Omega};\mathbb{R}^N)$, the following identity holds

$$\begin{aligned}&\int_\Omega [\mathrm{div}(h)F(x,v) - avf(x,v) + \langle \nabla_x F(x,v), h\rangle]\,dx\\ &= \int_\Omega \Big[(\frac{1}{2}\mathrm{div}(h) - a)|\nabla v|^2 - \langle Dh\nabla v, \nabla v\rangle\Big]\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 \langle h, \nu\rangle d\sigma,\end{aligned} \tag{5}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

An application of the method in [13] leads to the following result.

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a star-shaped domain with respect to 0. If $N \geq 3$, then for λ small, the stable solution u_λ is the unique solution of equation (4).*

Proof. Since u_λ is the minimal solution of (4) for $\lambda \in (0, \lambda^*)$, setting $v = u - u_\lambda$ equation (4) rewrites equivalently as

$$\begin{cases} -\Delta v = \lambda |x|^\alpha g_\lambda(x,v) & \text{in } \Omega\\ 0 \leq v < 1 - u_\lambda & \text{in } \Omega\\ v = 0 & \text{on } \partial\Omega,\end{cases} \tag{6}$$

where

$$g_\lambda(x,s) = \frac{1}{(1-u_\lambda(x)-s)^2} - \frac{1}{(1-u_\lambda(x))^2}\,. \tag{7}$$

It then suffices to prove that the solutions of (6) must be trivial for λ small enough. First compute $G_\lambda(x,s)$:

$$G_\lambda(x,s) = \int_0^s g_\lambda(x,t)\,dt = \frac{1}{1-u_\lambda(x)-s} - \frac{1}{1-u_\lambda(x)} - \frac{s}{(1-u_\lambda(x))^2}.$$

Since the validity of the relation

$$\nabla_x\Big(|x|^\alpha G_\lambda(x,s)\Big) = \alpha|x|^{\alpha-2}xG_\lambda(x,s) + |x|^\alpha \nabla_x G_\lambda(x,s),$$

for $h(x) = \frac{x}{N}$ and $f(x,v) = |x|^\alpha g_\lambda(x,v)$ we apply the Pohozaev identity (5) to a solution v of (6) to get

$$\begin{aligned}&\lambda\int_\Omega |x|^\alpha\big[(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - av(x)g_\lambda(x,v(x)) + \langle\nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle\big]\,dx\\ &= \int_\Omega \big[(\frac{1}{2}-a)|\nabla v|^2 - \langle D(\frac{x}{N})\nabla v, \nabla v\rangle\big]dx + \frac{1}{2N}\int_{\partial\Omega} |\nabla v|^2\langle x,\nu\rangle\,d\sigma\\ &\geq (\frac{1}{2} - a - \frac{1}{N})\int_\Omega |\nabla v|^2 dx.\end{aligned} \tag{8}$$

Since easy calculations show that

$$\frac{G_\lambda(x,s)}{g_\lambda(x,s)} = \frac{1-u_\lambda(x) - s - \frac{(1-u_\lambda(x)-s)^2(1-u_\lambda(x)+s)}{(1-u_\lambda(x))^2}}{1 - \frac{(1-u_\lambda(x)-s)^2}{(1-u_\lambda(x))^2}}$$

and

$$\frac{\nabla_x G_\lambda(x,s)}{g_\lambda(x,s)} = \frac{1-\frac{(1-u_\lambda(x)-s)^2(1-u_\lambda(x)+2s)}{(1-u_\lambda(x))^3}}{1-\frac{(1-u_\lambda(x)-s)^2}{(1-u_\lambda(x))^2}}\nabla u_\lambda(x),$$

we obtain

$$\Big|\frac{G_\lambda(x,s)}{g_\lambda(x,s)}\Big| \le C_0|1-u_\lambda(x)-s| \quad \text{and} \quad \Big|\frac{\nabla_x G_\lambda(x,s)}{g_\lambda(x,s)} - \nabla u_\lambda\Big| \le C_0|1-u_\lambda(x)-s|^2|\nabla u_\lambda| \tag{9}$$

for some $C_0>0$, provided λ is away from λ^*. Since $u_\lambda \to 0$ in $C^1(\bar\Omega)$ as $\lambda\to 0^+$, for $a>0$ from (9) we deduce that for any (x,s) satisfying $|1-u_\lambda(x)-s|\le\delta$

$$\begin{aligned}&(1+\frac{\alpha}{N})G_\lambda(x,s) - asg_\lambda(x,s) + \langle\nabla_x G_\lambda(x,s), \frac{x}{N}\rangle \qquad (10)\\ &\le g_\lambda(x,s)\Big[C_0(1+\frac{\alpha}{N})\delta - a(1-u_\lambda(x)-\delta) + \langle\nabla u_\lambda, \frac{x}{N}\rangle + \frac{C_0}{N}\delta^2|\nabla u_\lambda||x|\Big] \le 0,\end{aligned}$$

provided δ and λ are sufficiently small (depending on a). Since $N\ge 3$, we can pick $0<a<\frac12-\frac1N$, and then by (8), (10) get that

$$\begin{aligned}&\lambda\int_{\{0\le v\le 1-u_\lambda-\delta\}}|x|^\alpha\big[(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - av(x)g_\lambda(x,v(x)) + \langle\nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle\big]dx\\ &\ge (\frac12-a-\frac1N)\int_\Omega|\nabla v|^2\,dx \ge C_s(\frac12-a-\frac1N)\int_\Omega v^2\,dx \qquad (11)\end{aligned}$$

for δ and λ sufficiently small, where C_s is the best constant in the Sobolev embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.

On the other hand, since $G_\lambda(x,s)$, $sg_\lambda(x,s)$ and $\nabla_x G_\lambda(x,s)$ are quadratic with respect to s as $s\to 0$ (uniformly in λ away from λ^*), there exists a constant $C_\delta>0$ such that

$$(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - avg_\lambda(x,v(x)) + \langle\nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle \le C_\delta v^2(x) \tag{12}$$

for $x\in\{0\le v\le 1-u_\lambda-\delta\}$, uniformly for λ away from λ^*. Combining (11) and (12) we get that

$$C_s(\frac12-a-\frac1N)\int_{\{0\le v\le 1-u_\lambda-\delta\}}v^2dx \le \lambda C_\delta\int_{\{0\le v\le 1-u_\lambda-\delta\}}|x|^\alpha v^2dx.$$

Therefore, for λ sufficiently small we conclude that $v\equiv 0$ in $\{0\le v\le 1-u_\lambda-\delta\}$. This implies that $v\equiv 0$ in Ω for sufficiently small λ, and we are done. □

We now refine the above argument so as to cover other situations. To this aim, we consider the – potentially empty – set

$$H(\Omega) = \Big\{h\in\mathcal{C}^1(\bar\Omega,\mathbb{R}^N) : \mathrm{div}(h)\equiv 1 \text{ and } \langle h,\nu\rangle\ge 0 \text{ on } \partial\Omega\Big\},$$

and the corresponding parameter

$$M(\Omega) := \inf\Big\{\sup_{x\in\Omega}\bar\mu(h,x) :\ h\in H(\Omega)\Big\},$$

where

$$\bar{\mu}(h,x) = \frac{1}{2} \sup_{|\xi|=1} \langle (Dh(x) + Dh(x)^T)\xi, \xi \rangle.$$

The following is an extension of Theorem 3.1.

THEOREM 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $M(\Omega) < \frac{1}{2}$. Then, for λ small the minimal solution u_λ is the unique solution of problem (4), provided either $N \geq 3$ or $\alpha > 0$.*

Proof. As above, we shall prove that equation (6), with g_λ as in (7), has only trivial solutions for λ small. For a solution v of (6) the Pohozaev identity (5) with $h \in H(\Omega)$ yields

$$\begin{aligned} &\lambda \int_\Omega |x|^\alpha \big[G_\lambda(x, v(x))(1 + \alpha \langle \frac{x}{|x|^2}, h \rangle) - a v(x) g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h \rangle \big]\, dx \\ &= \int_\Omega \big[(\frac{1}{2} - a)|\nabla v|^2 - \frac{1}{2} \langle (Dh + Dh^T)\nabla v, \nabla v \rangle \big] dx + \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 \langle h, \nu \rangle\, d\sigma \qquad (13) \\ &\geq \int_\Omega (\frac{1}{2} - a - \bar{\mu}(h,x)) |\nabla v|^2\, dx. \end{aligned}$$

Fix $0 < a < \frac{1}{2} - M(\Omega)$ and choose $h \in H(\Omega)$ such that

$$\frac{1}{2} - a - \sup_{x \in \Omega} \bar{\mu}(h,x) > 0.$$

It follows from (9) that for any (x,s) satisfying $|1 - u_\lambda(x) - s| \leq \delta |x|$ there holds

$$\begin{aligned} &G_\lambda(x,s)(1 + \alpha \langle \frac{x}{|x|^2}, h \rangle) - a v g_\lambda(x,s) + \langle \nabla_x G_\lambda(x,s), h \rangle \qquad (14) \\ &\leq g_\lambda(x,s) \big[C_0 \delta |x| + \alpha C_0 \delta |h| - a(1 - u_\lambda - \delta|x|) + \langle \nabla u_\lambda, h \rangle + C_0 \delta^2 |x|^2 |\nabla u_\lambda||h| \big] \leq 0 \end{aligned}$$

provided λ and δ are sufficiently small. It then follows from (13) and (15) that

$$\begin{aligned} &\lambda \int_{\{0 \leq v \leq 1 - u_\lambda - \delta|x|\}} |x|^\alpha \Big[G_\lambda(x, v(x))(1 + \alpha \langle \frac{x}{|x|^2}, h \rangle) \\ &\qquad - a v(x) g_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), h \rangle \Big]\, dx \\ &\geq (\frac{1}{2} - a - \sup_{x \in \Omega} \bar{\mu}(h,x)) \int_\Omega |\nabla v|^2\, dx. \qquad (15) \end{aligned}$$

On the other hand, there exists a constant $C_\delta > 0$ such that

$$\begin{aligned} &G_\lambda(x, v(x))(1 + \alpha \langle \frac{x}{|x|^2}, h(x) \rangle) - a v(x) g_\lambda(x, v(x)) + < \nabla_x G_\lambda(x, v(x)), h(x) > \\ &= \frac{v^2(x)}{(1 - u_\lambda(x) - v(x))(1 - u_\lambda(x))^2} (1 + \alpha \langle \frac{x}{|x|^2}, h(x) \rangle) + \frac{a v^2(x)[v(x) - 2 + 2u_\lambda(x))]}{(1 - u_\lambda(x) - v(x))^2 (1 - u_\lambda(x))^2} \\ &+ \frac{v^2(x)(3 - 3u_\lambda(x) - 2v(x))}{(1 - u_\lambda(x) - v(x))^2 (1 - u_\lambda(x))^3} < \nabla u_\lambda(x), h(x) > \leq C_\delta \frac{v^2(x)}{|x|^2} \end{aligned}$$

for $x \in \{0 \le v \le 1 - u_\lambda - \delta|x|\}$, uniformly for λ away from λ^*.
If now $N \ge 3$, then Hardy's inequality combined with (15) implies

$$\frac{(N-2)^2}{4}\Big(\frac{1}{2} - a - \sup_{x\in\Omega}\bar\mu(h,x)\Big)\int_{\{0\le v\le 1-u_\lambda-\delta|x|\}}\frac{v^2}{|x|^2}dx \le \lambda C_\delta \int_{\{0\le v\le 1-u_\lambda-\delta|x|\}}\frac{v^2}{|x|^2}dx.$$

On the other hand, when $N = 2$ the space $H_0^1(\Omega)$ embeds continously into $L^p(\Omega)$ for every $p > 1$, and then, by Hölder inequality, for $\alpha > 0$ we get that

$$\int_\Omega \frac{v^2}{|x|^{2-\alpha}}\,dx \le \left(\int_\Omega |x|^{-(2-\alpha)\frac{p}{p-2}}\,dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |v|^p\,dx\right)^{\frac{2}{p}} \le C_{N,\alpha}^{-1}\int_\Omega |\nabla v|^2\,dx$$

provided $(2-\alpha)\frac{p}{p-2} < 2$, which is true for p large depending on α (see [6] for some very general Hardy inequalities). It combines with (15) to yield

$$C_{N,\alpha}\Big(\frac{1}{2} - a - \sup_{x\in\Omega}\bar\mu(h,x)\Big)\int_{\{0\le v\le 1-u_\lambda-\delta|x|\}}\frac{v^2}{|x|^{2-\alpha}}dx \le \lambda C_\delta \int_{\{0\le v\le 1-u_\lambda-\delta|x|\}}\frac{v^2}{|x|^{2-\alpha}}dx\,.$$

In both cases, we can conclude that for λ sufficiently small $v \equiv 0$ for $x \in \{0 \le v \le 1 - u_\lambda - \delta|x|\}$, for some $\delta > 0$ small. Since we can assume δ and λ sufficiently small to have

$$1 - u_\lambda - \delta|x| \ge \frac{1}{2} \quad \text{in} \quad \big\{x \in \Omega :\ |x| \ge \frac{1}{2}\mathrm{dist}(0,\partial\Omega)\big\},$$

we then have

$$v \equiv 0 \quad \text{in} \quad \big\{x \in \Omega :\ v(x) \le \frac{1}{2}\big\} \cap \big\{x \in \Omega :\ |x| \ge \frac{1}{2}\mathrm{dist}(0,\partial\Omega)\big\}.$$

Since $v = 0$ on $\partial\Omega$ and the domain $\{x \in \Omega :\ |x| \ge \frac{1}{2}\mathrm{dist}(0,\partial\Omega)\}$ is connected, the continuity of v gives that

$$v \equiv 0 \quad \text{in} \quad \big\{x \in \Omega :\ |x| \ge \frac{1}{2}\mathrm{dist}(0,\partial\Omega)\big\}.$$

Therefore, the maximum principle for elliptic equations implies $v \equiv 0$ in Ω, which completes the proof of Theorem 3.2. ◻

Remark 3.1. In [13] examples of dumbell shaped domains $\Omega \subset \mathbb{R}^N$ which satisfy condition $M(\Omega) < \frac{1}{2}$ are given for $N \ge 3$. When $N \ge 4$, there even exist topologically nontrivial domains with this property. Let us stress that in both cases Ω is not starlike, which means that the assumption $M(\Omega) < \frac{1}{2}$ on a domain Ω is more general than being shar-shaped.

The remaining case $N = 2$ and $\alpha = 0$, is a bit more delicate. We have the following result.

Theorem 3.3. *If Ω is either a strictly convex or a symmetric domain in $\mathbb{R}^2$, then $(S)_{\lambda,1}$ has the unique solution u_λ for small λ.*

Proof. The crucial point here is the following inequality: for every solution v of (6) there holds

$$\int_{\partial\Omega}|\nabla v|^2\,d\sigma \ge l(\partial\Omega)^{-1}\left(\int_\Omega |\Delta v|\,dx\right)^2.$$

Indeed, we have that

$$\int_{\partial\Omega} |\nabla v|^2 \, d\sigma \geq l(\partial\Omega)^{-1} \left(\int_{\partial\Omega} |\nabla v| \, d\sigma \right)^2 = l(\partial\Omega)^{-1} \left(\int_{\partial\Omega} \partial_\nu v \, d\sigma \right)^2$$
$$= l(\partial\Omega)^{-1} \left(\int_{\Omega} |\Delta v| \, dx \right)^2,$$

where $l(\partial\Omega)$ is the length of $\partial\Omega$. Note that $-\Delta v = \lambda g_\lambda(x,v) \geq 0$ for every solution $u_\lambda + v$ of $(S)_{\lambda,1}$, in view of the minimality of u_λ.

By Lemma 4 in [13] for λ small there exists $x_\lambda \in \Omega$ so that

$$\langle \nabla u_\lambda(x), x - x_\lambda \rangle \leq 0 \qquad \forall\, x \in \Omega. \tag{16}$$

In particular, for λ small x_λ lies in a compact subset of Ω and, when Ω is symmetric, coincides exactly with the center of symmetries. In both situations, then we have that there exists $c_0 > 0$ so that

$$\langle x - x_\lambda, \nu(x) \rangle \geq c_0 \qquad \forall\, x \in \partial\Omega.$$

We use now the Pohozaev identity (5) with $a = 0$ and $h(x) = \frac{x - x_\lambda}{2}$. For every solution v of (6) it yields

$$\lambda \int_\Omega \left[G_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \right] dx$$
$$= \frac{1}{4} \int_{\partial\Omega} |\nabla v|^2 \langle x - x_\lambda, \nu \rangle \, d\sigma \geq \frac{c_0}{4} \left(\int_\Omega |\Delta v| \, dx \right)^2. \tag{17}$$

Since

$$\nabla_x G_\lambda(x,s) = (1 - u_\lambda(x) - s)^{-2} \left[1 - \frac{(1 - u_\lambda(x) - s)^2 (1 - u_\lambda(x) + 2s)}{(1 - u_\lambda(x))^3} \right] \nabla u_\lambda(x),$$

by (16) we easily see that

$$\langle \nabla_x G_\lambda(x,s), x - x_\lambda \rangle \leq 0$$

for λ and δ small, provided (x,s) satisfies $|1 - u_\lambda(x) - s| \leq \delta$. Since $G_\lambda(x,s)$, $\nabla_x G_\lambda(x,s)$ are quadratic with respect to s as $s \to 0$ (uniformly in λ small), there exists a constant $C_\delta > 0$ such that

$$G_\lambda(x, v(x)) \leq C_\delta v^2(x)\,, \qquad \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \leq C_\delta v^2(x)$$

for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta\}$, uniformly for λ small.

Since on two-dimensional domains

$$\left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{p}} \leq C_p \int_\Omega |\Delta v| \, dx$$

for every $p \geq 1$ and $v \in W^{2,1}(\Omega)$ so that $v = 0$ on $\partial\Omega$, we get that

$$\lambda \int_\Omega \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \, dx \leq \lambda C_\delta \int_\Omega v^2 \, dx \leq \lambda C_\delta C_2^2 \left(\int_\Omega |\Delta v| \, dx \right)^2. \tag{18}$$

As far as the term with $G_\lambda(x,v(x))$, fix $b\in(0,1)$ and split Ω as the disjoint union of $\Omega_1=\{v\le b\}$ and $\Omega_2=\{v>b\}$. On Ω_1 we have that

$$\lambda\int_{\Omega_1} G_\lambda(x,v(x))\,dx \le \lambda C_\delta \int_\Omega v^2\,dx \le \lambda C_\delta C_2^2\left(\int_\Omega |\Delta v|\,dx\right)^2$$

provided λ and δ are small to satisfy $b\le 1-u_\lambda-\delta$ in Ω_1.
Since for λ small

$$\frac{G_\lambda(x,s)^2}{g_\lambda(x,s)}\le C \quad \forall\, b\le s\le 1,$$

we have that

$$\begin{aligned}\lambda\int_{\Omega_2} G_\lambda(x,v(x))\,dx &\le \lambda D_1\int_\Omega |v(x)|^{\frac32} g_\lambda^{\frac12}(x,v(x))\,dx\\ &\le \lambda D_2\left(\int_\Omega |v|^3\,dx\right)^{\frac12}\left(\int_\Omega g_\lambda(x,v(x))\,dx\right)^{\frac12}\\ &\le \lambda^{\frac12} D_3\left(\int_\Omega |\Delta v|\,dx\right)^2\end{aligned}$$

for some positive constants D_1, D_2 and D_3. So we get that

$$\lambda\int_\Omega G_\lambda(x,v(x))\,dx \le \left(\lambda C_\delta C_2^2+\lambda^{\frac12}D_3\right)\left(\int_\Omega |\Delta v|\,dx\right)^2. \tag{19}$$

Inserting (18)-(19) into (17) finally we get that

$$\left(2\lambda C_\delta C_2^2+\lambda^{\frac12}D_3-\frac{c_0}{4}\right)\left(\int_\Omega |\Delta v|\,dx\right)^2\ge 0,$$

and then $v\equiv 0$ for λ small. □

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