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ASYMPTOTICS FOR MULTIVARIATE LINEAR PROCESS WITH NEGATIVELY ASSOCIATED RANDOM VECTORS*

HYUN-CHULL KIM†, MI-HWA KO‡, AND TAE-SUNG KIM‡

Abstract. Let A_j be an $m \times m$ matrix such that $\sum_{j=0}^{\infty} \|A_j\| < \infty$ and $\sum_{j=0}^{\infty} A_j \neq O_{m\times m}$ where for any $m \times m,\ m \geq 1$, matrix $A = (a_{ij}),\ \|A\| = \sum_{i=1}^{m}\sum_{j=1}^{m} |a_{ij}|$ and $O_{m\times m}$ denotes the $m \times m$ zero matrix. For an m-dimensional linear process of the form $\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j}$, where $\{\mathbb{Z}_t\}$ is a sequence of stationary m-dimensional negatively associated random vectors with $E\mathbb{Z}_t = O$ and $E||\mathbb{Z}_t||^2 < \infty$, we prove the central limit theorems.

Key words. negatively associated random vector; multivariate linear process; central limit theorem

AMS subject classifications. Primary 60F05; Secondary 60G10

1. Introduction.

Define a linear process by

$$X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j},\ t = 1, 2, \cdot, \tag{1}$$

where $\{\epsilon_t\}$ is a centered sequence of random variables and $\{a_j\}$ is a sequence of real numbers. In time-series analysis, this process is of great importance. Many important time-series models,such as the casual ARMA process (Brockwell and Davis (1990)), have the type (1) with $\sum_{j=1}^{\infty} |a_j| < \infty$.

Kim and Baek (2001) established a central limit theorem for a linear process generated by linearly positive quadrant dependent random variables and Kim, Ko and Park (2004) also derived almost sure convergence for this linear process.

Let A_u be an $m \times m$ matrix such that $\sum_{u=0}^{\infty} \|A_u\| < \infty$ and $\sum_{u=0}^{\infty} A_u \neq O_{m\times m}$, where for any $m \times m,\ m \geq 1$, matrix $A = (a_{ij}),\ \|A\| := \sum_{i=1}^{m}\sum_{j=1}^{m} |a_{ij}|$ and $O_{m\times m}$ denotes the $m \times m$ zero matrix. Let $\mathbb{X}_t, t = 0, \pm 1, \cdots$, be an m-dimensional linear process of the form

$$\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j} \tag{2}$$

defined on a probability space (Ω, A, P), where $\{\mathbb{Z}_t, t = 0, \pm 1, \cdots\}$ is a sequence of strictly stationary m-dimensional random vectors with mean $\mathbb{O} : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$. The class of linear processes defined in (2) contains stationary multivariate autoregressive moving average processes (MARMA) that satisfy certain condition (See Brockwell and Davis (1990)).

Notions of negative dependence for collections of random variables have been much studied in recent years. The most prevalent negatively dependent notion is that of negative association. A finite collection $\{Y_i, 1 \leq i \leq m\}$ of random variables is said to be negatively associated (NA) if for any disjoint subsets A, B of $\{1, 2, \cdots, m\}$

*Received June 10, 2004; accepted for publication November 21, 2007.
†Department of Mathematics Education, Daebul University, Jeonnam 526-720, Korea (kimhc@mail.daebul.ac.kr).
‡Department of Mathematics, WonKwang University, Jeonbuk, 570-749, Korea (songhack@wonkwang.ac.kr; starkim@wonkwang.ac.kr).

and for all coordinatewise nondecreasing functions $f : \mathbb{R}^A \to \mathbb{R}$, $g : \mathbb{R}^B \to \mathbb{R}$ $Cov(f(Y_i : i \in A),\ g(Y_j : j \in B)) \le 0$, where the covariance is defined. An infinite collection of random variables is negatively associated if every finite subcollection is negatively associated. This negatively dependent notion was first defined by Joag-Dev and Proschan (1983). Negatively associated sequences are widely encountered in multivariate statistical analysis and reliability theory, and the notions of negative association have more attention recently. We refer to Joag-Dev and Proschan (1983) for fundamental properties of negatively associated sequences, Newman (1984), Birkel (1988) and Zhang (2000) for the central limit theorem, Matula (1992) for the three series theorem, Shao (2000) for the Rosenthal-type maximal inequality and the Kolmogorov exponential inequality.

In section 2 we extend this notion of negative association to the random vectors and derive a central limit theorem for a strictly stationary sequence of negative associated random vectors. We also prove the central limit theorems for a stationary multivariate linear process of the form (2) generated by negatively associated random vectors in section 3.

2. Preliminaries.

DEFINITION 2.1. A finite sequence $\{\mathbb{Z}_t, 1 \le t \le n\}$ of m-dimensional random vectors is said to be negatively associated if for any disjoint subsets A,B of $\{1, \cdots, n\}$ and for all coordinatewise nondecreasing functions f and g we have

$$Cov(f(\mathbb{Z}_i : i \in A),\ g(\mathbb{Z}_j : j \in B)) \le 0,$$

whenever this covariance is defined. An infinite collection of m-dimensional random vectors is negatively associated if every finite subcollection is negatively associated.

LEMMA 2.2. Let $\{Y_1, \cdots, Y_n\}$ be a strictly stationary sequence of negatively associated random variables with $EY_1 = 0, EY_1^2 < \infty$. Then

$$E(\max_{1 \le k \le n} |Y_1 + \cdots + Y_k|^2) \le CnEY_1^2$$

where C is a positive constant.

Proof. See the proof of Lemma 4 of Matula(1992).

LEMMA 2.3. Let $\{\mathbb{Z}_t : 1 \le t \le n\}$ be a strictly stationary sequence of negatively associated m-dimensional random vectors with $E\mathbb{Z}_1 = 0$ and $E \parallel \mathbb{Z}_1 \parallel^2 < \infty$, where for a vector $x \in \mathbb{R}^m$, denote its Euclidean norm by $\|x\|$. Then, there is a positive constant C such that

$$E \max_{1 \le k \le n} \parallel \sum_{t=1}^{k} \mathbb{Z}_t \parallel^2 \le Cm^2 nE \parallel \mathbb{Z}_1 \parallel^2 . \tag{3}$$

LEMMA 2.4 (NEWMAN (1984)). Let $\{Y_j, j \ge 1\}$ be a strictly stationary sequence of negative associated random variables with $EY_1 = 0$ and $EY_1^2 < \infty$. If

$$\sigma^2 = Var(Y_1) + 2\sum_{j=2}^{\infty} Cov(Y_1, Y_j) < \infty$$

holds then

$$n^{-\frac{1}{2}} \sum_{j=1}^{n} Y_j \to^{D} N(0, \sigma^2) \ as \ n \to \infty,$$

where $\to^{D}$ means convergence in distribution.

THEOREM 2.5. Let $\{\mathbb{Z}_t : t \geq 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}$ and $E \parallel \mathbb{Z}_1 \parallel^2 < \infty$. Let $S_n = \sum_{t=1}^{n} \mathbb{Z}_t$. If

$$E \parallel \mathbb{Z}_1 \parallel^2 + 2 \sum_{t=2}^{\infty} \sum_{j=1}^{m} E(Z_1^{(j)} Z_t^{(j)}) = \sigma^2 < \infty \tag{4}$$

holds, then, as $n \to \infty$,

$$n^{-\frac{1}{2}} S_n \xrightarrow{D} N(\mathbb{O}, \Gamma)$$

with covariance matrix $\Gamma = [\sigma_{kj}]$, $\;k = 1, \cdots, m$; $j = 1, \cdots, m$,

$$\sigma_{kj} = E(Z_1^{(k)} Z_1^{(j)}) + \sum_{t=2}^{\infty} \ [E(Z_1^{(k)} Z_t^{(j)}) + E(Z_1^{(j)} Z_t^{(k)})], \tag{5}$$

where $Z_t^{(j)}$ denotes the $j - th$ component of $\mathbb{Z}_t$.

Proof. By Lemma 2.4 it follows from (4) that, for each $j (1 \leq j \leq m)$,

$$n^{-\frac{1}{2}} \sum_{t=1}^{n} Z_t^{(j)} \to^{D} N(0, \sigma_j'^2) \ as \ n \to \infty,$$

where $\sigma_j'^2 = E(Z_1^{(j)})^2 + 2 \sum_{t=2}^{\infty} E(Z_1^{(j)} Z_t^{(j)}) < \infty$.
Hence by the Cramer-Wold device (See Billingsley (1968, page 48-49).) the desired result follows.

3. Result.

LEMMA 3.1. Let $\{\mathbb{Z}_t, t \geq 1\}$ be a strictly stationary sequence of negatively associated m-dimensional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}, E||\mathbb{Z}_1||^2 < \infty$. Let $\mathbb{X}_t = \sum_{j=1}^{\infty} A_j \mathbb{Z}_{t-j}$, $\mathbb{S}_k = \sum_{t=1}^{k} \mathbb{X}_t$, $\tilde{\mathbb{X}}_t = (\sum_{j=1}^{\infty} A_j) \mathbb{Z}_t$ and $\tilde{\mathbb{S}}_k = \sum_{t=1}^{k} \tilde{\mathbb{X}}_t$. Assume

$$\sum_{j=1}^{\infty} ||A_j|| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} A_j \neq \mathbb{O}_{m \times m} \ . \tag{6}$$

Then

$$n^{-\frac{1}{2}} \max_{1 \leq k \leq n} ||\tilde{\mathbb{S}}_k - \mathbb{S}_k|| = o_p(1).$$

Proof. First observe that

$$\begin{aligned}\tilde{\mathbb{S}}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} A_j\right)\mathbb{Z}_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} A_j \mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t\end{aligned}$$

and thus,

$$\begin{aligned}\tilde{\mathbb{S}}_k - \mathbb{S}_k &= -\sum_{t=1}^{k}\left(\sum_{j=t}^{\infty} A_j \mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= I_1 \ + \ I_2 \text{ (say)}.\end{aligned}$$

To prove

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_1\| \ = \ o_p(1), \tag{7}$$

note that

$$\begin{aligned}& n^{-1} E \max_{1\le k\le n} \left\|\sum_{t=1}^{k} \sum_{j=t}^{\infty} A_j \ \mathbb{Z}_{t-j}\right\|^2 \\ &= n^{-1} \ E \ \max_{1\le k\le n} \left\|\sum_{j=1}^{\infty} \sum_{t=1}^{j\wedge k} A_j \ \mathbb{Z}_{t-j}\right\|^2 \\ &\le n^{-1} \left(\sum_{j=1}^{\infty} \|A_j\| \left\{E \max_{1\le k\le n} \left\| \sum_{t=1}^{j\wedge k} \mathbb{Z}_{t-j}\right\|^2\right\}^{\frac{1}{2}}\right)^2 \\ &\qquad\qquad\qquad\qquad \text{by Minkowski inequality} \\ &\le Am^2 E \parallel Z_1 \parallel^2 \left[\sum_{j=1}^{\infty} \|A_j\| \left(\frac{j\wedge n}{n}\right)^{\frac{1}{2}}\right]^2\end{aligned}$$

by (3) and (6) and E $\parallel Z_1 \parallel^2 < \infty$. By the dominated convergence theorem the last term above tends to zero as $n \longrightarrow \infty$. Thus (7) is proved by the Markov inequality. Next, we show that

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_2\| \ = \ o_p(1). \tag{8}$$

Write

$$\begin{aligned}& I_2 = II_1 + II_2, \mathit{where} \\ & II_1 = A_1\mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)\end{aligned}$$

and

$$II_2 = (A_{k+1} + A_{k+2} + \cdots)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).$$

Let p_n be a sequence of positive integers such that

$$p_n \to \infty \ \text{ and } \ p_n/n \to 0 \ \text{ as } \ n \to \infty. \tag{9}$$

Then

$$\begin{aligned} n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_2\| &\le \left(\sum_{i=0}^{\infty} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le p_n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &\quad + \left(\sum_{i>p_n} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &= o_p(1) \end{aligned}$$

by (6), (9) and $E \parallel \mathbb{Z}_1 \parallel^2 < \infty$. It remains to prove that

$$Y_n \ : \ = n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_1\| \ = \ o_p(1).$$

To this end, define for each $l \ge 1$

$$II_{1,l} = \ B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + B_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l \\ \mathbb{O}_{m\times m}, & k > l. \end{cases}$$

Let $Y_{n,l} = n^{-\frac{1}{2}} \ \max_{1\le k\le n} \ \|II_{1,l}\|$. Clearly, for each $l \ge 1$,

$$Y_{n,l} = o_p(1). \tag{10}$$

On the other hand,

$$\begin{aligned} n(Y_{n,l} - Y_n)^2 &\le \max_{1\le k\le n} \left\| \sum_{i=1}^{k} (A_i - B_i)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \right\|^2 \\ &\le \max_{l<k\le n} \left(\sum_{i=l+1}^{k} \|A_i\| \cdot \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \right)^2 \\ &\le \left(\sum_{i>l} \|A_i\|\right)^2 \max_{l<k\le n} \max_{l<i\le k} \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\|^2 \\ &\le 4\left(\sum_{i>l} \|A_i\|\right)^2 \max_{l\le j\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|^2. \end{aligned}$$

From this result, (6) and $E \parallel Z_1 \parallel^2 < \infty$. , for any $\delta > 0$,

$$
\begin{aligned}
&\lim_{l\to\infty}\limsup_{n\to\infty}\ P(|Y_{n,l}-Y_n|^2>\delta)\\
&\le \lim_{l\to\infty}\limsup_{n\to\infty}\ 4\delta^{-1}\left(\sum_{i>l}\|A_i\|\right)^2\ n^{-1}E\max_{1\le j\le n}\ \|\mathbb{Z}_1+\cdots+\mathbb{Z}_j\|^2\\
&\le 4Am^2\delta^{-1}E\parallel Z_1\parallel^2\ \lim_{l\to\infty}\left(\sum_{i>l}\|A_i\|\right)^2=0. \qquad (11)
\end{aligned}
$$

In view of (10) and (11), it follows from Theorem 4.2 of Billingsley (1968, p.25) that $Y_n=\circ_p(1)$. This completes the proof of Lemma 3.1.

THEOREM 3.2. Let $\{\mathbb{Z}_t, t\ge 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1)=\mathbb{O}, E||\mathbb{Z}_t||^2<\infty$ and $\{\mathbb{X}_t\}$ an m-dimensional linear process defined in (2). Set $\mathbb{S}_n=\sum_{t=1}^n\mathbb{X}_t(\mathbb{S}_0=\mathbb{O})$, $\tilde{\mathbb{S}}_n=\sum_{t=1}^n\tilde{\mathbb{X}}_t$ as in Lemma 3.1. If (6) and (4) hold then

$$
n^{-\frac{1}{2}}\mathbb{S}_n\xrightarrow{\mathcal{D}}N(\mathbb{O},T)\quad\text{as}\quad n\to\infty, \qquad (12)
$$

where $T=(\sum_{j=1}^\infty A_j)\Gamma(\sum_{j=1}^\infty A_j)'$ and Γ is defined as in Theorem 2.5.

Proof. First note that $n^{-\frac{1}{2}}\tilde{\mathbb{S}}_n=n^{-\frac{1}{2}}(\sum_{j=1}^\infty A_j)\sum_{t=1}^n\mathbb{Z}_t$ and that $n^{-\frac{1}{2}}\tilde{\mathbb{S}}_n\xrightarrow{\mathcal{D}}N(\mathbb{O},T)$ according to Theorem 2.5. Hence, $n^{-\frac{1}{2}}\mathbb{S}_n\xrightarrow{\mathcal{D}}N(\mathbb{O},T)$ follows by applying Lemma 3.1 and Theorem 4.1 of Billingsley (1968).

We now introduce another central limit theorem.

THEOREM 3.3. Let $\{\mathbb{Z}_t, t\ge 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1)=\mathbb{O}, E||\mathbb{Z}_1||^2<\infty$ and let $\{\mathbb{X}_t\}$ be an m-dimensional linear process defined in (2). If

$$
\sum_{i=1}^\infty\sum_{j=i+1}^\infty||A_j||<\infty \qquad (13)
$$

holds, then

$$
n^{-\frac{1}{2}}\mathbb{S}_n\xrightarrow{\mathcal{D}}N(\mathbb{O},T)\ \ as\ \ n\to\infty,
$$

where $T=(\sum_{j=1}^\infty A_j)\Gamma(\sum_{j=1}^\infty A_j)'$ and Γ is defined as in Theorem 2.5.

Proof. Letting $\tilde{A}_i=\sum_{j=i+1}^\infty A_j$ and $\mathbb{Y}_t=\sum_{i=0}^\infty\tilde{A}_i\mathbb{Z}_{t-i}$, which is well defined since $\sum_{i=0}^\infty||\tilde{A}_i||<\infty$ by (13), we have

$$
\begin{aligned}
\mathbb{X}_t&=(\sum_{i=0}^\infty A_i)\mathbb{Z}_t-\tilde{A}_0\mathbb{Z}_t+\sum_{i=1}^\infty(\tilde{A}_i-\tilde{A}_{i-1})\mathbb{Z}_{t-i}\\
&=(\sum_{i=0}^\infty A_i)\mathbb{Z}_t+\mathbb{Y}_{t-1}-\mathbb{Y}_t
\end{aligned}
$$

which implies that

$$\mathbb{S}_n = \left(\sum_{i=0}^{\infty} A_i\right) \sum_{t=1}^{n} \mathbb{Z}_t + \mathbb{Y}_0 - \mathbb{Y}_n.$$

According to Theorem 2.5 we have $n^{-\frac{1}{2}} \sum_{t=1}^{n} \mathbb{Z}_t \to N(\mathbb{O}, \Gamma)$ as $n \to \infty$ and thus using this result on $n^{-\frac{1}{2}}(\sum_{i=0}^{\infty} A_i) \sum_{t=1}^{n} \mathbb{Z}_t$, we have $n^{-\frac{1}{2}}(\sum_{i=0}^{\infty} A_i) \sum_{t=1}^{n} \mathbb{Z}_t \to N(\mathbb{O}, T)$ *as* $n \to \infty$. Hence, this theorem is proved if

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \xrightarrow{p} \mathbb{O} \quad \text{as} \quad n \to \infty. \tag{14}$$

To prove (14) it is sufficient to show that

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \quad \to \quad \mathbb{O} \;\; a.s. \;\; \text{as} \quad n \to \infty. \tag{15}$$

But (15) follows from the fact that for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{|Y_{n,j}|}{\sqrt{n}} > \epsilon\right) = \sum_{n=1}^{\infty} P(|Y_{0,j}| > \sqrt{n}\epsilon) < \infty,$$

for all j, where $Y_{n,j}$ denotes the j-th component of $\mathbb{Y}_n$.

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