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SQUARE ROOTS OF HAMILTONIAN DIFFEOMORPHISMS

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In this paper, we prove that on any closed symplectic manifold there exists an arbitrarily C^∞-small Hamiltonian diffeomorphism not admitting a square root.

1. Introduction

Let (M,ω) be a closed symplectic manifold, i.e., $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \to \mathbb{R}$ we associate the Hamiltonian vector field X_L by setting

$$\omega(X_{L_t}, \cdot) = -dL_t(\cdot), \tag{1.1}$$

where $L_t(x) := L(t,x)$. The flow $\phi_L^t : M \to M$ of the vector field X_{L_t} is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi_L^1 \ . \tag{1.2}$$

The Hamiltonian diffeomorphisms form the Lie group $\mathrm{Ham}(M,\omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [**MS98**] for the basics in symplectic geometry.

In this paper, we prove the following Theorem.

Theorem 1. *In any C^∞-neighborhood of the identity in* $\mathrm{Ham}(M,\omega)$ *there exists a Hamiltonian diffeomorphism ϕ which has no square root, i.e., for all Hamiltonian diffeomorphism ψ (not necessarily close to the identity)*

$$\psi^2 \neq \phi \tag{1.3}$$

holds.

An immediate corollary of Theorem 1 is the following.

Corollary 2. *The exponential map*

$$\begin{aligned} \mathrm{Exp} : C^\infty(M,\mathbb{R})/\mathbb{R} &\to \mathrm{Ham}(M,\omega) \\ [L] &\mapsto \phi_L \end{aligned} \tag{1.4}$$

is not a local diffeomorphism.

In the proof of the Theorem, we use the following beautiful observation by Milnor [**Mil84, Warning 1.6**]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2k$-cycles, see the next section for details. The main work in this paper is to construct an example in the symplectic category.

2. Milnor's observation

We define

$$CM^k := M^k/(\mathbb{Z}/k), \tag{2.1}$$

where $\mathbb{Z}/k$ acts by cyclic shifts on M^k. We write elements of CM^k as

$$[x_1,\dots,x_k] \in CM^k\ . \tag{2.2}$$

The space of k-cycles of a diffeomorphism $\phi : M \to M$ is

$$\begin{aligned} &\mathscr{C}^k(\phi) \\ &\quad := \Big\{[x_1,\dots,x_k] \in CM^k \mid \phi^j(x_i) \neq x_i\, \forall j = 1,\dots,k-1,\ \phi(x_i) = x_{i+1}\Big\}\ . \end{aligned} \tag{2.3}$$

We point out that if $[x_1,\dots,x_k] \in \mathscr{C}^k(\phi)$ then $\phi^k(x_i) = x_i$ for $i = 1,\dots,k$.

Proposition 3 (Milnor [Mil84]). *If $\phi = \psi^2$ then $\mathscr{C}^{2k}(\phi)$ admits a free $\mathbb{Z}/2$-action. In particular, $\#\mathscr{C}^{2k}(\phi)$ is even if $\mathscr{C}^{2k}(\phi)$ is a finite set.*

For the convenience of the reader we include a proof of Milnor's ingenious observation.

Proof. We define

$$\begin{aligned} I : \mathscr{C}^{2k}(\phi) &\to \mathscr{C}^{2k}(\phi), \\ [x_1,\dots,x_{2k}] &\mapsto [\psi(x_1),\dots,\psi(x_{2k})]\ . \end{aligned} \tag{2.4}$$

Since $\psi\circ\phi = \phi\circ\psi$ and $\psi^2 = \phi$ the map I is well-defined and an involution. We assume by contradiction that $[x_1,\dots,x_{2k}]$ is a fixed point of I, i.e., there exists $0 \le r \le 2k-1$

$$\psi(x_i) = x_{i+r}, \tag{2.5}$$

where we read indices $\mathbb{Z}/2k$-cyclically. Using $x_{i+r} = \phi^r(x_i)$ we get

$$\psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \tag{2.6}$$

and thus

$$\psi^{2r-1}(x_i) = x_i \;. \tag{2.7}$$

In particular,

$$x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i) \;. \tag{2.8}$$

In summary, we have

$$x_i = \phi^{2r-1}(x_i) \quad \text{and} \quad x_i = \phi^{2k}(x_i) \;. \tag{2.9}$$

In general, if

$$z = \phi^{a}(z) \quad \text{and} \quad z = \phi^{b}(z) \tag{2.10}$$

for $a, b \in \mathbb{Z}$ then

$$z = \phi^{\mathrm{lcd}(a,b)}(z) \tag{2.11}$$

since by the Euclidean algorithm there exists $n_1, n_2 \in \mathbb{Z}$ with

$$\mathrm{lcd}(a,b) = n_1 a + n_2 b \;. \tag{2.12}$$

In our specific situation, $2r-1$ is odd and $2k$ is even and thus

$$1 \leq \mathrm{lcd}(2r-1, 2k) < 2k \tag{2.13}$$

contradicting the assumption $\phi^j(x_i) \neq x_i \,\forall j = 1, \ldots, 2k-1$. This proves the Proposition. $\square$

3. Proof of Theorem 1

Let (M, ω) be a closed symplectic manifold. We fix a Darboux chart $B^{2N}(R) \cong B \subset M$ where $B^{2N}(R)$ is the open ball of radius R in $\mathbb{R}^{2N}$. For an integer $k \geq 1$ and a positive number $\delta > 0$, we choose a smooth function $\rho : [0, R^2] \to \mathbb{R}$ satisfying the following:

$$\begin{cases} \dfrac{\pi}{2k} \geq \rho'(r) > 0 \;, \\ \rho'(r) = \dfrac{\pi}{2k} \quad \Longleftrightarrow \quad r = \frac{1}{2}R^2 \;, \\ \rho'|_{\left[\frac{8}{9}R^2, R^2\right]} = \delta > 0 \;. \end{cases} \tag{3.1}$$

We set for $1 \leq \nu \leq N$

$$\zeta(\nu) := \begin{cases} 1 & \nu = N, \\ \dfrac{9}{10} & \text{else} \end{cases} \tag{3.2}$$

and define

$$\begin{aligned} H : B^{2N}(R) &\to \mathbb{R} \\ z &\mapsto \rho\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) \ . \end{aligned} \tag{3.3}$$

We denote by $\phi_H^t : B^{2N}(R) \to B^{2N}(R)$ the induced Hamiltonian flow. We recall that the Hamiltonian flow of $z \mapsto |z|^2$ is given by $z \mapsto \exp(2it)z$ thus

$$\big(\phi_H^t(z)\big)_\nu = \exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)t\right] z_\nu \ . \tag{3.4}$$

We point out that ϕ_H^t preserves the quantities $|z_\nu|$, $\nu = 1, \ldots, N$.

Lemma 4. *The fixed points of ϕ_H^{2k} are precisely $z = 0$ and the circle*

$$C := \left\{(z_1, \ldots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \tfrac{1}{2}R^2 \text{ and } z_1 = \cdots = z_{N-1} = 0\right\} \ . \tag{3.5}$$

Moreover, ϕ_H acts on C by rotation of the last coordinate by an angle of $\frac{\pi}{k}$.

Proof. Assume that $\phi_H^{2k}(z) = z$, which is equivalent to

$$\exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)2k\right] z_\nu = z_\nu, \quad \nu = 1, \ldots, N \ , \tag{3.6}$$

thus, either $z_\nu = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 4k\zeta(\nu) \in 2\pi\mathbb{Z} \ . \tag{3.7}$$

From $\rho'(r) \leq \frac{\pi}{2k}$ we conclude that $z_1 = \cdots = z_{N-1} = 0$. Moreover, $z_N = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) = \rho'(|z_N|^2) = \frac{\pi}{2k} \tag{3.8}$$

holds. In summary, either $z = 0$ or $z \in C$. This together with (3.4) proves the Lemma. □

We now perturb H. For this we fix a smooth cut-off function $\beta : [0, R^2] \to [0, 1]$ satisfying

$$\beta|_{[\frac{1}{3}R^2, \frac{2}{3}R^2]} = 1 \quad \text{and} \quad \beta|_{[0, \frac{1}{9}R^2] \cup [\frac{8}{9}R^2, R^2]} = 0 \tag{3.9}$$

and set

$$F(z) := \beta(|z_N|^2) \cdot \mathrm{Re}\left(\frac{z_N^k}{|z_N|^k}\right) : B^{2N}(R) \to \mathbb{R}, \tag{3.10}$$

where Re is the real part. If we introduce new coordinates $(z_1, \ldots, z_{N-1}, r, \vartheta)$, where $z_N = r\exp(i\vartheta)$, the function F equals

$$F(z) = \beta(r^2)\cos(k\vartheta) \ . \tag{3.11}$$

We point out that the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F}$ maps $B^{2N}(R)$ into itself.

Lemma 5. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$*

$$\#\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F}) = 1 \ . \tag{3.12}$$

Proof. We set

$$D := \left\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k},\ j = 0, \ldots, 2k-1\right\}, \tag{3.13}$$

where C is defined in Lemma 4. The same lemma implies that ϕ_H acts on D as a cyclic permutation sending $\frac{j\pi}{k}$ to $\frac{(j+1)\pi}{k}$. Moreover, we have

$$\phi_{\epsilon F} z = z \tag{3.14}$$

for $z \in D$ since $D \subset \mathrm{Crit}F$. In particular, D corresponds precisely to a single element in $\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F})$. It remains to show that there are no other $2k$-cycles. We prove something stronger, namely that for sufficiently small $\epsilon > 0$ the only other fixed point of $(\phi_H \circ \phi_{\epsilon F})^{2k}$ is $z = 0$.

For $0 < a < b$, we set

$$A(a, b) := \left\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2]\right\} \ . \tag{3.15}$$

We observe that on $A(\frac{1}{3}, \frac{2}{3})$ we have $\beta = 1$ and thus the flow of ϵF is given by

$$(z_1, \ldots, z_{N-1}, r, \vartheta) \mapsto (z_1, \ldots, z_{N-1}, \sqrt{-2\epsilon k \sin(k\vartheta)t + r^2}, \vartheta) \ . \tag{3.16}$$

In particular, if we set

$$\bar{\epsilon} := \frac{7R^4}{324k^2} \tag{3.17}$$

then for $0 < \epsilon < \bar{\epsilon}$ we conclude that

$$(\phi_H \circ \phi_{\epsilon F})^{2k}\left(A(\tfrac{4}{9}, \tfrac{5}{9})\right) \subset A(\tfrac{1}{3}, \tfrac{2}{3}) \ , \tag{3.18}$$

since ϕ_H^t preserves the r coordinate. Fix $w \in A(\frac{4}{9}, \frac{5}{9})$ with $(\phi_H \circ \phi_{\epsilon F})^{2k}(w) = w$ and set for $j = 0, \ldots, 2k$

$$\begin{aligned} z_\nu^j &:= P_{z_\nu}\left((\phi_H \circ \phi_{\epsilon F})^j(w)\right), \quad \nu = 1, \ldots, N-1, \\ r^j &:= P_r\left((\phi_H \circ \phi_{\epsilon F})^j(w)\right), \\ \vartheta^j &:= P_\vartheta\left((\phi_H \circ \phi_{\epsilon F})^j(w)\right), \end{aligned} \tag{3.19}$$

where P_{z_ν}, P_r, and P_ϑ are the projections on the respective coordinates. It follows from equation (3.16) that

$$P_{z_\nu}\Big((\phi_H\circ\phi_{\epsilon F})^j(w)\Big)=P_{z_\nu}\Big(\phi_H^j(w)\Big),\quad \nu=1,\dots,N-1\ . \tag{3.20}$$

By the same argument as in the proof of Lemma 4, we conclude

$$z_\nu^j=0,\quad \forall\nu=1,\dots,N-1 \text{ and } \forall j=0,\dots,2k\ . \tag{3.21}$$

Next, it follows from the flow equations (3.4) and (3.16)

$$0<\vartheta_{j+1}-\vartheta_j\le\frac{\pi}{k}\mod 2\pi\ . \tag{3.22}$$

By (3.1) equality holds if and only if $r_{j+1}=\frac{1}{2}R^2$. Using again $(\phi_H\circ\phi_{\epsilon F})^{2k}(w)=w$ we deduce

$$\vartheta_{2k}-\vartheta_0=0\mod 2\pi \tag{3.23}$$

and therefore

$$r_0=r_1=\dots=r_{2k}=\tfrac{1}{2}R^2\ . \tag{3.24}$$

In summary

$$w=(0,\dots,0,\tfrac{1}{2}R^2,\vartheta_0) \tag{3.25}$$

with $\vartheta_0\in\frac{\pi}{k}\mathbb{Z}$, i.e., $w\in D$. Thus, we proved that the only $2k$-cycle of $\phi_H\circ\phi_{\epsilon F}$ in the region $A(\frac{4}{9},\frac{5}{9})$ is the one corresponding to the set D. Therefore it remains to prove that after possibly shrinking $\bar\epsilon$ there are no other fixed points of $(\phi_H\circ\phi_{\epsilon F})^{2k}$ outside $A(\frac{4}{9},\frac{5}{9})$ except for $z=0$. We argue by contradiction.

We assume that there exists a sequence $\epsilon_m\to 0$ and a sequence $(z^m)_{m\in\mathbb{N}}$ of points in $B^{2N}(R)\setminus A(\frac{4}{9},\frac{5}{9})$ with

$$(\phi_H\circ\phi_{\epsilon_m F})^{2k}(z^m)=z^m\quad\forall m\in\mathbb{N}\ . \tag{3.26}$$

By compactness we may assume that $z^m\to z^*\in B^{2N}(R)\setminus \mathrm{int}A(\frac{4}{9},\frac{5}{9})$ with

$$\phi_H^{2k}(z^*)=z^*\ . \tag{3.27}$$

It follows from Lemma 4 that $z^*=0$ and thus for M sufficiently large

$$z^m\in B^{2N}(\tfrac{1}{3}R)\quad\forall m\ge M\ . \tag{3.28}$$

Then by definition of β the restriction of $\phi_{\epsilon_m F}$ to the ball $B^{2N}(\frac{1}{3}R)$ equals the identity. Moreover, since ϕ_H fixes all balls centered at zero we have

$$z^m=(\phi_H\circ\phi_{\epsilon_m F})^{2k}(z^m)=\phi_H^{2k}(z^m)\quad\forall m\ge M\ . \tag{3.29}$$

Applying again Lemma 4 we conclude that $z^m=0$ for all $m\ge M$. This proves the Lemma. □

Remark 6. Proposition 3 together with Lemma 5 implies that for all $0 < \epsilon < \epsilon_0$ the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \to B^{2N}(R)$ has no square root.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. We choose $k \in \mathbb{Z}$, $\delta > 0$ and $0 < \epsilon < \epsilon_0$ (cp. Lemma 5) so that the Hamiltonian diffeomorphism

$$\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \to B^{2N}(R) \tag{3.30}$$

has precisely one $2k$-cycle. By construction $\phi_H \circ \phi_{\epsilon F}$ equals the map

$$(z_1, \ldots, z_N) \mapsto \left(e^{\frac{9i\delta}{5}} z_1, \ldots, e^{\frac{9i\delta}{5}} z_{N-1}, e^{2i\delta} z_N\right) \tag{3.31}$$

near the boundary of $B^{2N}(R)$. Indeed, if $z \in \partial B^{2N}(R)$, then we conclude

$$\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \geq \frac{9}{10} \sum_{\nu=1}^{N} |z_\nu|^2 = \frac{9}{10} R^2 > \frac{8}{9} R^2 \tag{3.32}$$

and therefore $\rho'(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2) = \delta$. Next, we extend the Hamiltonian function of $\phi_H \circ \phi_{\epsilon F}$ to $\widetilde{H} : S^1 \times M \to \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball B. If we choose $\delta > 0$ sufficiently small, then we can guarantee that outside B the only periodic orbits of $\widetilde{H}$ of period less or equal to $2k$ are critical points of $\widetilde{H}$, see [**HZ94**], in particular line 4 & 5 on page 185. In particular, $\phi_{\widetilde{H}}$ has still precisely one $2k$-cycle. Finally, by choosing k sufficiently large and δ and ϵ sufficiently small, $\phi_H \circ \phi_{\epsilon F}$ and thus $\phi_{\widetilde{H}}$ can be chosen to lie in an arbitrary C^∞-neighborhood of the identity on $B^{2N}(R)$ resp. M. Therefore, with Proposition 3 the Theorem follows. □

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