JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 11, Number 3, 343–361, 2013

ON EXOTIC MONOTONE LAGRANGIAN TORI IN $\mathbb{CP}^2$ AND $\mathbb{S}^2 \times \mathbb{S}^2$

Agnès Gadbled

In this note, we prove that two constructions of exotic monotone Lagrangian tori, namely the one by Chekanov and Schlenk (see [**7, 9**]) and the one obtained by the circle bundle construction of Biran (see [**5**]), are Hamiltonian isotopic in $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.

1. Introduction

Because of Darboux's theorem and because any open subset of $\mathbb{C}^n$ contains a Lagrangian torus, it is possible to construct a Lagrangian torus in any symplectic manifold. The construction of Lagrangian submanifolds can become far more difficult as soon as we require some conditions — we will be here interested in monotonicity — on the Lagrangian submanifold.

Recall that a Lagrangian submanifold L of a symplectic manifold (W, ω) is said to be monotone (see Oh [**13**]) if there exists a non-negative constant K_L such that for every disc $u \in \pi_2(W, L)$,

$$\int u^\star\omega = K_L \ \mu_L(u),$$

where $\mu_L(u)$ is the Maslov class of the disc u. Recall also that the existence of a monotone Lagrangian submanifold implies that the ambient symplectic manifold itself must be monotone, which means that there exists a non-negative constant K_W such that for every sphere $v \in \pi_2(W)$,

$$\int v^\star\omega = K_W \ c_1(v),$$

where c_1 is the first Chern class of (W, ω). Note that because of the relation between the first Chern class and the Maslov class of a sphere in $\pi_2(W)$, in the case the first Chern class of W does not vanish identically on $\pi_2(W)$, we have the following relation between the monotonicity constants:

$$K_W = 2K_L.$$

In particular in this case, the monotonicity constant of any Lagrangian submanifold is prescribed by the monotonicity constant of the ambient symplectic manifold, and this gives already many restrictions on the Lagrangian.

Even in the case of $\mathbb{C}^n$ endowed with its canonical symplectic structure, not many constructions of monotone Lagrangian tori are available. The first and simplest example is the Clifford (or split) torus, that is the product of n circles enclosing discs of the same area. As any construction of a Lagrangian torus gives rise to infinitely many tori with the same symplectic invariants by applying a Hamiltonian diffeomorphism, we are rather interested in equivalence classes of tori under the action of the group of Hamiltonian diffeomorphisms. It was only in 1995 that Chekanov [**6**] gave the first examples of monotone Lagrangian tori in $\mathbb{C}^n$, which are not Hamiltonian isotopic to the Clifford torus. Such tori are said exotic.

The Clifford torus can be embedded in the complex projective space and the product of spheres via the embedding of a ball or a polydisc of suitable size in $\mathbb{CP}^n$ or in $\times_n\mathbb{S}^2$. It is a monotone Lagrangian torus in $\mathbb{CP}^n$ or in $\times_n\mathbb{S}^2$ called again Clifford torus. The Clifford torus was also the only known example of monotone Lagrangian torus in $\mathbb{CP}^n$ and in $\times_n\mathbb{S}^2$ till Chekanov and Schlenk [**7, 9**] described in 2006 their families of exotic monotone Lagrangian tori.

Biran and Cornea [**5**] have also given recently a construction of monotone Lagrangian tori thanks to the circle bundle construction of Biran [**3**]. It is believed that one should recover the families of monotone Lagrangian tori of Chekanov and Schlenk with the constructions through circle bundles. The present note proves this fact for $\mathbb{CP}^2$ and $\mathbb{S}^2\times\mathbb{S}^2$ and we will deal with the higher dimensions in some future work.

The tori of Chekanov and Schlenk in $\mathbb{CP}^2$ and $\mathbb{S}^2\times\mathbb{S}^2$ are defined by modifying slightly the construction of the exotic monotone Lagrangian torus in $\mathbb{C}^2$ of Eliashberg and Polterovich in [**11**]: given a “suitable” (see Sections 3.1 and 4.1) curve $\gamma:[0,2\pi]\longrightarrow\mathbb{C}$, consider

$$\left\{\left(\gamma(s)\,\mathrm{e}^{\mathrm{i}\theta},\gamma(s)\,\mathrm{e}^{-\mathrm{i}\theta}\right)\middle|\,\theta\in[0,2\pi],s\in[0,2\pi]\right\}$$

and then embed it into $\mathbb{CP}^2$ or $\mathbb{S}^2\times\mathbb{S}^2$.

The monotone torus in $\mathbb{CP}^2$ (or analogously in $\mathbb{S}^2\times\mathbb{S}^2$) defined by Biran and Cornea in [**5**] is not coming from a torus in $\mathbb{C}^2$ and an embedding of a ball or a polydisc: one starts with a symplectic sphere Σ (coming from a polarization of $\mathbb{CP}^2$ or $\mathbb{S}^2\times\mathbb{S}^2$, see [**2**]), and constructs the torus as the restriction to an equator of the sphere Σ of a circle subbundle of a disc bundle E_Σ over Σ. In the case of $\mathbb{S}^2\times\mathbb{S}^2$, this construction is known to be equal to the first example in the literature of exotic monotone Lagrangian torus given by Entov and Polterovich in [**12**].

In this note we prove the following:

Theorem. *The construction of monotone exotic torus of Chekanov and Schlenk and the one given by the circle bundle construction of Biran are Hamiltonian isotopic in* $\mathbb{CP}^2$ *and* $\mathbb{S}^2 \times \mathbb{S}^2$*.*

The idea of the proof for the projective plane is to relate Biran and Cornea's torus to the first construction of exotic monotone torus in $\mathbb{C}^2$ given by Chekanov in [**6**]. This construction in $\mathbb{C}^2$ is known to be Hamiltonian isotopic to the monotone exotic torus of Eliashberg and Polterovich in [**11**]. A similar isotopy between these tori can be used in the case of $\mathbb{CP}^2$ to define an isotopy between Chekanov and Schlenk's torus and a modified Chekanov's torus. After embedding into $\mathbb{CP}^2$, one can prove that this modified Chekanov's torus is a circle subbundle of the disc bundle E_Σ over an equator of Σ and then isotope this torus inside the disc bundle to the torus of Biran and Cornea. The strategy for $\mathbb{S}^2 \times \mathbb{S}^2$ is analogous.

The article is organized the following way: in the first section, we recall the two constructions of exotic monotone Lagrangian torus in $\mathbb{C}^2$, namely the first one by Chekanov [**6**] and the one of Eliashberg and Polterovich [**11**], and we exhibit a Hamiltonian isotopy between the two. In the second section, we focus on the case of $\mathbb{CP}^2$, recall the constructions of Chekanov and Schlenk [**7, 9**], and Biran and Cornea [**5**], and prove the existence of the Hamiltonian isotopy in this case. The third section deals with the case of $\mathbb{S}^2 \times \mathbb{S}^2$.

2. The first constructions in $\mathbb{C}^2$

In this section, we recall the first constructions of monotone exotic tori in $\mathbb{C}^2$ as they will be useful to understand the constructions in the compact symplectic manifolds $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.

In order to simplify the computations and to avoid too many constants, we will use normalizations of the standard symplectic forms on T^*M for $M = \mathbb{R}^n$ or $M = \mathbb{S}^1$ (the circle $\mathbb{S}^1$ being identified with $\mathbb{R}/2\pi\mathbb{Z}$) such that the Liouville 1-form on T^*M is:

$$\lambda = \frac{1}{\pi}\Sigma p_i dq_i, \tag{2.1}$$

where (p, q) are the usual local coordinates on cotangent bundles, $q = (q_1, \dots, q_n)$ coordinates on the basis and $p = (p_1, \dots, p_n)$ coordinates in the fibers. For example, with such a normalization, the integral of the Liouville form along the circle centered at the origin and of radius r in $T^*\mathbb{R} \simeq \mathbb{R}^2$ is r^2.

2.1. The first description by Chekanov.

Chekanov has given a first description of exotic monotone tori in $\mathbb{R}^{2n}$ in [**6**]. We recall here his construction.

For any Lagrangian submanifold L in $\mathbb{R}^{2n}$, he has defined a Lagrangian submanifold $\Theta(L)$ in $\mathbb{R}^{2n+2}$ the following way. Consider the embedding

$$\begin{array}{rccc} i_n: & \mathbb{S}^1 \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^{n+1} \\ & (\theta, x_1, \dots, x_n) & \longmapsto & (\mathrm{e}^{x_1}\cos(\theta), \mathrm{e}^{x_1}\sin(\theta), x_2, \dots, x_n). \end{array}$$

Then $I_n = (i_n^*)^{-1}$ is a symplectic embedding of $T^*(\mathbb{S}^1 \times \mathbb{R}^n)$ into $T^*\mathbb{R}^{n+1}$. Denote by $\Theta(L)$ the image by the symplectic embedding I_n of the product of the zero section N_0 of T^*S^1 with the Lagrangian submanifold L. If L is monotone in $\mathbb{R}^{2n}$, then $\Theta(L)$ is monotone in $\mathbb{R}^{2n+2}$ with the same monotonicity constant.

We are here particularly interested in the case $n = 1$. We give the expression in coordinates of the map $I = I_1$ as it will be useful in the following. If $\theta \in \mathbb{S}^1, \tau \in T^*_\theta\mathbb{S}^1, x \in \mathbb{R}$ and $y \in T^*_x\mathbb{R}$, then

$$I((\tau, t), (y, x)) = (p_0, p_1, q_0, q_1)$$

with $(q_0, q_1) \in \mathbb{R}^2$ and $(p_0, p_1) \in T_{(q_0,q_1)}\mathbb{R}^2$ such that:

$$\begin{cases} q_0 &= \mathrm{e}^x\cos(\theta) \\ q_1 &= \mathrm{e}^x\sin(\theta) \\ p_0 &= \mathrm{e}^{-x}(-\tau\sin(\theta) + y\cos(\theta)) \\ p_1 &= \mathrm{e}^{-x}(\tau\cos(\theta) + y\sin(\theta)). \end{cases}$$

Identifying $T^*\mathbb{R}^2$ with $\mathbb{C}^2$ via the symplectomorphism

$$\begin{array}{ccc} T^*\mathbb{R}^2 & \longrightarrow & \mathbb{C}^2 \\ (q_0, q_1, p_0, p_1) & \longmapsto & (q_0 + \mathrm{i}p_0, q_1 + \mathrm{i}p_1), \end{array}$$

the map I can be written as the following embedding of $T^*(\mathbb{S}^1 \times \mathbb{R})$ into $\mathbb{C}^2$:

$$((\tau, \theta), (y, x)) \mapsto \begin{pmatrix} z_0 &= (\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y)\cos(\theta) - \mathrm{i}\tau\,\mathrm{e}^{-x}\sin(\theta) \\ z_1 &= (\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y)\sin(\theta) + \mathrm{i}\tau\,\mathrm{e}^{-x}\cos(\theta) \end{pmatrix}.$$

For $n = 1$, Chekanov's construction with L the circle centered at the origin of area $2r^2$ can be parameterized in $\mathbb{C}^2$ by:

$$\begin{cases} z_0 &= (\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y)\cos(\theta) \\ z_1 &= (\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y)\sin(\theta) \end{cases} \tag{2.2}$$

with $(x, y) \in L$. It is a monotone Lagrangian torus with monotonicity constant r^2. In [**6,** Theorem 4.2], Chekanov proved that this torus is not Hamiltonian isotopic to the Clifford torus (by versal deformations, using Ekeland–Hofer capacities and the displacement energy). In the following, we will call it Chekanov's torus and denote it $\Theta_{\mathrm{Ch}}(r^2)$.

2.2. The version by Eliashberg and Polterovich and its relation with the previous construction. Eliashberg and Polterovich have given in [**11**] an other description of an exotic monotone torus in $\mathbb{R}^4 \simeq \mathbb{C}^2$.

If D is a disc of $\mathbb{C}$ of area r^2, $r > 0$, which does not contain the origin, and $c = \{c(s) \mid s \in [0, 2\pi]\}$ is its boundary, parameterized by a smooth map $c\,:\,[0, 2\pi] \to \mathbb{C}$, then the torus defined as:

$$\left\{ \left(c(s)\,\mathrm{e}^{\mathrm{i}\theta}, c(s)\,\mathrm{e}^{-\mathrm{i}\theta}\right) \middle| \, \theta \in [0, 2\pi], s \in [0, 2\pi] \right\}$$

is a monotone torus of monotonicity constant r^2 denoted $\Theta_{\mathrm{EP}}(r^2)$. Eliashberg and Polterovich have proved that this torus is again not Hamiltonian isotopic to a split Lagrangian torus [**11,** Proposition 4.2.B] by counts of holomorphic discs with boundary along this torus.

Proposition 2.1. *For any positive radius r, the monotone torus $\Theta_{\mathrm{EP}}(r^2)$ is Hamiltonian isotopic to $\Theta_{\mathrm{Ch}}(r^2)$ in $\mathbb{C}^2$.*

We give here a detailed proof of this well known result as we will use it in the next sections. We fix a positive radius r and to simplify the notations, we will drop r from the notations in the proof.

This proposition can be deduced from a series of lemmata.

The exotic torus of Eliashberg and Polterovich is constructed as the orbit of some circle under a Hamiltonian circle action, so that it satisfies the following:

Lemma 2.2. *The torus Θ_{EP} is stable under the following action ρ_{EP} of the circle on $\mathbb{C}^2$: for $\theta \in [0; 2\pi]$ and $(z_0, z_1) \in \mathbb{C}^2$,*

$$\rho_{\mathrm{EP}}(\mathrm{e}^{\mathrm{i}\theta})(z_0, z_1) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} z_0 \\ \mathrm{e}^{-\mathrm{i}\theta} z_1 \end{pmatrix}$$

of Hamiltonian

$$H(z_0, z_1) = \frac{1}{2\pi}\left(|z_0|^2 - |z_1|^2\right).$$

More precisely, the torus Θ_{EP} is the orbit of the curve

$$C = \left\{ \begin{pmatrix} c(s) \\ c(s) \end{pmatrix} \middle| \, s \in [0, 2\pi] \right\}$$

under the action ρ_{EP}.

The exotic torus Θ_{Ch} satisfies a similar property:

Lemma 2.3. *The torus Θ_{Ch} is stable under the following action ρ_{Ch} of the circle on $\mathbb{C}^2$: for $\theta \in [0; 2\pi]$ and $(z_0, z_1) \in \mathbb{C}^2$,*

$$\rho_{\mathrm{Ch}}(\mathrm{e}^{\mathrm{i}\theta})(z_0, z_1) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

of Hamiltonian

$$H(z_0, z_1) = \frac{1}{\pi}\operatorname{Im}(z_0\bar{z}_1).$$

In particular, the parameterization (2.2) gives that the torus Θ_{Ch} is the orbit under the action ρ_{Ch} of the following curve of $\mathbb{C}^2$:

$$\left\{ \begin{pmatrix} \mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y \\ 0 \end{pmatrix} \middle| \, (x,y) \in L \right\}$$

where L is the circle of $\mathbb{C}$ centered in the origin of radius r. However, this torus can also be described as the orbit of the curve:

$$\Lambda = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}}(\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y) \\ -\frac{1}{\sqrt{2}}(\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y) \end{pmatrix} \middle| \, (x,y) \in L \right\}.$$

Lemma 2.4. *The two Hamiltonian actions ρ_{EP} and ρ_{Ch} are conjugate inside the special unitary group $SU(2)$.*

Proof. If we denote by P the matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{pmatrix},$$

then P is a matrix of the special unitary group such that for every $\theta \in [0; 2\pi]$,

$${}^t\bar{P} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

□

Now let Θ_{Ch}^P be the torus obtained as the orbit under the action ρ_{EP} of the curve $P\Lambda$ of $\mathbb{C}^2$. Then Θ_{Ch}^P is the image by the map P of the torus Θ_{Ch}.

Note moreover that the diffeomorphism P of $\mathbb{C}^2$ is a Hamiltonian diffeomorphism (because P is a matrix of $SU(2)$). This means that Θ_{Ch}^P is Hamiltonian isotopic to Θ_{Ch} and it is now sufficient for Proposition 2.1 to prove that Θ_{Ch}^P is Hamiltonian isotopic to Θ_{EP}.

Lemma 2.5. *The tori Θ_{Ch}^P and Θ_{EP} are Hamiltonian isotopic inside $\mathbb{C}^2$.*

To prove this, we will use the following lemma:

Lemma 2.6 (See [15]). *A Lagrangian isotopy is exact if and only if it can be extended to an ambient Hamiltonian isotopy.*

Recall [**15,** Section 6.1] that given a Lagrangian isotopy of a closed manifold N into a symplectic manifold (W, ω):

$$\Phi : N \times [0; 1] \to W,$$

the pull-back $\Phi^\star\omega$ is of the form $\alpha_s \wedge ds$ where $\{\alpha_s\}$ is a family of closed 1-forms on N. The Lagrangian isotopy is said to be exact if α_s is exact for all s.

Proof of Lemma 2.5. Thanks to Lemma 2.6, it is enough to prove that the tori Θ^P_{Ch} and Θ_{EP} are exact Lagrangian isotopic.

We note first that

$$\Lambda^P = P\Lambda = \left\{ \frac{1}{\sqrt{2}}(\mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y)\frac{1}{\sqrt{2}}(1+i)\begin{pmatrix}1\\1\end{pmatrix} \middle| (x,y)\in L \right\},$$

so that C and Λ^P are both curves lying in the diagonal of $\mathbb{C}^2$.

Moreover, as c is the boundary of the disc D, the integral of the Liouville form of $\mathbb{C}$ is equal to r^2 on the curve c.

On the other hand, let f be the map

$$\begin{array}{rccc} f: & T^*\mathbb{R} & \longrightarrow & \mathbb{C} \\ & (x,y) & \longmapsto & \mathrm{e}^x + \mathrm{i}\,\mathrm{e}^{-x}y. \end{array}$$

The map f is a symplectomorphism from $T^*\mathbb{R}$ onto its image

$$\{z\in\mathbb{C} \mid \Re e(z) > 0\}.$$

In particular, the integral of the Liouville form on the curve $f(L)$ is equal to the integral of the Liouville form on L. The rotation $z \mapsto \frac{1}{\sqrt{2}}(1+i)z$ preserves the Liouville form of $\mathbb{C}$ so that the integral of the Liouville form is also equal to r^2 on the curve

$$L^P = \frac{1}{\sqrt{2}}f(L)\frac{1}{\sqrt{2}}(1+i).$$

As L^P and c are two closed curves in $\mathbb{C}$ on which the integral of the Liouville form takes the same value (that is they are the boundary of domains of the same area), these two curves can be Hamiltonianly isotoped one into the other in $\mathbb{C}$ (this can also be seen by applying Lemma 2.6). We denote by $\varphi : \mathbb{C}\times[0;1] \longrightarrow \mathbb{C}$ a Hamiltonian isotopy of $\mathbb{C}$ such that:

$$\varphi_0 = \mathrm{Id} \text{ and } \varphi_1\big(L^P\big) = c.$$

As L^P and c are boundary of domains that do not contain the origin, the Hamiltonian isotopy can be chosen so that:

$$\forall\, t\in[0;1], \quad \varphi_t(0) = 0, \tag{2.3}$$

(in other words, $\varphi_t\big(L^P\big)$ never crosses the origin).

The two curves Λ^P and C are then Hamiltonian isotopic inside the diagonal $\Delta_{\mathbb{C}^2}$ of $\mathbb{C}^2$ via the isotopy (φ,φ), and we use this Hamiltonian isotopy and the circle action ρ_{EP} to construct an exact Lagrangian isotopy from Θ^P_{Ch}

to Θ_{EP}. This Lagrangian isotopy $\Phi : \Theta_{\mathrm{Ch}}^P \times [0;1] \longrightarrow \mathbb{C}^2$ is defined for $t \in [0;1]$ by:

$$\begin{array}{rccc} \Phi_t : & \Theta_{\mathrm{Ch}}^P & \longrightarrow & \mathbb{C}^2 \\ & \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} z \\ \mathrm{e}^{-\mathrm{i}\theta} z \end{pmatrix} & \longmapsto & \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} \varphi(z,t) \\ \mathrm{e}^{-\mathrm{i}\theta} \varphi(z,t) \end{pmatrix} \end{array}$$

and it is a well defined Lagrangian isotopy because of property (2.3). As φ is a Hamiltonian isotopy and ρ_{EP} is a Hamiltonian circle action, one can check that Φ is exact. Thanks to Lemma 2.6, this ends the proof of Lemma 2.5 and proves Proposition 2.1. ☐

3. In the complex projective plane

In this section, the projective complex plane $\mathbb{CP}^2$ will be endowed with the Fubini–Study symplectic form normalized so that the area of any complex projective line is

$$\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = 2.$$

This implies in particular that:

- The monotonicity constant of $\mathbb{CP}^2$ is $\frac{2}{3}$ and any monotone Lagrangian submanifold in $\mathbb{CP}^2$ has monotonicity constant $\frac{1}{3}$.
- With the normalization (2.1) of the symplectic form of $\mathbb{R}^4 \simeq \mathbb{C}^2$, the open ball $B(2)$ of radius $\sqrt{2}$ is symplectically embedded into $\mathbb{CP}^2$ via the canonical embedding:

$$\begin{array}{rccc} E_2 : & B(2) & \longrightarrow & \mathbb{CP}^2 \\ & (z_0, z_1) & \longmapsto & \left[z_0 : z_1 : \sqrt{2 - |z_0|^2 - |z_1|^2} \right] \end{array} \tag{3.1}$$

Because of the symplectic embedding (3.1), a way to construct a monotone torus into $\mathbb{CP}^2$ is to begin with a monotone torus of monotonicity constant $\frac{1}{3}$ in $B(2)$ and embed this torus into $\mathbb{CP}^2$ via E_2. This is for example a possible construction of the Clifford torus: one begins with the product of the two circles centered in the origin and of radius $\sqrt{\frac{2}{3}}$ in each factor of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$.

One could try to embed also Chekanov's torus as a monotone Lagrangian torus of $\mathbb{CP}^2$ via E_2. In order to get the right monotonicity constant, one should begin with L a circle of area $\frac{2}{3}$. Unfortunately, one can check that the torus $\Theta_{\mathrm{Ch}}(\frac{2}{3})$ does not sit in the open ball $B(2)$ (for example with the parameterization (2.2), for $s = 0$ and any θ, one has $\|z\| > \sqrt{2}$).

Eliashberg and Polterovich's torus Θ_{EP} as it has been defined in Section 2 is also not lying in the ball $B(2)$, so that it cannot be embedded inside the complex projective plane $\mathbb{CP}^2$. But this construction can be modified in order to define an exotic torus in $\mathbb{CP}^2$.

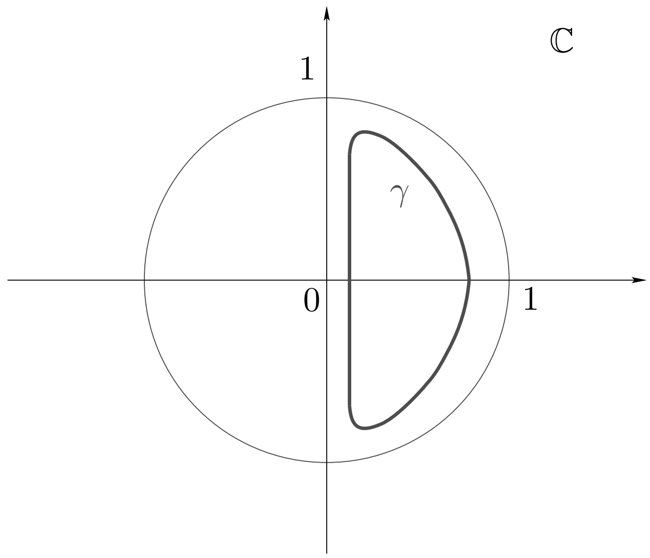

Figure 1. The curve γ.

3.1. Chekanov and Schlenk's torus in the complex projective plane. Instead of defining a torus as the orbit under ρ_{EP} of the boundary of a disc that does not contain the origin, one can also define a torus as the orbit of any closed embedded curve, which is the boundary of a domain that does not contain the origin of $\mathbb{C}$. The first torus of the family of monotone tori in $\mathbb{CP}^n$ defined by Chekanov and Schlenk [**7, 9**] is constructed this way.

To be precise, replace the circle c in the Eliashberg–Polterovich construction by the curve γ, which is the boundary of a domain of area $\frac{1}{3}$ sitting inside the disc of radius 1 centered in the origin of $\mathbb{C}$ and inside the half-plane of complex numbers of positive real part (see Figure 1).

Then the curve

$$\Gamma = \begin{pmatrix} \gamma \\ \gamma \end{pmatrix}$$

is lying inside the ball $B(2)$ and as the action of the cercle ρ_{EP} is by multiplication by matrices of $SU(2)$, the orbit of this curve is also entirely contained in $B(2)$. The torus constructed this way can then be embedded in $\mathbb{CP}^2$ through the canonical embedding (3.1) and will be denoted Θ_{CS}. Chekanov and Schlenk have proved (see [**8, 9**])) that this torus is not Hamiltonian isotopic to the Clifford torus in $\mathbb{CP}^2$ and is not Hamiltonianly displaceable in $\mathbb{CP}^2$.

3.2. The torus of Biran and Cornea. The Biran–Cornea torus is defined using the Lagrangian circle bundle construction of Biran (see [**3, 4**]).

Let Σ be the quadric of $\mathbb{CP}^2$, given by the homogeneous equation:

$$z_0^2 + z_1^2 + z_2^2 = 0.$$

Then $(\mathbb{CP}^2, \omega, J; \Sigma)$ (where J is the standard complex structure on $\mathbb{CP}^2$) is a polarized Kähler manifold of degree 1 in the sense of [**2, 3**], which means that Σ is a smooth and reduced complex hypersurface of the Kähler manifold (W, ω, J) (with $W = \mathbb{CP}^2$), whose homology class $[\Sigma] \in H_2(W, \mathbb{Z})$ represents the Poincaré dual of $[\omega]$.

In the case of such a polarization, Biran proved in [**2**] that there exists an isotropic CW-complex $\Delta \in (W, \omega)$ whose complement $(W \backslash \Delta, \omega)$ is symplectomorphic to a standard symplectic disc bundle (E, ω_0) modeled on the normal bundle N_Σ of Σ in W and whose fibers have area 1.

The standard symplectic disc bundle is by definition the open unit disc subbundle of the complex line bundle $\pi : N_\Sigma \to \Sigma$ (with respect to some Hermitian metric), endowed with the symplectic structure ω_0 given by the following formula:

$$\omega_0 = \pi^\star \omega_{|\Sigma} + d(r^2 \alpha),$$

where r is the radial coordinate on the fibers of $E_\Sigma \to \Sigma$ defined using the Hermitian metric and α is a connection 1-form on $E_\Sigma \backslash \Sigma$ with curvature

$$d\alpha = -\pi^\star(\omega_\Sigma).$$

In the case Σ is the quadric above, Δ is $\mathbb{RP}^2$.

Biran and Cornea's torus is then constructed the following way (see [**5,** Section 6.4.1]): the quadric Σ is topologically a sphere, so that we can consider an equator $\mathcal{E}$ of Σ, that is a circle that divides the sphere into two discs of equal areas. Let now Θ_{BC} be the restriction to the equator $\mathcal{E}$ of the circle subbundle of radius $\sqrt{\frac{2}{3}}$ inside the disc bundle $E_\Sigma \to \Sigma$. This defines an exotic monotone torus in $\mathbb{CP}^2$ whose Floer homology (with $\mathbb{Z}/2$ coefficients) is trivial.

The following remark of Biran on the description of E_Σ for the quadric will be crucial in the proof that Θ_{BC} is Hamiltonian isotopic to Θ_{CS} in $\mathbb{CP}^2$. It was also used recently by Opshtein to prove a symplectic embedding Theorem (see [**14,** Theorem 4]).

Proposition 3.1. *Let a, b, and c be three real numbers, not identically zero, and let $D_{a,b,c}$ denote the projective line of $\mathbb{CP}^2$ defined by the homogeneous equation*

$$az_0 + bz_1 + cz_2 = 0. \tag{3.2}$$

The line $D_{a,b,c}$ intersects the quadric Σ in two complex conjugate points x and $\bar{x}$ and $\mathbb{RP}^2$ cuts this line along a circle in two complex conjugate discs that are the fibers of E_Σ in x and $\bar{x}$, respectively.

Proof. The projective line $D_{a,b,c}$ is a curve of degree 1 and intersects the curve Σ of degree 2 in two points. As the line and the quadric are defined by equations with real coefficients, their intersection points are complex conjugate. Moreover, they are distinct because the quadric has no real point.

Let $x = [x_0, x_1, x_2]$ be an intersection point of $D_{a,b,c}$ and Σ. The tangent space T_x at the quadric in this point x is defined by the homogeneous equation:

$$x_0 z_0 + x_1 z_1 + x_2 z_2 = 0. \tag{3.3}$$

In the complex linear space $\mathbb{C}^3$, the two equations (3.2) and (3.3) define two complex hyperplanes $\tilde{D}$ and $\tilde{T}$ respectively. They are distinct as (x_0, x_1, x_2) is not a multiple of a real vector. As a consequence, they have an intersection of complex dimension 1. However, the vector (x_0, x_1, x_2) of $\mathbb{C}^3$ is a non-zero vector in the intersection of $\tilde{D}$ and $\tilde{T}$ so that

$$\tilde{D} \cap \tilde{T} = \mathrm{Vect}_{\mathbb{C}}(x_0, x_1, x_2).$$

Moreover the vector $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$ belongs to $\tilde{D}$ and is orthogonal to (x_0, x_1, x_2) so that $\tilde{D}$ is the direct orthogonal sum of the two vector spaces spanned by (x_0, x_1, x_2) and $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$, respectively. Note also that the vector $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$ is in the Hermitian-orthogonal to the vector space $\tilde{T}$. Consequently, the complex projective lines $D_{a,b,c}$ and T_x are orthogonal for the induced Hermitian product on $\mathbb{CP}^2$.

The complex projective line $D_{a,b,c}$ is stable under complex conjugation and its invariant submanifold $D_{a,b,c} \cap \mathbb{RP}^2$ is a circle, which divides $D_{a,b,c}$ in two discs that are images of each other by the complex conjugation, and thus have the same area (equal to 1 with our conventions). Moreover, each disc contains only one of the intersection points with the quadric x or $\bar{x}$.

To be sure that the two discs above are exactly the fibers of E_Σ, one should be sure that the discs are "centered" in the intersection points. One way to prove this is to see that it is true in the case $a=1$, $b=0$ and $c=0$ and it can be extended to the other cases thanks to a transformation of $U(3)$. $\square$

Remark 3.2. Through any point of $\mathbb{C}^3$ passes a complex line defined by an equation with real coefficients (for dimensional reason, the Hermitian orthogonal supplement of a point must intersect $\mathbb{R}^3$). As a consequence, Proposition 3.1 describes the bundle E_Σ in any point of the quadric Σ.

3.3. Biran and Cornea's torus is Hamiltonian isotopic to Chekanov and Schlenk's torus.

Theorem 3.3. *The Biran and Cornea's torus is Hamiltonian isotopic to Chekanov and Schlenk's torus in* $\mathbb{CP}^2$.

Proof. First we also define a modified Chekanov torus using the curve γ. Let $\widetilde{\Gamma}$ be the curve

$$\widetilde{\Gamma} = \begin{pmatrix} \gamma \\ -\gamma \end{pmatrix}$$

and $\widetilde{\Theta}_{\mathrm{Ch}}$ the torus defined as the orbit of the curve $\widetilde{\Gamma}$ under the action of ρ_{Ch}. Note that as the curve γ is Hamiltonian isotopic to the curve L in $\mathbb{C}$, the same proof as the one of Lemma 2.5 proves that $\widetilde{\Theta}_{\mathrm{Ch}}$ is Hamiltonian isotopic to Θ_{Ch} in $\mathbb{C}^2$.

The same way as for Θ_{CS}, because $\widetilde{\Gamma}$ is contained in $B(2)$ and because the action of the cercle ρ_{Ch} is by multiplication by matrices of $SU(2)$, the torus $\widetilde{\Theta}_{\mathrm{Ch}}$ is entirely contained in $B(2)$ and can be embedded in $\mathbb{CP}^2$ as a monotone torus.

Let also $\widetilde{\Theta}^P_{\mathrm{Ch}}$ be the torus defined as the orbit under the action ρ_{EP} of the curve $P\Gamma$. As before,

$$\widetilde{\Theta}^P_{\mathrm{Ch}} = P\widetilde{\Theta}_{\mathrm{Ch}},$$

so that $\widetilde{\Theta}^P_{\mathrm{Ch}}$ and $\widetilde{\Theta}_{\mathrm{Ch}}$ are Hamiltonian isotopic. Moreover, the Hamiltonian isotopy is given by a family of matrices of $U(2)$ so that this isotopy is preserving the ball $B(2)$. As a consequence, this isotopy enables to define an isotopy between $\widetilde{\Theta}^P_{\mathrm{Ch}}$ and $\widetilde{\Theta}_{\mathrm{Ch}}$ inside the complex projective plane.

Now note that in $B(2)$,

$$\Theta_{\mathrm{CS}} = R_{-\frac{\pi}{4}}\widetilde{\Theta}^P_{\mathrm{Ch}}$$

where $R_{-\frac{\pi}{4}}$ is the rotation $\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$ in each factor of $\mathbb{C}^2$, so that Θ_{CS} is the image of $\widetilde{\Theta}^P_{\mathrm{Ch}}$ by a Hamiltonian diffeomorphism.

To prove the theorem, we will prove that $\widetilde{\Theta}_{\mathrm{Ch}}$ is Hamiltonian isotopic to Θ_{BC}.

The torus $\widetilde{\Theta}_{\mathrm{Ch}}$ can be defined in homogeneous coordinates of $\mathbb{CP}^2$ as the set:

$$\widetilde{\Theta}_{\mathrm{Ch}} = \left\{ [\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : \sqrt{2-2|\gamma(s)|^2}] \;\middle|\; \theta, s \in [0, 2\pi] \right\}.$$

It is Hamiltonian isotopic in $\mathbb{CP}^2$ to any torus obtained rotation inside the plane of the torus $\widetilde{\Theta}_{\mathrm{Ch}}$, and in particular to the torus

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\widetilde{\Theta}_{\mathrm{Ch}}$$

described in $\mathbb{CP}^2$ as the set

$$\widetilde{\Theta}'_{\mathrm{Ch}} = \left\{ [\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : i\sqrt{2-2|\gamma(s)|^2}] \;\middle|\; \theta, s \in [0, 2\pi] \right\}.$$

The latter can also be seen as the image of $\widetilde{\Theta}_{\mathrm{Ch}}$ by the following modified embedding of the ball $B(2)$ inside $\mathbb{CP}^2$:

$$\begin{array}{cccc} \widetilde{E}_2 : & B(2) & \longrightarrow & \mathbb{CP}^2 \\ & (z_0, z_1) & \longmapsto & \left[z_0 : z_1 : i\sqrt{2 - |z_0|^2 - |z_1|^2} \right]. \end{array} \tag{3.4}$$

It is thus enough to prove that the torus Θ_{BC} is Hamiltonian isotopic to $\widetilde{\Theta}'_{\mathrm{Ch}}$ in $\mathbb{CP}^2$ and for that purpose, it will be more convenient to work with the embedding $\widetilde{E}_2$.

Let Z be the cylinder of $T^*\mathbb{S}^1 \times \{0\} \subset T^*(\mathbb{S}^1 \times \mathbb{R})$ defined as:

$$Z = \left\{(\theta, \tau, 0, 0) \in T^*(\mathbb{S}^1 \times \mathbb{R}^n) \mid |\tau| < 1\right\}.$$

The image of this cylinder by Chekanov's map I can be parameterized in $T^*\mathbb{R}^2$ by:

$$\begin{cases} q_0 = \cos(\theta), \\ q_1 = \sin(\theta), \\ p_0 = -\tau \sin(\theta), \\ p_1 = \tau \cos(\theta), \end{cases}$$

for $|\tau| < 1$ and in $\mathbb{C}^2$ by:

$$\begin{pmatrix} z_0 = \cos(\theta) - \mathrm{i}\tau \sin(\theta) \\ z_1 = \sin(\theta) + \mathrm{i}\tau \cos(\theta) \end{pmatrix}.$$

As $|\tau| < 1$, any point in the image by I of the cylinder Z is lying inside the ball $B(2)$. Its image $I(Z)$ under the embedding (3.4) is parameterized by:

$$\{[\cos(\theta) - \mathrm{i}\tau \sin(\theta) : \sin(\theta) + \mathrm{i}\tau \cos(\theta) : \mathrm{i}\sqrt{1 - \tau^2}] \mid s \in [0, 2\pi], \tau \in (-1, 1)\}$$

Lemma 3.4. *The quadric Σ is the union of the image $I(Z)$ and the two points "at infinity" $[1 : i : 0]$ and $[1 : -i : 0]$.*

Proof. Any point of $I(Z)$ satisfies the equation of the quadric. Moreover, when τ tends to $\varepsilon \in \{-1; 1\}$, the point of homogeneous coordinates

$$\left[\cos(\theta) - \mathrm{i}\tau \sin(\theta) : \sin(\theta) + \mathrm{i}\tau \cos(\theta) : \mathrm{i}\sqrt{1 - \tau^2}\right]$$

tends to the point $[1 : \varepsilon : 0]$, so that $I(Z)$ can be compactified in the closed surface Σ. □

Lemma 3.5. *The image by I of the zero section $\tau = 0$ is an equator of the quadric.*

Indeed, the zero section $\tau = 0$ of $T^*\mathbb{S}^1$ cuts the cylinder Z in two pieces of equal area so that its image cuts the quadric in two discs of same area 1, one disc containing the point $[1 : i : 0]$ and the other one the point $[1 : -i : 0]$. In the homogeneous coordinates of $\mathbb{CP}^2$, this equator can be parameterized by:

$$[\cos(\theta) : \sin(\theta) : i],\ \theta \in [0; 2\pi].$$

Lemma 3.6. *For a fixed $\theta \in \mathbb{R}$, the curve*

$$\{[\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : i\sqrt{2 - 2|\gamma(s)|^2}] \mid s \in [0, 2\pi]\}$$

lies inside the fiber of E_Σ at the point $[\cos(\theta) : \sin(\theta) : i]$.

Moreover, this curve enclose a domain of area $\frac{2}{3}$ inside the fiber at the base point $[\cos(\theta) : \sin(\theta) : i]$.

Proof. Thanks to Proposition 3.1, we know how to describe the disc bundle E_Σ at a point of the quadric as soon as we have the equation of a real projective line containing this point. In the case of a point $[\cos(\theta) : \sin(\theta) : i]$ on the equator of the Σ, it lies on the real line of homogeneous equation

$$\sin(\theta)z_0 - \cos(\theta)z_1 = 0.$$

Because of Proposition 3.1, this means that the fibers of the disc bundle E_Σ in the points

$$[\cos(\theta) : \sin(\theta) : i] \text{ and } [\cos(\theta) : \sin(\theta) : -i] = [\cos(\theta + \pi) : \sin(\theta + \pi) : i]$$

are the two discs of the projective line N_θ of equation

$$\sin(\theta)z_0 - \cos(\theta)z_1 = 0$$

separated by the curve $N_\theta \cap \mathbb{RP}^2$.

Any point of the curve

$$\{[\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : i\sqrt{2 - 2|\gamma(s)|^2}] \mid s \in [0, 2\pi]\}$$

belongs to the line N_θ. Moreover, this curve does not intersect $\mathbb{RP}^2$ as the curve γ does not intersect neither the imaginary axes nor the circle of radius 1. Therefore, this curve is entirely contained in one of the fiber of the disc bundle E_Σ. Actually, the image of the half disc

$$\{z \in \mathbb{C} \mid \Re e(z) > 0 \text{ and } |z| < 1\}$$

by the map

$$z \mapsto \{[\cos(\theta)\sqrt{2}z : \sin(\theta)\sqrt{2}z : i\sqrt{2 - 2|z|^2}]\}$$

is contained in N_θ and does not intersect $\mathbb{RP}^2$. Moreover the area of the image of the half disc is equal to 1 (that is twice the area of the original half disc) so that it is the entire fiber of E_Σ at the point $[\cos(\theta) : \sin(\theta) : i]$, image of the complex number $\frac{1}{\sqrt{2}}$. The curve γ is enclosing a domain of area $\frac{1}{3}$, and consequently the curve

$$\{[\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : i\sqrt{2 - 2|\gamma(s)|^2}] \mid s \in [0, 2\pi]\}$$

is enclosing a domain of area $\frac{2}{3}$. $\square$

To end the proof, one can see that in any fiber of the normal bundle, the curve

$$\{[\cos(\theta)\sqrt{2}\gamma(s) : \sin(\theta)\sqrt{2}\gamma(s) : i\sqrt{2-2|\gamma(s)|^2}] \mid s \in [0, 2\pi]\}$$

is isotopic to the circle of area $\frac{2}{3}$ (used to construct the torus of Biran and Cornea). In order to construct a Lagrangian isotopy between $\widetilde{\Theta}'_{\mathrm{Ch}}$ and Θ_{BC}, one can use the isotopy defined for the fiber of the point corresponding to $\theta = 0$ and then extend this isotopy by the action of the circle ρ_{Ch}. Lemma 2.6 enables then to extend this exact Lagrangian isotopy into a Hamiltonian isotopy of $\mathbb{CP}^2$. $\square$

4. In the product of two two-dimensional spheres

Let us now consider $W = \mathbb{S}^2 \times \mathbb{S}^2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with the symplectic form

$$\omega_W = \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}}$$

where ω_{FS} is normalized such that

$$\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = 1,$$

which means that $\mathbb{CP}^1$ is symplectomorphic to the sphere of radius $\frac{1}{2}$ in $\mathbb{R}^3$ with our conventions. With this normalization, the product of the two spheres is monotone with monotonicity constant

$$K_W = \frac{1}{2}.$$

As a consequence, the monotonicity constant of any monotone Lagrangian submanifold in this product is

$$K_L = \frac{1}{4}.$$

In this symplectic manifold, one has also the two corresponding constructions of exotic monotone Lagrangian torus.

4.1. Chekanov and Schlenk's torus in the product of spheres.

For the construction by Chekanov and Schlenk, one has this time to begin with a curve γ enclosing a domain of area $\frac{1}{4}$ in the complex plane inside the half-disc of radius 1 and positive real part.

With this choice of curve γ the torus of $\mathbb{C}^2$ obtained by action of ρ_{EP}

$$\left\{ \left(\gamma(s)\,\mathrm{e}^{\mathrm{i}\theta}, \gamma(s)\,\mathrm{e}^{-\mathrm{i}\theta}\right) \middle| \, \theta \in [0, 2\pi], s \in [0, 2\pi] \right\}$$

is contained into the product $B(1) \times B(1)$, where $B(1)$ is the ball of radius 1 in $\mathbb{C}$. As this product can be symplectically embedded into the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ via

$$
\begin{array}{rccc}
E_{1,1}: & B(1) \times B(1) & \longrightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\
& (z_0, z_1) & \longmapsto & \left(\left[z_0 : \sqrt{1-|z_0|^2} \right], \left[z_1 : \sqrt{1-|z_1|^2} \right] \right),
\end{array} \tag{4.1}
$$

this construction produces a monotone Lagrangian torus still denoted Θ_{CS} in $\mathbb{CP}^1 \times \mathbb{CP}^1$, which is not Hamiltonian isotopic to the Clifford torus and non-displaceable (see [**8, 9**]).

4.2. The torus constructed by Biran's circle bundle construction. The construction of Biran involves again a polarization. This time, the symplectic hypersurface Σ is the diagonal sphere described as

$$\Sigma = \{(x, x) \in \mathbb{S}^2 \times \mathbb{S}^2 |\ x \in \mathbb{S}^2\}$$

in $\mathbb{S}^2 \times \mathbb{S}^2$, or as the hypersurface of $\mathbb{CP}^1 \times \mathbb{CP}^1$ satisfying the homogeneous equation:

$$z_0 w_1 = z_1 w_0,$$

where $[z_0 : w_0]$ are the homogeneous coordinates on the first copy of $\mathbb{CP}^1$ and $[z_1 : w_1]$ on the second.

The total space of the standard symplectic disc bundle (E_Σ, ω_0) modeled on the normal bundle N_Σ of Σ in W and whose fibers have area 1 is in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ symplectomorphic to the complement of the antidiagonal:

$$\Delta = \{(x, -x) \in \mathbb{S}^2 \times \mathbb{S}^2 |\ x \in \mathbb{S}^2\}$$

described in $\mathbb{CP}^1 \times \mathbb{CP}^1$ as the Lagrangian submanifold of homogeneous equation:

$$z_0 \bar{z}_1 + w_0 \bar{w}_1 = 0.$$

The exotic torus, still denoted Θ_{BC}, is now defined as the restriction of the circle subbundle of radius $\frac{1}{\sqrt{2}}$ over an equator of Σ.

This torus is known to be equal to the exotic monotone Lagrangian torus of Entov and Polterovich ([**12,** Example 1.22], see also the study in [**10,** Section 2]):

$$K = \left\{ (x, y)) \in \mathbb{S}^2 \times \mathbb{S}^2 \;\middle|\; x_3 + y_3 = 0, x_1 y_1 + x_2 y_2 + x_3 y_3 = -\frac{1}{2} \right\},$$

with $\mathbb{S}^2$ being in their conventions the sphere of radius 1 of $\mathbb{R}^3$, the symplectic form being rescaled.

It is also equal to the torus defined in the cotangent bundle of $\mathbb{S}^2$ by the geodesic flow (see [**1**]) and embedded in a suitable way in a Weinstein's neighborhood of the Lagrangian sphere Δ.

4.3. The two constructions are Hamiltonian isotopic.

Theorem 4.1. *Biran's exotic torus is Hamiltonian isotopic to Chekanov and Schlenk's exotic torus in* $\mathbb{S}^2 \times \mathbb{S}^2$.

Proof. In the case of the product of spheres, we cannot use the corresponding modified Chekanov's torus as in the case of the complex projective plane because this torus does not embed into the product of balls of radius 1.

However, in $\mathbb{C}^2$, the modified Chekanov's torus $\widetilde{\Theta}_{\mathrm{Ch}}$ sits in the normal bundle of the image $I(Z)$ of the cylinder Z of the original description, and more precisely in the fibers over the image of the zero section. We also know from Section 3.1 that Θ_{CS} is the image of $\widetilde{\Theta}_{\mathrm{Ch}}$ by the composition of the rotations $R_{-\frac{\pi}{4}}$ and the Hamiltonian diffeomorphism described by the matrix P. Therefore, it also sits in the normal bundle of a surface Q, namely the image of $I(Z)$ by $R_{-\frac{\pi}{4}}P$.

This surface Q can be parameterized, with the coordinates (τ, θ) coming from the original coordinates on $T^*\mathbb{S}^1$ by:

$$\begin{pmatrix} z_0 & = & \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tau\right) \mathrm{e}^{\mathrm{i}(\theta+\pi/4)} \\ z_1 & = & \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tau\right) \mathrm{e}^{-\mathrm{i}(\theta+\pi/4)} \end{pmatrix}.$$

This surface is, as expected, invariant by the action of ρ_{EP} and the torus Θ_{CS} (for the moment considered in $\mathbb{C}^2$) lies in the normal bundle of the surface along the curve $\tau = 0$. Moreover, as $\tau \in (0, 1)$, Q is contained in the product $B(1) \times B(1)$ and can be embedded into $\mathbb{S}^2 \times \mathbb{S}^2$.

We would like to relate this surface to the diagonal Σ as in the proof of the $\mathbb{CP}^2$ case. However, the diagonal Σ is not invariant by the action of the cercle ρ_{EP} but by the following action of the circle, which can be written on $\mathbb{C}^2$:

$$\rho_{\mathrm{BC}}(\mathrm{e}^{\mathrm{i}\theta})(z_0, z_1) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

This action cannot be conjugate into $SU(2)$, but it will be enough to conjugate the two circle actions inside $SO(3) \times SO(3)$ using the description of W as the product of two spheres of $\mathbb{R}^3$. As the rotation $\mathrm{e}^{\mathrm{i}\theta}$ on $\mathbb{CP}^1$ corresponds to the rotation of matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

one sees that ρ_{EP} and ρ_{BC} are conjugate in $SO(3) \times SO(3)$, for example by the pair (P_1, P_2) where P_1 is the identity matrix and

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In order to see the action of the diffeomorphism of matrix P_2 on $\mathbb{S}^2$ we need to embed symplectically the ball $B(1)$ in the sphere of radius $\frac{1}{2}$ of $\mathbb{R}^3$. Such a map can be given for example by

$$\begin{array}{rccc} E_1 : & B(1) & \longrightarrow & S^2 \\ & z = a + \mathrm{i}b & \longmapsto & \begin{pmatrix} \sqrt{1-|z|^2}\, a \\ \sqrt{1-|z|^2}\, b \\ \frac{1}{2} - |z|^2 \end{pmatrix}. \end{array}$$

The image by (P_1, P_2) of $(E_1, E_1)(Q)$ in $\mathbb{S}^2 \times \mathbb{S}^2$ is now invariant by the action ρ_{BC}, but is not the diagonal Σ (which was to be expected as the cylinder has an area equal to 4):

$$\widetilde{Q} = (P_1, P_2)((E_1, E_1)(Q)) = \left\{(X, Y) \in \mathbb{S}^2 \times \mathbb{S}^2\right\}$$

with X parameterized by

$$\begin{pmatrix} \sqrt{\frac{1}{2} + \tau - \frac{1}{2}\tau^2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tau\right)\cos(\theta + \pi/4) \\ \sqrt{\frac{1}{2} + \tau - \frac{1}{2}\tau^2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tau\right)\sin(\theta + \pi/4) \\ \tau - \frac{1}{2}\tau^2 \end{pmatrix}$$

and Y parameterized by

$$\begin{pmatrix} \sqrt{\frac{1}{2} - \tau - \frac{1}{2}\tau^2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tau\right)\cos(\theta + \pi/4) \\ \sqrt{\frac{1}{2} - \tau - \frac{1}{2}\tau^2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tau\right)\sin(\theta + \pi/4) \\ \tau + \frac{1}{2}\tau^2 \end{pmatrix}$$

for $s \in [0, 2\pi]$ and $\tau \in (-1, 1)$.

Now note that the image of the zero section (the curve $\tau = 0$) of the cylinder is an equator of the diagonal sphere. Moreover, the tangent bundle of $\widetilde{Q}$ along the equator is equal to the tangent bundle of the sphere Σ. This means that the original curve γ that was sitting in the normal bundle of Q is, after these Hamiltonian isotopies, a curve on which the integral of the Liouville form is equal to $\frac{1}{2}$ sitting in the normal bundle of Σ along the equator. It is lying in the disc bundle of area one as it does not intersect the antidiagonal Δ after the Hamiltonian diffeomorphism. One finishes the proof using Lemma 2.6 as in the case of $\mathbb{CP}^2$. $\square$

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Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge
CB3 0WB
UK
E-mail address: A.Gadbled@dpmms.cam.ac.uk

Received 09/20/2011, accepted 11/10/2011

This work follows from a question raised by Felix Schlenk. I wish to thank him, Paul Biran, Octav Cornea and Leonid Polterovich for their explanations and remarks on these constructions. I also want to thank Kai Cieliebak, Ivan Smith and Maksim Maydanskiy for helpful discussions. This work was carried out while the author held a postdoctoral fellowship attached to the SNF project "Complexity and recurrence in Hamiltonian systems" at the University of Neuchâtel, a Mathematical Sciences Research Institute postdoctoral fellowship, and a postdoctoral fellowship at the University of Cambridge funded by the European Research Council grant ERC-2007-StG-205349. The author wishes to acknowledge these three institutions for their hospitality and stimulating research atmosphere.