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TAMED SYMPLECTIC FORMS AND STRONG KÄHLER WITH TORSION METRICS

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Symplectic forms taming complex structures on compact manifolds are strictly related to Hermitian metrics having the fundamental form $\partial\overline{\partial}$-closed, i.e., to strong Kähler with torsion (SKT) metrics. It is still an open problem to exhibit a compact example of a complex manifold having a tamed symplectic structure but non-admitting Kähler structures. We show some negative results for the existence of symplectic forms taming complex structures on compact quotients of Lie groups by discrete subgroups. In particular, we prove that if M is a nilmanifold (not a torus) endowed with an invariant complex structure J, then (M, J) does not admit any symplectic form taming J. Moreover, we show that if a nilmanifold M endowed with an invariant complex structure J admits an SKT metric, then M is at most 2-step. As a consequence we classify eight-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric.

1. Introduction

Let (M, Ω) be a compact $2n$-dimensional symplectic manifold. An almost complex structure J on M is said to be *tamed* by Ω if

$$\Omega(X, JX) > 0$$

for any non-zero vector field X on M. When J is a complex structure (i.e., J is integrable) and Ω is tamed by J, the pair (Ω, J) has been called a *Hermitian-symplectic structure* in [**39**]. Although any symplectic structure always admits tamed almost complex structures, it is still an open problem to find an example of a compact complex manifold admitting a Hermitian-symplectic structure, but no Kähler structures. From [**28**, **39**] there exist no examples in dimension 4. Moreover, the study of tamed symplectic structures in dimension 4 is related to a more general conjecture of Donaldson (see for instance [**7**, **28**, **44**]).

A natural class where to search examples of Hermitian-symplectic manifolds is provided by compact quotients of Lie groups by discrete subgroups, since this set typically contains examples of manifolds admitting both complex structures and symplectic structures, but no Kähler structures. Indeed, it is well known that a *nilmanifold*, i.e., a compact quotient of a nilpotent Lie group by a discrete subgroup, cannot admit any Kähler metric unless it is a torus (see for instance [**3**, **23**]). Moreover, in the case of *solvmanifolds*, i.e., compact quotients of solvable Lie groups by discrete subgroups, Hasegawa showed in [**24**, **25**] that a solvmanifold has a Kähler structure if and only if it is covered by a finite quotient of a complex torus, which has the structure of a complex torus bundle over a complex torus.

In this paper we show some negative results for the existence of Hermitian-symplectic structures on compact quotients of Lie groups by discrete subgroups. First of all we show that Hermitian-symplectic structures are strictly related to a "special" type of Hermitian metric, the Kähler with torsion (SKT) one. We recall (see [**17**]) that a J-Hermitian metric g on a complex manifold (M, J) is called SKT (*Strong Kähler with Torsion*) or *pluriclosed* if the fundamental form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ satisfies

$$\partial\overline{\partial}\omega = 0.$$

For complex surfaces a Hermitian metric satisfying the SKT condition is *standard* in the terminology of Gauduchon [**19**] and one standard metric can be found in the conformal class of any given Hermitian metric on a compact manifold. However, the theory is different in higher dimensions. SKT metrics have a central role in type II string theory, in two-dimensional supersymmetric σ-models (see [**17**, **40**]) and they also have relations with generalized Kähler geometry (see for instance [**1**, **5**, **14**, **17**, **22**, **26**]).

SKT metrics have been recently studied by many authors. For instance, new simply connected compact strong KT examples have been recently constructed by Swann in [**41**] via the twist construction. Moreover, in [**15**] it has been shown that the blow-up of an SKT manifold M at a point or along a compact submanifold admits an SKT metric.

For real Lie groups admitting SKT metrics there are only some classification results in dimensions 4 and 6. More precisely, six-dimensional SKT nilpotent Lie groups have been classified in [**13**] and a classification of SKT solvable Lie groups of dimension 4 has been recently obtained in [**29**].

We outline our paper. In Section 2, we show that the existence of a Hermitian-symplectic structure on a complex manifold (M, J) is equivalent to the existence of a J-compatible SKT metric g whose fundamental form ω satisfies $\partial\omega = \overline{\partial}\beta$ for some ∂-closed $(2,0)$-form β (see Proposition 2.1). This result allows us to write down a natural obstruction to the existence of a Hermitian-symplectic structure on a compact complex manifold (see Lemma 2.1).

Let G/Γ a compact quotient of a simply connected Lie group G by a discrete subgroup Γ. By an invariant complex (resp. symplectic) structure on G/Γ we will mean one induced by a complex (resp. symplectic) structure on the Lie algebra $\mathfrak{g}$ of G. In Section 3, we start by showing that the existence of a symplectic form taming an invariant complex structure J on G/Γ implies the existence of an invariant one (see Proposition 3.2). This can be used to prove the following:

Theorem 1.1. *Let $(M = G/\Gamma, J)$ be a compact quotient of a Lie group by a discrete subgroup endowed with an invariant complex structure J. Assume that the center ξ of the Lie algebra $\mathfrak{g}$ associated to G satisfies*

$$J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}, \tag{1.1}$$

then (M, J) does not admit any symplectic form taming J.

In the case of a simply connected nilpotent Lie group G if the structure equations of its Lie algebra $\mathfrak{g}$ are rational, then there exists a discrete subgroup Γ of G for which the quotient G/Γ is compact [**31**]. A general classification of real nilpotent Lie algebras exists only for dimension less or equal than 7 and 6 is the highest dimension in which there do not exist continuous families [**20**, **30**]. It turns out that in dimension 6 by [**13**] any six-dimensional non-abelian nilpotent Lie algebra admitting an SKT metric is 2-step nilpotent. Moreover, by [**41**] any six-dimensional SKT nilmanifold is the twist of the Kähler product of the two torus $\mathbb{T}^4$ and $\mathbb{T}^2$ by using two integral 2-forms supported on $\mathbb{T}^4$. In this paper, we extend the previous result to any dimension.

Theorem 1.2. *Let $(M = G/\Gamma, J)$ be a nilmanifold (not a torus) endowed with an invariant complex structure J. Assume that there exists a J-Hermitian* SKT *metric g on M. Then $\mathfrak{g}$ is 2-step nilpotent and M is a total space of a principal holomorphic torus bundle over a torus.*

Moreover, we show that an invariant SKT metric compatible with an invariant complex structure on a nilmanifold is always standard (Corollary 3.1).

Theorems 1.1 and 1.2 will be used to prove the following:

Theorem 1.3. *Let $(M = G/\Gamma, J)$ be a nilmanifold (not a torus) with an invariant complex structure. Then (M, J) does not admit any symplectic form taming J.*

It can be observed that there exist examples of nilpotent Lie algebras not satisfying condition (1.1) (see Example 3.1). Hence, Theorem 1.3 cannot be directly deduced from Theorem 1.1.

In dimension 8, some examples of nilpotent Lie algebras endowed with an SKT metric have been found in [**16**, **32**, **36**], but there is no general classification result. In the last part of the paper, we describe the eight-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric. In dimension 6 by [**13**], the existence of a strong KT structure on a nilpotent Lie algebra $\mathfrak{g}$ depends only on the complex structure of $\mathfrak{g}$. We show that in dimension 8 the previous property is no longer true.

A good quaternionic analog of Kähler geometry is given by *hyper-Kähler with torsion* (shortly HKT) geometry. This geometry was introduced by Howe and Papadopoulos [**27**] and later studied for instance in [**21**]. Nilmanifolds also provides examples of compact HKT manifolds (see [**9**, **21**]). In the last section, we investigate eight-dimensional SKT nilmanifolds that also admit HKT structures.

2. Link with SKT metrics

The study of SKT metrics is strictly related to the study of the geometry of the Bismut connection. Indeed, any Hermitian structure (J, g) admits a unique connection ∇^B preserving g and J and such that the tensor

$$c(X, Y, Z) = g(X, T^B(Y, Z))$$

is totally skew-symmetric, where T^B denote the torsion of ∇^B (see [**18**]). This connection was used by Bismut in [**4**] to prove a local index formula for the Dolbeault operator for non-Kähler manifolds. The torsion 3-form c is related to the fundamental form ω of g by

$$c(X, Y, Z) = -d\omega(JX, JY, JZ)$$

and it is well known that $\partial\overline{\partial}\omega = 0$ is equivalent to the condition

$$dc = 0.$$

By [**38**] it turns out that Hermitian-symplectic structures are related to static solutions of a new metric flow on complex manifolds called *Hermitian curvature flow*. Indeed, Streets and Tian constructed a flow using the Ricci tensor associated to the Chern connection instead of the Levi–Civita connection. In this way they obtained an elliptic flow and proved some results on short-time existence of solutions to this flow and on stability of Kähler–Einstein metrics. A modified Hermitian curvature flow was used in [**39**] to study the evolution of SKT metrics, showing that the existence of some particular type of static SKT metrics implies the existence of a Hermitian-symplectic structure on the complex manifold. Static SKT metrics on Lie groups have also been recently studied in [**11**].

The aim of this section is to point out a natural link between Hermitian-symplectic structures (also called *holomorphic-tamed* by de Bartolomeis and Tomassini [**42**]) and SKT metrics. We have the following

Proposition 2.1. *Let (M, J) be a complex manifold. Giving a Hermitian-symplectic structure Ω on (M, J) is equivalent to assigning an* SKT *metric*

g such that the associated fundamental form ω satisfies

$$\partial\omega = \overline{\partial}\beta \tag{2.1}$$

for some ∂-closed $(2,0)$-form β.

Proof. Let Ω be a Hermitian-symplectic structure on (M, J), then:

$$d\Omega = 0, \quad \Omega^{1,1} > 0,$$

where $\Omega^{1,1}$ denotes the $(1,1)$-component of Ω. We can write the real 2-form Ω as

$$\Omega = \omega - \beta - \overline{\beta}.$$

where $\beta = -\Omega^{2,0}$ is the opposite of the $(2,0)$-component of Ω and $\omega = \Omega^{1,1}$ is the $(1,1)$-component. Then

$$d\Omega = 0 \Longleftrightarrow \begin{cases} \partial\omega = \overline{\partial}\beta, \\ \partial\beta = 0, \end{cases}$$

and ω is the fundamental form associated to an SKT metric.

Conversely if g is an SKT metric whose fundamental form ω satisfies (2.1) for some ∂-closed $(2,0)$-form β, then $\Omega = \omega - \beta - \overline{\beta}$ defines a Hermitian-symplectic structure on (M, J). □

In order to write down a natural obstruction to the existence of a Hermitian-symplectic structure, we recall some basic facts about Hermitian Geometry. Let (M, J, g) be a $2n$-dimensional compact Hermitian manifold and let ω be the associated fundamental form. The complex structure J is extended to act on r-forms by

$$J\alpha(X_1, \ldots, X_r) = (-1)^r \alpha(JX_1, \ldots, JX_r).$$

With respect to this extension J commutes with the duality induced by g. On M the Hodge star operator

$$*\colon \Lambda^{r,s}(M) \to \Lambda^{n-s,n-r}(M)$$

is defined by the relation

$$\alpha \wedge *\overline{\beta} = g(\alpha, \overline{\beta})\,\frac{\omega^n}{n!}.$$

Moreover, g induces the L^2 product

$$(\alpha, \beta) = \int_M \alpha \wedge *\overline{\beta}\,\frac{\omega^n}{n!}$$

and the differential operators

$$\partial^*\colon \Lambda^{r,s}(M) \to \Lambda^{r-1,s}(M), \quad \overline{\partial}^*\colon \Lambda^{r,s}(M) \to \Lambda^{r,s-1}(M)$$

defined as

$$\partial^* = - * \overline{\partial} *, \quad \overline{\partial}^* = - * \partial * .$$

It is well known that

$$(\partial^*\alpha, \beta) = (\alpha, \partial\beta), \quad (\overline{\partial}^*\alpha, \beta) = (\alpha, \overline{\partial}\beta).$$

Lemma 2.1. *Let (M, Ω, J) be a compact Hermitian-symplectic manifold and let η be a $(2,1)$-form. Fix an arbitrary Hermitian metric g on (M, J) with associated Hodge star operator $*$. Then*

$$(\partial^*\eta, \Omega^{1,1}) \neq 0 \Longrightarrow \overline{\partial}^*\eta \neq 0.$$

Proof. We have

$$(\partial^*\eta, \Omega^{1,1}) = (\eta, \partial\Omega^{1,1}).$$

By Proposition 2.1

$$\partial\Omega^{1,1} = \overline{\partial}\beta,$$

for some ∂-closed $(2,0)$-form. Therefore,

$$(\partial^*\eta, \Omega^{1,1}) = (\eta, \overline{\partial}\beta) = (\overline{\partial}^*\eta, \beta),$$

which implies the statement. □

3. Main results

In this section, we prove the results stated in the introduction.

We recall that an almost complex structure J on a real Lie algebra $\mathfrak{g}$ is said to be *integrable* if the Nijenhuis condition

$$[X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] = 0$$

holds for all $X, Y \in \mathfrak{g}$. By a complex structure on a Lie algebra, we will always mean an integrable one. We can give the following:

Definition 3.1. Let $(\mathfrak{g}, J)$ be a real Lie algebra endowed with a complex structure J. A Hermitian-symplectic structure on $(\mathfrak{g}, J)$ is a symplectic form Ω, which tames J, that is a real 2-form Ω such that

$$\Omega(X, JX) > 0,$$

for any non-zero vector $X \in \mathfrak{g}$.

In order to prove Theorem 1.1, we consider the following:

Lemma 3.1. *Let $\mathfrak{g}$ be a real Lie algebra endowed with a complex structure J such that $J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, where ξ denotes the center of $\mathfrak{g}$. Then $(\mathfrak{g}, J)$ cannot admit any Hermitian-symplectic structure.*

Proof. Suppose that $(\mathfrak{g}, J)$ admits a Hermitian-symplectic structure Ω, then by definition Ω satisfies

$$d\Omega(X, Y, Z) = -\Omega([X, Y], Z) - \Omega([Y, Z], X]) - \Omega([Z, X], Y) = 0,$$

for every $X, Y, Z \in \mathfrak{g}$ and

$$\Omega(X, JX) > 0, \quad \forall \ X \neq 0.$$

Therefore, in particular one has:

$$d\Omega(X, Y, Z) = -\Omega([Y, Z], X) = 0, \quad \forall X \in \xi, \forall \ Y, Z \in \mathfrak{g},$$

or, equivalently, that

$$\Omega(X, W) = 0,$$

for any $X \in \xi$ and $W \in [\mathfrak{g}, \mathfrak{g}]$. Assume that there exists a non-zero $W \in [\mathfrak{g}, \mathfrak{g}] \cap J\xi$; then $JW \in \xi$ and one has $\Omega(W, JW) = 0$, which is not possible since J is tamed by Ω. □

If G is a real Lie group with Lie algebra $\mathfrak{g}$, then assigning a left-invariant almost complex structure on G is equivalent to choosing an almost complex structure J on $\mathfrak{g}$. Such a J is integrable if and only if it is integrable as an almost complex structure on G. Therefore, a complex structure on $\mathfrak{g}$ induces a complex structure on G by the Newlander–Nirenberg theorem and G becomes a complex manifold. The elements of G act holomorphically on G by multiplication on the left but G is not a complex Lie group in general.

Let now $M = G/\Gamma$ be a compact quotient of a simply connected Lie group G by a uniform discrete subgroup Γ. By an *invariant* complex structure (resp. an *invariant* Hermitian-symplectic structure) on G/Γ we will mean a complex structure (resp. Hermitian-symplectic structure) induced by one on the Lie algebra $\mathfrak{g}$.

We can prove the following:

Lemma 3.2. *If* $(M = G/\Gamma, J)$ *with* J *an invariant complex structure admits a Hermitian-symplectic structure* Ω*, then it admits an invariant Hermitian symplectic structure* $\tilde{\Omega}$.

Proof. Suppose that there exists a non-invariant Hermitian-symplectic structure Ω, then by using the property that G has a bi-invariant volume form $d\mu$ (see [**33**]) and by applying the symmetrization process of [**12**] we can construct a new symplectic invariant form $\tilde{\Omega}$, defined by

$$\tilde{\Omega}(X, Y) = \int_{m \in M} \Omega_m(X_m, Y_m) \ d\mu,$$

for any left-invariant vector field X, Y. The 2-form $\tilde{\Omega}$ tames the complex structure J. □

Now are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $(M = G/\Gamma, J)$ be a compact quotient of a Lie group with a left-invariant complex structure. Assume that there exists a J-compatible Hermitian-symplectic structure on M. Then using Lemma 3.2, we can construct a left-invariant Hermitian-symplectic structure Ω on (G, J). In view of Lemma 3.1, we have $J\xi \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, as required. □

Consider a Lie algebra $(\mathfrak{g}, J, g)$ with a complex structure and a J-compatible inner product. Let ∇^B be the Bismut connection associated to (J, g). Then the inner product g is SKT if and only if the torsion 3-form $c(X, Y, Z) = g(X, T^B(Y, Z))$ of the Bismut connection ∇^B is closed. In view of [**9**], the Bismut connection ∇^B and the torsion 3-form c can be written in terms of Lie brackets as

$$(\nabla^B_X Y, Z) = \frac{1}{2}\{g([X, Y] - [JX, JY], Z) - g([Y, Z] + [JY, JZ], X) + g([Z, X] - [JZ, JX], Y)\}, \tag{3.1}$$

$$c(X, Y, Z) = -g([JX, JY], Z) - g([JY, JZ], X) - g([JZ, JX], Y), \tag{3.2}$$

for any $X, X, Z \in \mathfrak{g}$. We have the following:

Lemma 3.3. *Let $(\mathfrak{g}, J, g)$ be a Lie algebra with a complex structure and a J-compatible inner product. Let ξ be the center of $\mathfrak{g}$ and let $X \in \xi$. Then for any $Y \in \mathfrak{g}$*

$$dc(X, Y, JX, JY) = 2\left(\|[Y, JX]\|^2 - g([[JX, Y], JX], Y) - g([[Y, JY], JX], X)\right). \tag{3.3}$$

Proof. Let $X \in \xi$. Then since J is integrable and X belongs to the center, we have

$$[JX, JY] = J[JX, Y]. \tag{3.4}$$

Using (3.1) and (3.2), we have

$$\begin{aligned} dc(X, Y, JX, JY) &= -c([Y, JX], X, JY) + c([Y, JY], X, JX) \\ &\quad - c([JX, JY], X, Y) \\ &= -c([Y, JX], X, JY) + c([Y, JY], X, JX) \\ &\quad - c(J[JX, Y], X, Y) \\ &= g([J[Y, JX], JX], JY) + g([Y, JX], [Y, JX]) \\ &\quad - g([Y, J[Y, JX]], X) \\ &\quad - g([J[Y, JY], JX], JX) - g([[JX, Y], JX], Y) \\ &\quad + g([JX, JY], [JX, JY]) - g([JY, [JX, Y]], X). \end{aligned}$$

Now, by (3.4) we get $[J[Y, JX], JX] = J[[Y, JX], JX]$, so

$$g([J[Y, JX], JX], JY) - g([[JX, Y], JX], Y) = -2g([[JX, Y], JX], Y).$$

Moreover,

$$\begin{aligned} &g([Y, J[Y, JX]], X) + g([J[Y, JY], JX], JX) \\ &\quad + g([JY, [JX, Y]], X) = 2g([[Y, JY], JX], X) \end{aligned}$$

and therefore

$$dc(X,Y,JX,JY) = 2g([Y,JX],[Y,JX]) - 2g([[JX,Y],JX],Y) - 2g([[Y,JY],JX],X),$$

as required. $\square$

Now we consider the nilpotent case. We recall that a real Lie algebra $\mathfrak{g}$ is nilpotent if there exists a positive s such that for the descending central series $\{\mathfrak{g}^k\}_{k\geq 0}$ defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}], \quad \dots, \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1},\mathfrak{g}],$$

one has $\mathfrak{g}^s = \{0\}$ and $\mathfrak{g}^{s-1} \neq \{0\}$. The Lie algebra $\mathfrak{g}$ is therefore called s-step nilpotent and one has that $\mathfrak{g}^{s-1}$ is contained in the center ξ of $\mathfrak{g}$.

The next proposition gives an obstruction to the existence of an SKT inner product on a nilpotent Lie algebra with a complex structure.

Proposition 3.1. *Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra endowed with a complex structure J. If the center ξ is not J-invariant, then $(\mathfrak{g}, J)$ does not admit any* SKT *metric.*

Proof. Let g be an arbitrary J-compatible inner product on $\mathfrak{g}$ and let ∇^B be the associated Bismut connection. Assume that the center ξ is not J-invariant, then there exists $X \in \xi$ such that $JX \notin \xi$, or equivalently there exists $X \in \xi$ such that the map $ad_{JX} : \mathfrak{g} \longrightarrow \mathfrak{g}$ is not identically zero. But certainly the restriction $ad_{JX}|_{\mathfrak{g}^{s-1}}$ is zero because $\mathfrak{g}^{s-1} \subseteq \xi$. So there exists an integer k such that $ad_{JX}|_{\mathfrak{g}^k}$ is not identically zero and $ad_{JX}|_{\mathfrak{g}^{k+1}}$ is zero. Then we can choose $Y \in \mathfrak{g}^k$ such that $[JX,Y] \neq 0$ and $[[Y,Z],JX] = 0$, for each $Z \in \mathfrak{g}$; so, using (3.3) we have

$$dc(X,Y,JX,JY) = 2\,\|[Y,JX]\|^2 \neq 0$$

and g cannot be SKT. $\square$

Let J be a complex structure on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$. We introduce the ascending series $\{\mathfrak{g}_l^J\}_{l\geq 0}$ defined inductively by

$$\begin{cases} \mathfrak{g}_0^J = \{0\}, \\ \mathfrak{g}_l^J = \{X \in \mathfrak{g} \mid [J^kX, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J, \;\; k = 1,2\}, \quad l > 1. \end{cases}$$

We say that J is *nilpotent* if $\mathfrak{g}_h^J = \mathfrak{g}$ for some positive integer h. Equivalently by [**6**, **37**], a complex structure on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ is nilpotent if there is a basis $\{\alpha^1,\dots,\alpha^n\}$ of $(1,0)$-forms satisfying $d\alpha^1 = 0$ and

$$d\alpha^j \in \Lambda^2\langle \alpha^1,\dots,\alpha^{j-1},\overline{\alpha}^1,\dots,\overline{\alpha}^{j-1}\rangle,$$

for $j = 2,\dots,n$.

Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra with a left-invariant complex structure admitting an SKT metric. Since the center ξ of $\mathfrak{g}$ is J-invariant, J induces a complex structure $\hat{J}$ on the quotient $\hat{\mathfrak{g}} = \mathfrak{g}/\xi$ by the relation

$$\hat{J}(X + \xi) = JX + \xi, \tag{3.5}$$

for any $X \in \mathfrak{g}$. Moreover, if $\hat{J}$ is nilpotent, then J is also nilpotent. Indeed, we have

$$\mathfrak{g}_1^J = \xi, \quad \mathfrak{g}_2^J = \mathrm{span}\langle \hat{\mathfrak{g}}_1^J, \xi \rangle$$

and more in general $\mathfrak{g}_k^J = \mathrm{span}\langle \hat{\mathfrak{g}}_{k-1}^J, \xi \rangle$, for any $k \geq 1$. Therefore, if J is non-nilpotent, $\hat{J}$ has to be non-nilpotent.

We have the following:

Proposition 3.2. *Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra endowed with an* SKT *inner product g. Then there exists an* SKT *inner product $\hat{g}$ on $(\hat{\mathfrak{g}} = \mathfrak{g}/\xi, \hat{J})$, where $\hat{J}$ is defined by* (3.5).

Proof. We have the decomposition

$$\mathfrak{g} = \xi \oplus \xi^{\perp},$$

where $\xi^{\perp}$ denotes the orthogonal complement of ξ with respect to the SKT metric g. So any $X \in \mathfrak{g}$ splits in

$$X = X^{\xi} + X^{\perp}.$$

Then we can identify at the level of vector spaces $\mathfrak{g}/\xi$ with $\xi^{\perp}$ in the following way:

$$\mathfrak{g}/\xi \cong \{X + \xi \quad | \quad X \in \xi^{\perp}\}.$$

We claim that the inner product $\hat{g}$ on $\hat{\mathfrak{g}}$ defined by

$$\hat{g}(X + \xi, Y + \xi) = g(X^{\perp}, Y^{\perp}), \quad X, Y \in \mathfrak{g}$$

is SKT. We note that $[X, Y]^{\perp} = [X^{\perp}, Y^{\perp}]$, and applying Proposition 3.1 we obtain $(JX)^{\perp} = JX^{\perp}$. Then by a direct calculation the torsion 3-form of the Bismut connection of $(\hat{J}, \hat{g})$ satisfy

$$\hat{c}(X + \xi, Y + \xi, Z + \xi) = c(X^{\perp}, Y^{\perp}, Z^{\perp}),$$

for any $X, Y, Z \in \mathfrak{g}$. Therefore the closure of c implies the closure of $\hat{c}$ since $dc(X, Y, Z, W) = 0$ for every $X, Y, Z, W \in \xi^{\perp}$. □

Now are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $(M = G/\Gamma, J)$ be a nilmanifold with an invariant complex structure J. Assume that there exists a J-Hermitian SKT metric g on M. We can suppose that g is left-invariant (see [**12**, **43**]). Then (J, g) can be regarded as an SKT structure on the Lie algebra $\mathfrak{g}$ associated to G.

Since J preserves the center ξ of $\mathfrak{g}$ and $(\mathfrak{g}/\xi, \hat{J})$ admits an SKT metric, the center of $\mathfrak{g}/\xi$

$$\hat{\xi} = \{X^\perp + \xi \mid [X^\perp, \mathfrak{g}] \in \xi\}$$

is $\hat{J}$-invariant. Then the ideal of $\mathfrak{g}$

$$\pi^{-1}\hat{\xi} = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \xi\}$$

is J-invariant.

We prove that $\mathfrak{g}$ is 2-step nilpotent by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 6$, then the claim follows from [**13**]. Suppose now that the statement is true for $\dim \mathfrak{g} < 2n$ and assume $\dim \mathfrak{g} = 2n$. Let ξ be the center of $\mathfrak{g}$ and let $\hat{\mathfrak{g}} = \mathfrak{g}/\xi$. In view of Proposition 3.2 we have that $(\hat{\mathfrak{g}}, \hat{J})$ admits an SKT inner product. Hence, since $\dim \hat{\mathfrak{g}} < 2n$ by our assumption we get that $\hat{\mathfrak{g}}$ is at most 2-step nilpotent. This implies that $\mathfrak{g}$ is at most 3-step nilpotent. If $\mathfrak{g}$ is 3-step, then there exist $X, Y \in \mathfrak{g}$ such that

$$[X, Y] \neq 0.$$

with $Y = [X_1, X_2] \in \mathfrak{g}^1 \subseteq \pi^{-1}\hat{\xi}$. Since $J(\pi^{-1}\hat{\xi}) = \pi^{-1}\hat{\xi}$, we have that $JY \in \pi^{-1}\hat{\xi}$. Now

$$\begin{aligned} dc(X, Y, JX, JY) &= -c([X, Y], JX, JY) + c([X, JX], Y, JY) \\ &\quad - c([X, JY], Y, JX) - c([Y, JX], X, JY) \\ &\quad + c([Y, JY], X, JX) - c([JX, JY], X, Y) \\ &= g([X, Y], [X, Y]) - g([J[X, JX], JY], JY) \\ &\quad + g([Y, J[X, JX]], Y) - 2g([Y, JY], [X, JX]) \\ &\quad + g([X, JY], [X, JY]) + g([Y, JX], [Y, JX]) \\ &\quad + g([JX, JY], [JX, JY]). \end{aligned}$$

As a consequence of the Jacobi identity we get

$$[[W_1, W_2], Z] = 0, \tag{3.6}$$

for any $W_1, W_2 \in \mathfrak{g}$ and $Z \in \pi^{-1}\hat{\xi}$. Therefore, $[Y, JY]$ and $[Y, J[X, JX]]$ vanish. Now we can also show that $[J[X, JX], JY]$ is zero. Indeed, by using the integrability of J and (3.6) we get

$$[J[X, JX], JY] = [[X, JX], Y] + J[J[X, JX], Y] + J[[X, JX], JY] = 0,$$

and so

$$0 = dc(X, Y, JX, JY) = \|[X, Y]\|^2 + \|[X, JY]\|^2 + \|[Y, JX]\|^2 + \|[JX, JY]\|^2,$$

which is a contradiction with the assumption that $\mathfrak{g}$ is 3-step nilpotent.

Since the center $Z(G)$ of G is J-invariant and $\Gamma \cap Z(G)$ is a uniform discrete subgroup of $Z(G)$, we have that the surjective homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}/\xi$ induces a holomorphic principal torus bundle $\tilde{\pi} : G/\Gamma \to \mathbb{T}^{2p}$ over a $2p$-dimensional torus $\mathbb{T}^{2p}$ with $2p = \dim \mathfrak{g} - \dim \xi$. □

Remark 3.1. (1) Since by [**35**] on a 2-step nilpotent Lie algebra every complex structure is nilpotent, Theorem 1.2 implies that if a nilmanifold endowed with a left-invariant complex structure J admits a J-compatible SKT metric, then the complex structure has to be nilpotent.

(2) By [**41**] any six-dimensional SKT nilmanifold is the twist of the Kähler product of the two torus $\mathbb{T}^4$ and $\mathbb{T}^2$ by using two integral 2-forms supported on $\mathbb{T}^4$. In a similar way, as a consequence of Theorem 1.2, we have that any SKT nilmanifold is the twist of a torus.

Corollary 3.1. *Let $(M = G/\Gamma, J)$ be a $2n$-dimensional nilmanifold endowed with an invariant complex structure J. If there exists a J-compatible* SKT *metric g on M and $n > 2$, then $b_1(M) \geq 4$ and every invariant J-compatible metric on M is standard.*

Proof. By Theorem 1.2 we have the Lie algebra $\mathfrak{g}$ of G has to be 2-step nilpotent. Since $\mathfrak{g}^1 \subseteq \xi$, the annihilator of ξ in $\mathfrak{g}^*$ is contained in the annihilator of $\mathfrak{g}^1$, therefore its elements are all d-closed. If $\dim \xi = 2n - 2p$ we can choose a basis of $(1,0)$-forms $\{\alpha^1, \ldots, \alpha^n\}$ such that

$$\begin{cases} d\alpha^j = 0, \quad j = 1, \ldots, p, \\ d\alpha^l \in \Lambda^2\langle \alpha^1, \ldots, \alpha^p, \overline{\alpha}^1, \ldots, \overline{\alpha}^p \rangle, \quad l = p+1, \ldots, n, \end{cases}$$

with $\{\alpha^1, \ldots, \alpha^p\}$ in the annihilator of ξ and $\{\alpha^{p+1}, \ldots, \alpha^n\}$ in the complexification of the dual ξ^* of ξ. Using the previous basis we can show that there exist at least two d-closed $(1,0)$-forms. Indeed, if $p = 1$, then we have

$$\begin{cases} d\alpha^1 = 0, \\ d\alpha^l \in \Lambda^2\langle \alpha^1, \overline{\alpha}^1 \rangle, \quad l = 2, \ldots, n, \end{cases}$$

and therefore $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{h}_3^{\mathbb{R}} \oplus \mathbb{R}^{2n-3}$ of the three-dimensional real Heisenberg Lie algebra $\mathfrak{h}_3^{\mathbb{R}}$ with $\mathbb{R}^{2n-3}$ and $b_1(\mathfrak{g}) = 2n-1 \geq 4$ since $n > 2$.

By using Nomizu's Theorem [**34**] we have that the de Rham cohomology of M is isomorphic to the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$. Since there exist at least four real d-closed 1-forms we get that $b_1(M) = b_1(\mathfrak{g}) \geq 4$.

We recall that by [**19**] a J-compatible metric h on (M, J) is standard if and only if the Lee form $\theta = Jd^*\omega$ is co-closed, where ω denotes the fundamental form of (J, h). Since $\mathfrak{g}$ is nilpotent, it is also unimodular, so any $(2n-1)$-form is d-closed and in particular, if θ is the Lee form of an invariant J-compatible metric h, then $d(*\theta) = 0$. □

Proof of Theorem 1.3. Let $(M, G/\Gamma, J)$ be a nilmanifold with an invariant complex structure and let $(\mathfrak{g}, J)$ be its associated Lie algebra. Assume that there exists a Hermitian-symplectic structure Ω on (M, J). Using

Lemma 3.2, we may assume that Ω is left-invariant. Hence, Ω can be regarded as a Hermitian-symplectic structure on $(\mathfrak{g}, J)$. Using Lemma 3.2, we have that Ω induces an SKT inner product on $(\mathfrak{g}, J)$. Then Theorem 1.2 implies that $\mathfrak{g}$ is 2-step nilpotent and therefore $\mathfrak{g}^1$ is contained in the center ξ. But since ξ is J-invariant, $J(\mathfrak{g}^1) \subseteq \xi$ and Theorem 1.1 implies the statement. □

Theorem 1.3 cannot be directly deduced from Theorem 1.1, since there exist examples of nilpotent Lie algebras not satisfying (1.1). For instance, we can consider the following:

Example 3.1. Let $\mathfrak{g}$ be the ten-dimensional nilpotent Lie algebra with structure equations

$$\begin{cases} de^j = 0, \quad j = 1, \dots, 7, \\ de^8 = e^1 \wedge e^5 + e^1 \wedge e^6 + e^3 \wedge e^5 + e^3 \wedge e^6, \\ de^9 = e^2 \wedge e^5 + e^2 \wedge e^6 + e^4 \wedge e^5 + e^4 \wedge e^6, \\ de^{10} = e^1 \wedge e^8 + e^3 \wedge e^8 + e^2 \wedge e^9 + e^4 \wedge e^9 \end{cases}$$

(see [**35**, Example 3.13]) endowed with the non-nilpotent complex structure

$$Je_1 = e_2, \;\; Je_3 = e_4, \;\; Je_5 = e_7, \;\; Je_8 = e_9, \;\; Je_6 = e_{10},$$

where $\{e_1, \dots, e_{10}\}$ denotes the dual basis of $\{e^1, \dots, e^{10}\}$. We have

$$\xi = \operatorname{span}\langle e_7, e_{10}\rangle, \quad \mathfrak{g}^1 = \operatorname{span}\langle e_8, e_9, e_{10}\rangle$$

and thus $J\xi \cap \mathfrak{g}^1 = \{0\}$. In this case, a basis of $(1,0)$-forms is then given by

$$\alpha^1 = e^1 + ie^2, \;\; \alpha^2 = e^3 + ie^4, \;\; \alpha^3 = e^5 + ie^7, \;\; \alpha^4 = e^8 + ie^9, \;\; \alpha^5 = e^6 + ie^{10}$$

with complex structure equations

$$\begin{cases} d\alpha^j = 0, \quad j = 1, 2, 3, \\ d\alpha^4 = \dfrac{1}{2}(\alpha^{13} + \alpha^{1\bar{3}} + \alpha^{23} + \alpha^{2\bar{3}} + \alpha^{15} + \alpha^{1\bar{5}} + \alpha^{25} + \alpha^{2\bar{5}}), \\ d\alpha^5 = \dfrac{i}{2}(\alpha^{\bar{1}4} + \alpha^{1\bar{4}} + \alpha^{\bar{2}4} + \alpha^{2\bar{4}}), \end{cases}$$

where by $\alpha^{i\bar{j}}$ we denote $\alpha^i \wedge \overline{\alpha}^j$.

4. A classification of eight-dimensional SKT nilmanifolds

In this section, we classify eight-dimensional nilpotent Lie algebras admitting an SKT structure. We begin describing the following two families of nilpotent Lie algebras.

1. ***First family*:** Consider the family of eight-dimensional nilpotent Lie algebras with complex structure equations

$$\begin{cases} d\alpha^j = 0, j = 1, 2, \\ d\alpha^3 = B_1\alpha^{12} + B_4\alpha^{1\overline{1}} + B_5\alpha^{1\overline{2}} + C_3\alpha^{2\overline{1}} + C_4\alpha^{2\overline{2}}, \\ d\alpha^4 = F_1\alpha^{12} + F_4\alpha^{1\overline{1}} + F_5\alpha^{1\overline{2}} + G_3\alpha^{2\overline{1}} + G_4\alpha^{2\overline{2}}, \end{cases} \tag{4.1}$$

where the capital letters are arbitrary complex numbers and by $\alpha^{i\overline{j}}$ we denote $\alpha^i \wedge \overline{\alpha}^j$. Then the inner product

$$g = \sum_{k=1}^{4} \alpha^k \otimes \overline{\alpha}^k$$

is SKT if and only if the complex numbers satisfy the equation

$$|B_1|^2 + |F_1|^2 + |G_3|^2 + |B_5|^2 + |C_3|^2 + |F_5|^2 = 2\ \mathfrak{Re}(C_4\overline{B}_4 + F_4\overline{G}_4). \tag{4.2}$$

Note that in terms of g and of the derivatives of the α^k's, this last equation can be rewritten as

$$\sum_{j=3}^{4}\left(\|d\alpha^j\|^2 + \mathfrak{Re}\left[g(d\alpha^j, \alpha^{1\overline{1}})g(d\overline{\alpha}^j, \alpha^{2\overline{2}})\right] - \sum_{k=1}^{2}|g(d\alpha^j, \alpha^{k\overline{k}})|^2\right) = 0$$

and by a direct computation we have that the Lee form $\theta = Jd^*\omega$ is in general not closed.

Note that if both $|B_4|^2 + |C_4|^2$ and $|F_4|^2 + |G_4|^2$ vanish, then the Lie algebra is abelian. So we can suppose for instance that $|B_4|^2 + |C_4|^2 \neq 0$ (otherwise we consider the change of basis)

$$\alpha^1 \mapsto \alpha^1, \quad \alpha^2 \mapsto \alpha^2, \quad \alpha^3 \mapsto \alpha^4, \quad \alpha^4 \mapsto \alpha^3$$

and arguing as in the proof of Lemma 2.14 in [**43**] we can change the basis in order to have

$$\begin{cases} d\alpha^j = 0, j = 1, 2, \\ d\alpha^3 = \rho\alpha^{12} + \alpha^{1\overline{1}} + B\alpha^{1\overline{2}} + D\alpha^{2\overline{2}}, \\ d\alpha^4 = F_1\alpha^{12} + F_4\alpha^{1\overline{1}} + F_5\alpha^{1\overline{2}} + G_3\alpha^{2\overline{1}} + G_4\alpha^{2\overline{2}}, \end{cases}$$

with $\rho \in \{0, 1\}$, $B, D \in \mathbb{C}$. With the new change

$$\alpha^1 \mapsto \alpha^1, \quad \alpha^2 \mapsto \alpha^2, \quad \alpha^3 \mapsto \alpha^3, \quad \alpha^4 \mapsto \alpha^4 - F_4\alpha^3,$$

we then get

$$\begin{cases} d\alpha^j = 0, j = 1, 2, \\ d\alpha^3 = \rho\alpha^{12} + \alpha^{1\overline{1}} + B\alpha^{1\overline{2}} + D\alpha^{2\overline{2}}, \\ d\alpha^4 = F_1'\alpha^{12} + F_5'\alpha^{1\overline{2}} + G_3'\alpha^{2\overline{1}} + G_4'\alpha^{2\overline{2}}. \end{cases}$$

Therefore, if $(F_1')^2 + (F_5')^2 + (G_3')^2 + (G_4')^2 = 0$, the Lie algebra is a direct sum of $\mathbb{R}^2$ with a six-dimensional SKT nilpotent Lie algebra. So for instance the direct sum $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}^2$ of the three-dimensional complex Heisenberg Lie algebra $\mathfrak{h}_3^{\mathbb{C}}$ with $\mathbb{R}^2$ admits an SKT structure. If $(F_1')^2 + (F_5')^2 + (G_3')^2 + (G_4')^2 \neq 0$, then Z_3 and Z_4 belong to the complexification of the center ξ, so $\dim \xi \geq 4$. Note that $\dim \mathfrak{g}^1 = 1$ if and only if the Lie algebra is isomorphic to the direct sum $\mathfrak{h}_3^{\mathbb{R}} \oplus \mathbb{R}^5$. Moreover $\dim \mathfrak{g}^1 = 3$ if and only one of the following cases occur:
(a) $(F_1')^2 + (F_5')^2 + (G_3')^2 \neq 0$, $\rho = B = 0$ and D real;
(b) $F_1' = F_5' = G_3' = 0$, $G_4' \neq 0$ and $\rho^2 + B^2 \neq 0$.

In case (a), we get the Lie algebra

$$\begin{cases} d\alpha^j = 0, j = 1, 2, \\ d\alpha^3 = \alpha^{1\bar{1}} + D\alpha^{2\bar{2}}, \\ d\alpha^4 = F_1'\alpha^{12} + F_5'\alpha^{1\bar{2}} + G_3'\alpha^{2\bar{1}} + G_4'\alpha^{2\bar{2}}, \end{cases}$$

with $(F_1')^2 + (F_5')^2 + (G_3')^2 \neq 0$ and D real. In particular, for $D = 1$, $F_1' = \sqrt{2}$, $F_5' = G_3' = G_4' = 0$, we get the Lie algebra with structure equations

$$\begin{cases} d\alpha^j = 0, j = 1, 2, \\ d\alpha^3 = \alpha^{1\bar{1}} + \alpha^{2\bar{2}}, \\ d\alpha^4 = \sqrt{2}\alpha^{12}, \end{cases}$$

which is isomorphic to the direct sum $\mathfrak{h}_7^{Q} \oplus \mathbb{R}$, where $\mathfrak{h}_7^{Q}$ is the real seven-dimensional Lie algebra of Heinseberg type and with three-dimensional center.

In case (b) we get a Lie algebra isomorphic to

$$\begin{cases} d\alpha^j = 0, \quad j = 1, 2, \\ d\alpha^3 = \rho\alpha^{12} + \alpha^{1\bar{1}} + B'\alpha^{1\bar{2}}, \\ d\alpha^4 = \alpha^{2\bar{2}}, \end{cases}$$

with $\rho^2 + (B')^2 \neq 0$.

2. ***Second family*:** Consider the family of eight-dimensional nilpotent Lie algebras equipped with a complex structure having structure equations

$$\begin{cases} d\alpha^j = 0, \quad j = 1, 2, 3, \\ d\alpha^4 = F_1\alpha^{12} + F_2\alpha^{13} + F_4\alpha^{1\bar{1}} + F_5\alpha^{1\bar{2}} + F_6\alpha^{1\bar{3}} + G_1\alpha^{23} \\ \qquad\quad + G_3\alpha^{2\bar{1}} + G_4\alpha^{2\bar{2}} + G_5\alpha^{2\bar{3}} + H_2\alpha^{3\bar{1}} + H_3\alpha^{3\bar{2}} + H_4\alpha^{3\bar{3}}, \end{cases} \tag{4.3}$$

where the capitol letters are arbitrary complex numbers and $H_4 \neq 0$. In this case requiring that the inner product $g = \sum_{k=1}^{4} \alpha^k \otimes \overline{\alpha}^k$ is SKT is equivalent to require that the following equations are satisfied:

$$\begin{cases} -H_3\overline{F}_4 + H_2\overline{G}_3 + F_5\overline{F}_6 - F_4\overline{G}_5 + F_2\overline{F}_1 = 0, \\ -H_3\overline{F}_5 + G_4\overline{F}_6 + H_2\overline{G}_4 - G_3\overline{G}_5 + G_1\overline{F}_1 = 0, \\ -H_4\overline{F}_5 + G_5\overline{F}_6 + H_2\overline{H}_3 - G_3\overline{H}_4 + G_1\overline{F}_2 = 0, \\ |F_2|^2 + |F_6|^2 + |H_2|^2 = 2\ \mathrm{Re}(H_4\overline{F}_4), \\ |F_1|^2 + |F_5|^2 + |G_3|^2 = 2\ \mathrm{Re}(F_4\overline{G}_4), \\ |G_1|^2 + |G_5|^2 + |H_3|^2 = 2\ \mathrm{Re}(H_4\overline{G}_4). \end{cases} \tag{4.4}$$

Every Lie algebra $\mathfrak{g}$ of this family splits in $\mathfrak{g} = V_1 \oplus V_2$, where V_i are J-invariant vector subspaces such that

$$\dim V_1 = 6, \quad \dim V_2 = 2, \quad [V_1, V_1] \subseteq V_2, \quad V_2 \subseteq \xi$$

and there exists a real J-invariant two-dimensional vector subspace V_3 (generated by Z_3 and $\overline{Z}_3$) of V_1 such that $[V_3, V_3] \neq 0$. In particular, any Lie algebra of this family is 2-step nilpotent, has $\dim \mathfrak{g}^1 = 2$ and by a direct computation we have that the Lee form $\theta = Jd^*\omega$ is in general not closed.

Now we are ready to classify eight-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric.

Theorem 4.1. *Let $M^8 = G/\Gamma$ be an eight-dimensional nilmanifold (not a torus) with an invariant complex structure J. There exists an* SKT *metric g on M compatible with J if and only if the Lie algebra $(\mathfrak{g}, J)$ belongs to one of the two families described above.*

Proof. Assume that there exists an SKT metric g on (M^8, J). In view of [**12**, **43**], we may assume that the metric g is invariant, and so that it is induced by an inner product g on the Lie algebra $\mathfrak{g}$.

In the proof of Theorem 1.2, we have already shown that there exists a basis of $(1,0)$-forms $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ such that

$$\begin{cases} d\alpha^j = 0, \quad j = 1, \dots, p, \\ d\alpha^l \in \Lambda^2\langle \alpha^1, \dots, \alpha^p, \overline{\alpha}^1, \dots, \overline{\alpha}^p \rangle, \quad l = p+1, \dots, 4, \end{cases}$$

with $\{\alpha^1, \dots, \alpha^p\}$ in the annihilator of ξ and $\{\alpha^{p+1}, \dots, \alpha^4\}$ in the complexification of the dual ξ^* of the center. Applying the Gram-Schmidt process to the previous basis $\{\alpha^1, \dots, \alpha^4\}$, we get a unitary coframe $\{\tilde{\alpha}^1, \dots, \tilde{\alpha}^4\}$ such that

$$\begin{cases} d\tilde{\alpha}^j = 0, \quad j = 1, \dots, p, \\ d\tilde{\alpha}^l \in \Lambda^2\langle \tilde{\alpha}^1, \dots, \tilde{\alpha}^p, \overline{\tilde{\alpha}^1}, \dots, \overline{\tilde{\alpha}^p} \rangle, \quad l = p+1, \dots, 4, \end{cases}$$

since $\operatorname{span}\langle\tilde{\alpha}^1,\ldots,\tilde{\alpha}^j\rangle = \operatorname{span}\langle\alpha^1,\ldots,\alpha^j\rangle$, for any $j = 1,\ldots,4$. Then it is not restrictive to assume that $\{\alpha^1,\ldots,\alpha^4\}$ is a g-unitary coframe, i.e., that the fundamental form ω of g with respect to $\{\alpha^1,\ldots,\alpha^4\}$ takes the standard expression:

$$\omega = -\frac{i}{2}\sum_{j=1}^{4}\alpha^j\wedge\alpha^{\bar{j}}.$$

We can distinguish three cases according to the dimension $2n-2p$ of the center ξ.

(a) If $p = 1$, then $\mathfrak{g} \cong \mathfrak{h}_3^{\mathbb{R}} \oplus \mathbb{R}^5$ and it belongs to the first family.
(b) For $p = 2$ we get the first family (4.1).
(c) For $p = 3$ we obtain the second family (4.3).

□

Remark 4.1. In dimension 8, it is no longer true that the existence of a strong KT structure on a nilpotent Lie algebra $\mathfrak{g}$ depends only on the complex structure of $\mathfrak{g}$. Indeed, for any of two families of eight-dimensional nilpotent Lie algebras with complex structure equations (4.1) and (4.3) consider a generic J-Hermitian metric g. The fundamental form ω associated to the Hermitian structure (J,g) can be then expressed as

$$\begin{aligned}\omega = {} & a_1\alpha^{1\bar{1}} + a_2\alpha^{2\bar{2}} + a_3\alpha^{3\bar{3}} + a_4\alpha^{4\bar{4}} + a_5\alpha^{1\bar{2}} - \overline{a}_5\alpha^{2\bar{1}} + a_6\alpha^{1\bar{3}} - \overline{a}_6\alpha^{3\bar{1}}\\ & + a_7\alpha^{1\bar{4}} - \overline{a}_7\alpha^{4\bar{1}} + a_8\alpha^{2\bar{3}} - \overline{a}_8\alpha^{3\bar{2}} + a_9\alpha^{2\bar{4}} - \overline{a}_9\alpha^{4\bar{2}} + a_{10}\alpha^{3\bar{4}} - \overline{a}_{10}\alpha^{4\bar{3}},\end{aligned}$$

where a_l, $l = 1,\ldots,10$, are arbitrary complex numbers (with $\overline{a}_l = -a_l$, for any $l = 1,\ldots,4$) such that ω is positive definite.

For the first family the SKT equation for a generic J-Hermitian metric g is:

$$\begin{aligned}& -a_3C_4\overline{B}_4 - 2a_{10}B_4\overline{G}_4 - a_3B_4\overline{C}_4 + a_{10}B_1\overline{F}_1 + a_3|B_1|^2 - \overline{a}_{10}\overline{B}_1F_1 + a_4|F_5|^2\\ & \quad + a_4|F_1|^2 - \overline{a}_{10}G_3\overline{C}_3 + a_4|G_3|^2 + a_3|B_5|^2 - \overline{a}_{10}\overline{B}_5F_5 - a_4\overline{G}_4F_4 + a_3|C_3|^2\\ & \quad + \overline{a}_{10}\overline{C}_4F_4 + a_{10}\overline{F}_5B_5 - a_4\overline{F}_4G_4 + \overline{a}_{10}G_4\overline{B}_4 + a_{10}\overline{G}_3C_3 - a_{10}C_4\overline{F}_4 = 0,\end{aligned}$$

so it is no longer true that equation (4.2) implies that any J-Hermitian metric is SKT.

For the second family, the SKT equations for the generic J-Hermitian metric g are (4.4) and so as in the six-dimensional case the SKT condition depends only on the complex structure.

Nilmanifolds also provide examples of compact HKT manifolds. We recall that a hyperHermitian manifold (M, J_1, J_2, J_3, g) is called HKT if the fundamental forms $\omega_l(\cdot,\cdot) = g(J_l\cdot,\cdot)$, $l = 1,2,3$, satisfy

$$J_1d\omega_1 = J_2d\omega_2 = J_3d\omega_3.$$

This is equivalent to the property that the three Bismut connections associated to the Hermitian structures (J_l, g) coincide and this connection is said to be an HKT connection. The HKT structure is called strong or weak depending on whether the torsion 3-form of the HKT connection is closed or not.

By [**2**, Theorem 4.2] if a $4n$-dimensional nilmanifold $M^{4n} = G/\Gamma$ is endowed with an HKT structure (J_1, J_2, J_3, g) induced by a left-invariant HKT structure on G, then the hypercomplex structure has to be abelian, i.e.,

$$[J_l X, J_l Y] = [X, Y], \quad l = 1, 2, 3,$$

for any X, Y in the Lie algebra $\mathfrak{g}$ of G. By [**9**] every abelian hypercomplex structure on a (non-abelian) nilpotent Lie algebra gives rise to a weak HKT structure.

In [**10**] a description of hypercomplex eight-dimensional nilpotent Lie algebras was given. More precisely, it was shown that a eight-dimensional hypercomplex nilpotent Lie algebra $\mathfrak{g}$ has to be 2-step nilpotent and with $b_1(\mathfrak{g}) \geq 4$.

In particular, by [**8**] there exist only three (non-abelian) nilpotent Lie algebras of dimension 8 admitting an abelian hypercomplex structure and they are abelian extensions of a Lie algebra of Heinsenberg type, i.e., are isomorphic to one of the following:

$$\mathfrak{h}_5 \oplus \mathbb{R}^3, \quad \mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}^2, \quad \mathfrak{h}_7^{Q} \oplus \mathbb{R},$$

where $\mathfrak{h}_5$ is the five-dimensional Lie algebra of Heinseberg type with one-dimensional center.

As a consequence of Theorem 4.1 we have the following:

Corollary 4.1. *The nilpotent Lie algebra $\mathfrak{h}_5 \oplus \mathbb{R}^3$ has weak* HKT *structures and it does not admit any* SKT *structure. The Lie algebras $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}^2$ and $\mathfrak{h}_7^{Q} \oplus \mathbb{R}$ have* SKT *structures and weak* HKT *structures.*

Proof. The nilpotent Lie algebra $\mathfrak{h}_5 \oplus \mathbb{R}^3$ has one-dimensional commutator, but by the description of the two families a eight-dimensional SKT nilpotent Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}^1 = 1$ is isomorphic to $\mathfrak{h}_3^{\mathbb{R}} \oplus \mathbb{R}^5$. In the first family, there are Lie algebras isomorphic to $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}^2$ and $\mathfrak{h}_7^{Q} \oplus \mathbb{R}$. □

References

[1] V. Apostolov and M. Gualtieri, *Generalized Kähler manifolds with split tangent bundle*, Commun. Math. Phys. **271** (2007), 561–575.

[2] M. L. Barberis, I. Dotti and M. Verbitsky, *Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry*, Math. Res. Lett. **16**(2) (2009), 331–347.

[3] C. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), 513–518.

[4] J. M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284** (1989), 681–699.

[5] G. R. Cavalcanti and M. Gualtieri, *Generalized complex structures on nilmanifolds*, J. Symplectic Geom. **2** (2004), 393–410.

[6] L. A. Cordero, M. Fernández, A. Gray and L. Ugarte, *Compact nilmanifolds with nilpotent complex structure: Dolbeault cohomology*, Trans. Amer. Math. Soc. **352** (2000), 5405–5433.

[7] S. K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, in 'Inspired by S.S. Chern', ed. P. A. Griffiths, World Scientific, 2006.

[8] I. Dotti and A. Fino, *Abelian hypercomplex 8-dimensional nilmanifolds*, Ann. Global Anal. Geom. **18**(1) (2000), 47–59.

[9] I. Dotti and A. Fino, *Hyperkähler torsion structures invariant by nilpotent Lie groups*, Classical Quantum Gravity **19**(3) (2002), 551–562.

[10] I. Dotti and A. Fino, *Hypercomplex 8-dimensional nilpotent Lie groups*, J. Pure Appl. Algebra **184**(1) (2003), 41–57.

[11] N. Enrietti, *Static SKT metrics on Lie groups*, `arXiv:1009.0620`.

[12] A. Fino and G. Grantcharov, *On some properties of the manifolds with skew-symmetric torsion and holonomy $SU(n)$ and $Sp(n)$*, Adv. Math. **189** (2004), 439–450.

[13] A. Fino, M. Parton and S. Salamon, *Families of strong KT structures in six dimensions*, Comm. Math. Helv. **79**(2) (2004), 317–340.

[14] A. Fino and A. Tomassini, *Non Kähler solvmanifolds with generalized Kähler structure*, J. Symplectic Geom. **7**(2) (2009), 1–14.

[15] A. Fino and A. Tomassini, *Blow ups and resolutions of strong Kähler with torsion metrics*, Adv. Math. **221**(3) (2009), 914–935.

[16] A. Fino and A. Tomassini, *On Astheno–Kähler metrics*, J. Lond. Math. Soc. (2) **83**(2) (2011), 290–308, `arXiv:0806.0735`.

[17] S. J. Gates, C. M. Hull and M. Rǒcek, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nucl. Phys. **B248** (1984), 157–186.

[18] P. Gauduchon, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B **11** (1997), 257–288.

[19] P. Gauduchon, *La 1-forme de torsione dune variete hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[20] M. Goze and Y. Khakimdjanov, *Nilpotent Lie Algebras*, Math. Appl. **361**, Kluwer, 1996.

[21] G. Grantcharov and Y. S. Poon, *Geometry of hyper-Kähler connections with torsion*, Comm. Math. Phys. **213** (2000), 19–37.

[22] M. Gualtieri, *Generalized complex geometry*, DPhil thesis, University of Oxford, 2003, `arXiv:0401221`.

[23] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), 65–71.

[24] K. Hasegawa, *Complex and Kähler structures on compact solvmanifolds*, J. Symplectic Geom. **3** (2005), 749–767.

[25] K. Hasegawa, *A note on compact solvmanifolds with Kähler structures*, Osaka J. Math. **43**(1) (2006), 131–135.

[26] N. J. Hitchin, *Instantons and generalized Kähler geometry*, Comm. Math. Phys. **265** (2006), 131–164.

[27] P. S. Howe and G. Papadopoulos, *Twistor spaces for hyper-Kähler manifolds with torsion*, Phys. Lett. **B379** (1996), 80–86.

[28] T.-J. Li and W. Zhang, *Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds*, Comm. Anal. Geom. **17**(4) (2009), 651–683.

[29] T. B. Madsen and A. Swann, *Invariant strong KT geometry on four-dimensional solvable Lie groups*, `arXiv:0911.0535`.

[30] L. Magnin, *Sur les algébres de Lie nilpotentes de dimension* ≤ 7, J. Geom. Phys. **3** (1986), 119–144.

[31] A. I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation Ser. 1 **9** (1962), 276–307.

[32] R. Mejldal, *Complex manifolds and strong geometries with torsion*, Master thesis, Department of Mathematics and Computer Science, University of Southern Denmark, July 2004.

[33] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.

[34] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. Math. **59** (1954), 531–538.

[35] S. Rollenske, *Geometry of nilmanifolds with left-invariant complex structure and deformations in the large*, Proc. Lond. Math. Soc. (3) **99**(2) (2009), 425–460.

[36] F. A. Rossi and A. Tomassini, *On strong Kähler and Astheno–Kähler metrics on nilmanifolds*, preprint (2010).

[37] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.

[38] J. Streets and G. Tian, *Hermitian curvature flow*, J. Eur. Math. Soc. (JEMS) **13**(3) (2011), 601–634, `arXiv:0804.4109`.

[39] J. Streets and G. Tian, *A Parabolic flow of pluriclosed metrics*, Int. Math. Res. Notices **2010** (2010), 3101–3133.

[40] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986), 253–284.

[41] A. Swann, *Twisting Hermitian and hypercomplex geometries*, Duke Math. J. **155** (2010), 403–431.

[42] A. Tomassini, 2011, private communication.

[43] L. Ugarte, *Hermitian structures on six dimensional nilmanifolds*, Transf. Groups **12** (2007), 175–202.

[44] B. Weinkove, *The Calabi–Yau equation on almost-Kähler four manifolds*, J. Differential Geom. **76**(2) (2007), 317–349.

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