

Motion of Non-Convex Polygons by Crystalline Curvature and Almost Convexity Phenomena

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The behavior of solution polygons to generalized crystalline curvature flow is discussed. The conditions to guarantee that the solution polygon keeps its admissibility as long as enclosed area of solution polygon is positive are clarified. We also show that the solution polygon becomes “almost convex” before the extinction time.

Key words: motion by crystalline curvature, non-convex polygon, edge-disappearing, convexity phenomena

1. Introduction

Recently curvature-dependent interface motions under several kinds of interfacial energies are studied by many authors. Mean curvature flow is a typical motion for smooth energies. This flow is obtained as the gradient flow for the total interfacial energy when the interfacial energy is a constant. For non-smooth energies, there is a special class of interfacial energies which are called *crystalline* energies. For these energies, S. Angenent and M.E. Gurtin [2] and J.E. Taylor [12] independently investigated the motion of polygonal plane curves by its *crystalline* curvature. They also introduce *admissibility* of polygonal curves. After these pioneer works, many authors discuss more general motions and higher dimensional problems.

In this paper we consider motion of admissible polygons in the plane by crystalline curvature. Let $\sigma = \sigma(\mathbf{n}) > 0$ be an interfacial energy function defined on S^1 . Here S^1 denotes the set of all unit vectors in \mathbb{R}^2 . When the Wulff shape $\mathcal{W}_\sigma = \{\mathbf{z} \in \mathbb{R}^2 \mid \mathbf{z} \cdot \mathbf{n} \leq \sigma(\mathbf{n}) \text{ for all } \mathbf{n} \in S^1\}$ is a polygon, σ is called *crystalline*. By definition, \mathcal{W}_σ is always convex. This shape originally comes from crystal physics and describes the equilibrium shape of crystal. The concept of admissibility of polygons is based on the Wulff shape. Roughly speaking, admissible polygons have the same normal vectors as that of the Wulff shape and the change of normal angle at each corner between two edges is the similar to that of the Wulff shape. We denote by H_j crystalline curvature of j -th edge of an admissible polygon. The detailed definitions are mentioned in Section 2. The motion equation $V_j = H_j$ is called *crystalline curvature flow*. Here V_j denotes the inward normal velocity of j -th edge. An example motion by this flow is shown in Fig. 1. We see that each

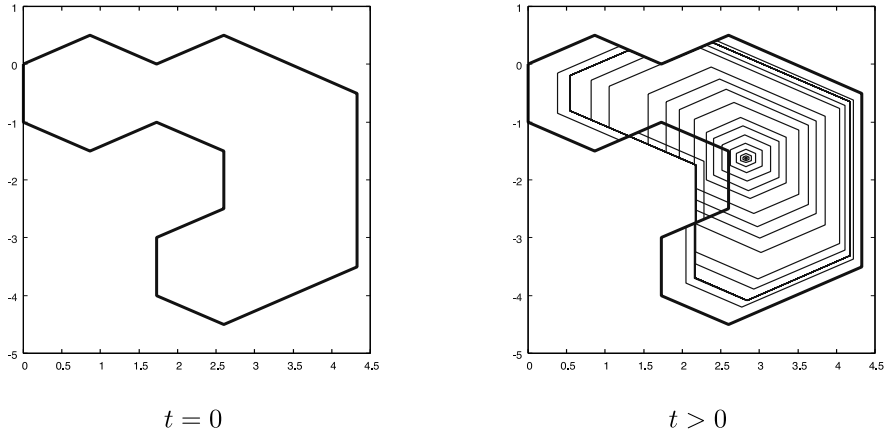


Fig. 1. Numerical simulation for crystalline curvature flow $V_j = H_j$ (left: initial polygon, right: time evolution of $\Omega(t)$).

edge of solution polygon keeps its orientation and the solution polygon eventually shrinks to a point.

In this paper we consider generalized crystalline curvature flow:

$$V_j = g(\mathbf{N}_j, H_j), \quad (1)$$

with an initial admissible polygon Ω_0 and discuss the behavior of the solution polygon $\Omega(t)$. Here the function g is a continuous function from $S^1 \times \mathbb{R}$ to \mathbb{R} and \mathbf{N}_j denotes the outward unit normal vector of j -th edge. The detailed conditions on g are shown in Section 3.

When the initial polygon is convex, solution polygon keeps their convexity and any edges never disappear as long as an enclosed area of solution polygon is positive. For this case, asymptotic behavior of solution polygons are studied by several authors (see [1, 9, 10, 11] and their references).

On the other hand, there are few results for non-convex case. K. Ishii and H.M. Soner [7] consider $V_j = H_j$ when the Wulff shape is a regular polygon and discuss the motion of solution polygon. M.-H. Giga and Y. Giga [4] extend Ishii–Soner’s results for more general motions (1). (Unfortunately, there are not precise proofs.) They show that the solution polygon from non-convex initial polygon keeps its admissibility till the extinction time and becomes convex strictly before the extinction time. The later property is called “convexity phenomena” and well known for smooth plane curves moving by the curvature flow [5] and its anisotropic version [3]. However, this assertion is not valid for crystalline flow (1). Because examples of non-convex self-similar solutions are shown in [8]. Namely, it is needed to discuss the motion of non-convex polygons again.

In this paper, we clarify the assumptions on g and the Wulff shape to guarantee that the solution polygon belongs to the admissible class as long as enclosed area of solution polygon is positive. We also show that a crystalline curvature of each

edges becomes non-negative and the solution polygon becomes star-shaped strictly before the extinction time.

The paper is organized as follows: In next section, we introduce some definitions and notation. In Section 3, the conditions on g and the interfacial energy are given and show our results. In Section 4, we prove theorems.

We finally mention that the first result in this paper has an important meaning from viewpoint of numerical studies. Recently, many authors use crystalline motions as the approximating method for motion of smooth curves. From our result, we can guarantee that such numerical procedures do not break down till the extinction time even if some edges disappear during time evolution.

2. Notation and definitions

Let σ be crystalline and \mathcal{W}_σ N_σ -sided convex polygon.

We first define admissibility of polygons. Let Ω and $p_j = (x_j, y_j)$ ($j = 0, 1, \dots, N - 1$) be a N -sided polygon and the j -th vertex of Ω , respectively. Here and hereafter the subscripts are in anticlockwise and mod N . We denote by \mathcal{S}_j the j -th edge which is line segment from p_j to p_{j+1} and also denote by \mathcal{P} the boundary of Ω , that is, $\mathcal{P} = \partial\Omega = \bigcup_{j=0}^{N-1} \mathcal{S}_j$. We denote by $\mathcal{N}_{\mathcal{W}_\sigma}$ (resp. \mathcal{N}_Ω) the set of all outward unit normal vectors of \mathcal{W}_σ (resp. Ω), respectively.

The polygon Ω is said to be *admissible* if Ω satisfies the following two conditions: (1) $\mathcal{N}_{\mathcal{W}_\sigma} = \mathcal{N}_\Omega$ and (2) $(s\mathbf{N}_j + (1-s)\mathbf{N}_{j+1})/|s\mathbf{N}_j + (1-s)\mathbf{N}_{j+1}| \notin \mathcal{N}_{\mathcal{W}_\sigma}$ for any $0 < s < 1$ and $j = 0, 1, \dots, N - 1$. This is an analogical concept of smooth curves. Namely, Ω has all normal angles of the Wulff shape and if $\mathbf{N}_j = \mathbf{n}_k \in \mathcal{N}_{\mathcal{W}_\sigma}$, then $\mathbf{N}_{j\pm 1} \in \{\mathbf{n}_{k-1}, \mathbf{n}_{k+1}\}$, where \mathbf{n}_k is the outward unit normal vector of k -th edge of the Wulff shape.

The quantity $H_j := \chi_j l_\sigma(\mathbf{N}_j)/d_j$ defined for an admissible polygon is called *crystalline curvature* of the j -th edge. Here d_j and $l_\sigma(\mathbf{n})$ are the length of the j -th edge of the solution polygon and the length of the edge of the Wulff shape with normal $\mathbf{n} \in \mathcal{N}_{\mathcal{W}_\sigma}$, respectively. The quantity χ_j is called transition number and given by $\chi_j := (\text{sgn}(\theta_j - \theta_{j-1}) + \text{sgn}(\theta_{j+1} - \theta_j))/2$. Here θ_j denotes the outward normal angle of \mathcal{S}_j . This takes $+1$ (resp. -1) if \mathcal{P} is convex (resp. concave) at \mathcal{S}_j in the outward normal direction; otherwise it takes zero. In this paper we call a edge with zero transition number “inflection edge.” Note that if Ω is convex, then $\chi_j = +1$ for all j and a crystalline curvature of each edge of the Wulff shape is always one. The later means that the Wulff shape plays a role like as the unit circle in the usual sense.

3. Edge-disappearing and almost convexity phenomena

3.1. Finite time edge-disappearing

We first assume the following:

ASSUMPTION (G1). $\lambda \mapsto g(\mathbf{N}_j, \lambda)$ is locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$ for all $\mathbf{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$.

ASSUMPTION (G2). For all $\mathbf{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$, $g(\mathbf{N}_j, \lambda)$ is monotone non-decreasing on λ and satisfies $g(\mathbf{N}_j, 0) = 0$, $g(\mathbf{N}_j, \lambda) > 0$ ($\lambda > 0$), $g(\mathbf{N}_j, -\lambda) = -g(\mathbf{N}_j, \lambda)$ and $\lim_{\lambda \rightarrow \infty} g(\mathbf{N}_j, \lambda) = +\infty$.

Notice that (1) can be written as the following:

$$\begin{aligned} \dot{d}_j(t) &= (\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1}))V_j \\ &\quad - \frac{1}{\sin(\theta_j - \theta_{j-1})}V_{j-1} - \frac{1}{\sin(\theta_{j+1} - \theta_j)}V_{j+1} \end{aligned} \tag{2}$$

for all j (see [2]). From the standard theory of a system of ordinary differential equations, we have short time existence and uniqueness of the solution polygon under (G1) (see [4] and its references). We also have that the solution of (2) exists as long as all d_j 's are positive. The assumption (G2) means that equation (1) includes crystalline curvature flow $V_j = H_j$. We later show that at least one edge disappears in finite time. On the other hand, there is a possibility that two parts of the boundary $\mathcal{P}(t)$ contact each other before the edge-disappearing. We call this phenomenon ‘‘self-contacting of the boundary’’ or ‘‘self-contacting’’ in short. If the self-contacting happens, the admissibility and simply-connectedness of solution polygon may break down even if all d_j 's are positive. We will show that such a phenomena never occur as long as all d_j 's are positive under some assumptions mentioned later. Namely, the admissibility of solution polygon is preserved until just before the edge-disappearing time. Let T_1 be the edge-disappearing time. Then, we have two possibility: The first is that all edges disappear at the same time. This means that the solution polygon shrinks to a single point and the solution polygon vanishes at $t = T_1$. The second is that at least one edge remains at $t = T_1$. In this case, if the limit $\Omega(T_1) = \lim_{t \rightarrow T_1} \Omega(t)$ exists and is admissible, then we can continue a time evolution of solution polygon in the class of admissible polygons beyond $t = T_1$. For this procedure, we add the following three conditions:

ASSUMPTION (W1). The Wulff shape \mathcal{W}_σ is point symmetric with respect to the origin.

ASSUMPTION (G3). $g(-\mathbf{N}_j, \lambda) = g(\mathbf{N}_j, \lambda)$ for any $\mathbf{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$ and $\lambda \in \mathbb{R}$.

ASSUMPTION (G4). For any $\mathbf{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$, the function $g(\mathbf{N}_j, \lambda)$ satisfies

$$\int_1^\infty g(\mathbf{N}_j, \lambda)\lambda^{-2} d\lambda = \int_{-1}^{-\infty} g(\mathbf{N}_j, \lambda)\lambda^{-2} d\lambda = \infty.$$

From the assumption (W1), if $\mathbf{N} \in \mathcal{N}_{\mathcal{W}_\sigma}$, then $-\mathbf{N} \in \mathcal{N}_{\mathcal{W}_\sigma}$ and $l_\sigma(\mathbf{N}) = l_\sigma(-\mathbf{N})$. For example, regular even-sided polygons satisfy (W1). The condition (G3) means a symmetry of mobility. The assumption (G4) describes a growth rate of $g(\cdot, \lambda)$ as $\lambda \rightarrow \pm\infty$. It is shown in [4] that if g is linear or superlinear, then the degenerate pinching singularity never occur. If (G4) fails, the degenerate pinching may occur. For the convex case, Andrews [1] show the existence of the

degenerate pinching singularities. In [6], some examples of the degenerate pinching for non-convex case are shown.

Under all assumptions, we can guarantee to solve the problem in the class of admissible polygons till enclosed area of solution polygon becomes zero.

THEOREM 1. *Let $\Omega(0)$ be a N_0 -sided admissible polygon. Assume (G1)–(G4) and (W1). Then, there exists a finite time $T_1 > 0$ such that $\Omega(t)$ is N_0 -sided admissible polygon for $0 \leq t < T_1$ and at least one edge disappears at $t = T_1$.*

Moreover, one of the following two phenomena occurs exclusively at $t = T_1$:

1. *$\Omega(t)$ shrinks to a single point, that is, $\lim_{t \rightarrow T_1} d_j(t) = 0$ for all j .*
2. *If at least one edge remains, then only inflection edges may disappear and the limit $\Omega(T_1) = \lim_{t \rightarrow T_1} \Omega(t)$ exists and is admissible. In addition, locally at most two consecutive edges disappear at $t = T_1$.*

REMARK 1. These three assumptions (G3), (G4) and (W1) are essential for the assertion. If one of them fails, then we can find counter examples. Schematic examples are shown in Figs. 2, 3 and 4. Fig. 2 shows that example motion of splitting phenomena when (W1) does not hold. In this case, the Wulff shape is a pentagon. The two center above edges have zero crystalline curvature and thus never move in each normal direction. The bottom edge moves upward and eventually vertex-edge type self-contacting happens. We see that the solution polygon splits into two particles and each polygon is not admissible. Fig. 3 shows that example motion of splitting phenomena via edge-edge type self-contacting when (G3) does not hold. In this case, the Wulff shape is a octagon. If $g((0, 1), \cdot) \ll g((0, -1), \cdot)$, the bottom edge moves upward faster than the above edge and eventually the both edges touch each other. We see that the solution polygon splits into two particles and each polygon is not admissible. Fig. 4 shows that example motion of degenerate pinching singularity when (G4) does not hold. In this case, the Wulff shape is a octagon. Because of the lack of (G4), two parts of the boundary collapse to line segments. The limit $\Omega(T_1)$ has line segments and is not admissible.

3.2. Almost convexity phenomena

We here characterize properties of solution polygon near the extinction time. Since N_0 is finite and new edges never be generated, the number of edges is monotone non-increasing in time. By Theorem 1, the edge-disappearing happens in a finite time. Thus, we have a finite sequence of edge-disappearing times:

$$0 < T_1 < T_2 < \cdots < T_m < \infty.$$

We also put $T_0 = 0$. From the previous theorem, the solution polygon shrinks to a single point at the extinction time $t = T_m$. We have the following:

THEOREM 2. *Suppose the same assumptions as in the previous theorem. Then, the number of edges with negative curvature is zero and the number of inflection edges is zero or two at $t = T_{m-1}$. Moreover, if there are two inflection edges at $t = T_{m-1}$, then each inflection edges are adjacent each other.*

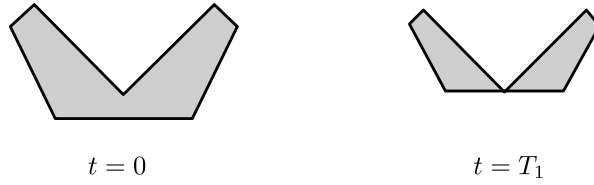


Fig. 2. Example motion when (W1) does not hold.

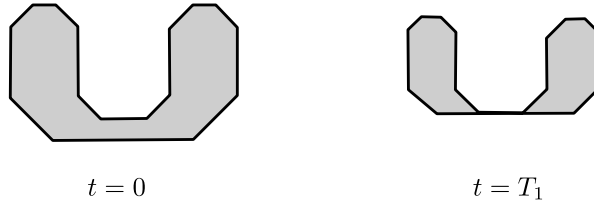


Fig. 3. Example motion when (G3) does not hold.

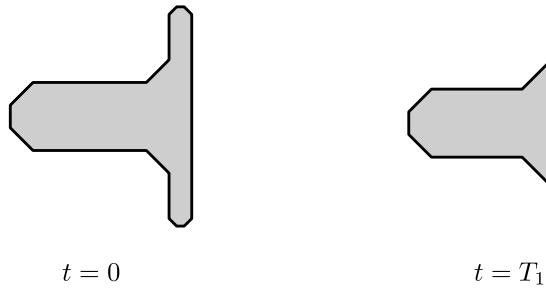


Fig. 4. Example motion when (G4) does not hold.

REMARK 2. If there are no inflection edges at $t = T_{m-1}$, $\Omega(t)$ is convex and all $H_j(t)$'s are positive for $t \geq T_{m-1}$. Non-convex self-similar solutions are typical examples for the case where there are two inflection edges at $t = T_{m-1}$ with $m = 1$.

From this theorem, we easily obtain that there exists the first time $T_* \leq T_{m-1}$ such that all curvature become non-negative for $t \geq T_*$. We can also characterize a shape of solution polygon near the extinction time in the case where there remain two inflection edges. Let S_k and S_{k+1} be inflection edges at $t = T_{m-1}$. Put $D_1(t) := \{z \in \Omega(t) \mid (p_k(t) - z) \cdot \mathbf{N}_k > 0\}$ and $D_2(t) := \{z \in \Omega(t) \mid (p_{k+1}(t) - z) \cdot \mathbf{N}_{k+1} > 0\}$. Noting that since $0 < |\theta_k - \theta_{k+1}| < \pi$ and the set $D(t) := D_1(t) \cap D_2(t)$ is not empty for $t \geq T_{m-1}$, we can choose a point $\mathbf{r}(t) \in D(t)$ for $t \geq T_{m-1}$. Then, the line segment from $\mathbf{r}(t)$ to any boundary point $\mathbf{p} \in \mathcal{P}(t)$ is contained in $\Omega(t)$. Because $\Omega(t) \cap D_1(t)$ (resp. $D_2(t)$) is convex. Thus, the solution polygon is star-shaped for $t \geq T_{m-1}$.

We summarize the above assertions as follows:

COROLLARY 1 (Almost convexity phenomena). *For any $t \geq T_{m-1}$, the solution polygon $\Omega(t)$ is star-shaped and $H_j(t) \geq 0$ for all j .*

REMARK 3. There exists the minimum time $T^* \leq T_{m-1}$ such that the solution polygon $\Omega(t)$ is star-shaped for any $t > T^*$. The order of T_* and T^* depends on the initial shape.

We finally mention the sufficient condition for the convexity phenomena. By Theorem 2, there is only one pair of adjacent inflection edges $t \geq T_{m-1}$ even if inflection edges exist. If Ω_0 is symmetric for some point $\mathbf{r}^* \in \mathbb{R}^2$, the solution polygon is also symmetric for \mathbf{r}^* since the flow is symmetric. Thus, the number of pairs of adjacent inflection edges are even and this leads that there are no inflection edges at $t = T_{m-1}$. Thus, we have the following:

COROLLARY 2 (Sufficient condition for convexity phenomenon). *If the initial polygon $\Omega(0)$ is point symmetric, then the solution polygon becomes convex at $t = T_{m-1}$.*

REMARK 4. In Proposition 6 in [4], it is shown that the convexity phenomena always appears for any initial polygons under the same assumptions as in our theorems. However, this result is based on Lemma 2 in [4] and the convexity statement in this lemma is overstated because there are some examples of non-convex self-similar solutions [8]. Thus, the sufficient condition on g and \mathcal{W}_σ under which the convexity phenomena always appears for any initial polygons is still open.

4. Proofs of theorems

Let $T_1 := \sup\{t \mid \inf_j d_j(t) > 0\}$ ($\leq \infty$). We first show that $d_j(t)$ as the solution of (2) has the limit as $t \rightarrow T_1$ when T_1 is finite.

LEMMA 1. *Assume that $T_1 < \infty$. Under the assumptions (G1) and (G2), there exist $\lim_{t \rightarrow T_1} d_j(t)$ for all j .*

Proof. Let $A := \liminf_{t \rightarrow T_1} d_j(t)$ and $B := \limsup_{t \rightarrow T_1} d_j(t)$. We will show $A = B$.

Case 1: $\chi_j = 0$.

By $\theta_{j+1} - \theta_j = -(\theta_j - \theta_{j-1})$, we have

$$\dot{d}_j(t) = \frac{1}{\sin(\theta_j - \theta_{j-1})} V_{j+1} - \frac{1}{\sin(\theta_j - \theta_{j-1})} V_{j-1}.$$

If $\theta_j - \theta_{j-1} > 0$ (resp. < 0), we have $\chi_{j-1} \geq 0$ (resp. \leq) and $\chi_{j+1} \leq 0$ (resp. \geq). Thus, we have $\dot{d}_j(t) \leq 0$ and we have $A = B$.

Case 2: $\chi_j = 1$.

Since $\chi_{j\pm 1} \geq 0$ and $0 < \theta_{j+1} - \theta_j, \theta_j - \theta_{j-1} < \pi$, we have

$$\dot{d}_j(t) \leq (\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1})) V_j. \tag{3}$$

If $\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1}) \leq 0$, then we have $\dot{d}_j(t) \leq 0$ since $V_j > 0$. Thus, $A = B$.

Let $\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1}) > 0$. Suppose that $A < B$. Then, since $d_j(t)$ is continuous in $[0, T_1)$, there exist time sequences $\{s_n\}$ and $\{t_n\}$ converging to T_1 as $n \rightarrow \infty$ such that $d_j(s_n) = (2A+B)/3$, $d_j(t_n) = (A+2B)/3$ and $s_n < t_n < s_{n+1}$. By mean value theorem, there is a time sequence $\{\tau_n\}$ such that $\tau_n \in (s_n, t_n)$,

$$d_j(\tau_n) \geq \frac{2A+B}{3} \quad (4)$$

and

$$\dot{d}_j(\tau_n) \geq \frac{B-A}{3(t_n - s_n)}. \quad (5)$$

By (3) and (4), we have

$$\dot{d}_j(\tau_n) \leq (\cot(\theta_{j+1} - \theta_j) + \cot(\theta_j - \theta_{j-1}))g\left(N_j, \frac{3l_\sigma(N_j)}{2A+B}\right) < +\infty. \quad (6)$$

However, from (5), we have $\lim_{n \rightarrow \infty} \dot{d}_j(\tau_n) = +\infty$. This leads a contradiction to (6). We can also prove the case where $\chi_j = -1$ in the same way. \square

From the above lemma, we can define $d_j(T_1) := \lim_{t \rightarrow T_1} d_j(t)$ for all j , if T_1 is finite. Then, $d_j(t)$'s are continuous on $[0, T_1]$. Therefore, the limit $\Omega(T_1) = \lim_{t \rightarrow T_1} \Omega(t)$ exists in the Hausdorff topology. We next show that self-contacting never occur for $0 < t < T_1$. Note that the strategy of the proof is similar to that in [7]. We extend their argument. For the later arguments, we introduce some notation. Let $\mathcal{L}_k(t)$ and $M_k(t)$ be a line which contains $\mathcal{S}_k(t)$ and a region where $\mathcal{S}_k(t)$ can move into, respectively. Namely,

$$\begin{aligned} \mathcal{L}_k(t) &:= \{z \in \mathbb{R}^2 \mid (p_k(t) - z) \cdot \mathbf{N}_k = 0\}, \\ M_k(t) &:= \{z \in \mathbb{R}^2 \mid \chi_k(p_k(t) - z) \cdot \mathbf{N}_k > 0\} \cup \mathcal{L}_k(t). \end{aligned}$$

We denote by $N(t)$ and $J(t)$ a number of edges of $\Omega(t)$ and a set of integers given by $J(t) := \{0, 1, \dots, N(t) - 1\}$, respectively. Notice that $N(t) = N_0$ and $J(t) = J(0)$ for $0 \leq t < T_1$. On transition number, we introduce the following new notation: Let $D \subset \{0, 1, \dots, N - 1\}$ and $m \in \{1, 0, -1\}$. If $\chi_j = m$ for all $j \in D$, we denote $\chi_D = m$.

LEMMA 2. *Assume that (G1)–(G3) and (W1). Then, the self-contacting never occur before $t = T_1$.*

Proof. Let $T < T_1$ be the first self-contacting time. Note that $d_j(T) > 0$ for all j . Then, There are three possible patterns of the self-contacting: (1) vertex-edge contact, (2) edge-edge contact and (3) vertex-vertex contact.

Case (1): vertex-edge contact.

Suppose that p_i contacts \mathcal{S}_j and \mathcal{S}_{i-1} and \mathcal{S}_i do not contact \mathcal{S}_j (see Fig. 5). From admissibility of $\Omega(t)$, we have $(s\mathbf{N}_i + (1-s)\mathbf{N}_{i-1})/|s\mathbf{N}_i + (1-s)\mathbf{N}_{i-1}| \notin \mathcal{N}_{\mathcal{W}_\sigma}$ for any $s \in (0, 1)$. By the assumption (W1), we have $-\mathbf{N}_j \in \mathcal{N}_{\mathcal{W}_\sigma}$.

However, by geometry, it is obvious that there exists $s^* \in (0, 1)$ such that $(s^* \mathbf{N}_i + (1 - s^*) \mathbf{N}_{i-1}) / |s^* \mathbf{N}_i + (1 - s^*) \mathbf{N}_{i-1}| = -\mathbf{N}_j$, which leads a contradiction (see Fig. 5).

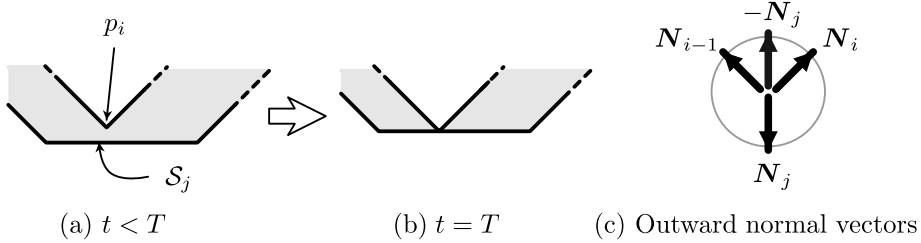


Fig. 5. Vertex-edge self contacting: $p_i(T) \in S_j(T)$.

Case (2): edge-edge contact.

S_i and S_j are parallel and contact each other at $t = T$ (see Fig. 6). The cases where $(\chi_i, \chi_j) = (0, 0), (1, 1), (-1, -1), (1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ are impossible because the distance between S_i and S_j does not decrease or a self-intersection of $\mathcal{P}(t)$ occurs before $t = T$ in these cases. Thus, there remain two cases: $(\chi_i, \chi_j) = (1, -1), (-1, 1)$. We only consider the case where $(\chi_i, \chi_j) = (1, -1)$ since the argument is symmetric.

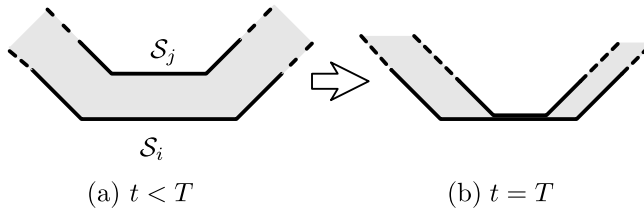


Fig. 6. Edge-edge self contacting: $S_j(T) \subseteq S_i(T)$.

Let $w(t)$ be a distance between S_i and S_j . Then $w(t) > 0$ for $0 \leq t < T$ and $w(T) = 0$. If $d_i(T) < d_j(T)$, then self-intersect must occur before $t = T$ and this contradicts the definition of T . Thus, we have $d_i(T) \geq d_j(T)$. There are two subcases (2-i): $d_i(T) > d_j(T)$ and (2-ii): $d_i(T) = d_j(T)$.

Subcase (2-i): $d_i(T) > d_j(T)$.

There exists $\delta > 0$ such that $d_i(t) > d_j(t)$ for all $t \in (T - \delta, T)$. By (W1), we have $l_\sigma(\mathbf{N}_i) = l_\sigma(\mathbf{N}_j)$ since $\mathbf{N}_j = -\mathbf{N}_i$. We then have

$$\begin{aligned} \dot{w}(t) &= -V_i - V_j \\ &= -g\left(\mathbf{N}_i, \frac{l_\sigma(\mathbf{N}_i)}{d_i}\right) - g\left(\mathbf{N}_j, -\frac{l_\sigma(\mathbf{N}_j)}{d_j}\right) \\ &= -g\left(\mathbf{N}_i, \frac{l_\sigma(\mathbf{N}_i)}{d_i}\right) + g\left(\mathbf{N}_i, \frac{l_\sigma(\mathbf{N}_i)}{d_j}\right) > 0. \end{aligned}$$

Here we have used the monotonicity and symmetry of g from (G2) and (G3). This contradicts $w(T) = 0$. Therefore, the subcase (2-i) does not arise.

Subcase (2-ii): $d_i(T) = d_j(T)$.

In this case, the both edges completely overlap each other at $t = T$ and thus the adjacent edges are also contact each other. If one of the adjacent edge-edge contacts is in subcase (2-i), we can apply the previous argument. If both of the adjacent edge-edge contacts are in subcase (2-ii), we again consider the next adjacent edge-edge contacts and finally, we can find the edge-edge contact in subcase (2-i) because if we can not find it, the solution polygon must collapse to a polygonal line, that is, some edges disappear at $t = T$ and thus this contradicts the fact that all $d_j(T)$ are positive. Therefore, the subcase (2-ii) also does not arise and the case (2) does not occur.

Case (3): vertex-vertex contact.

p_i and p_j contact each other. In this case, we have $\mathcal{S}_i \parallel \mathcal{S}_j$, $\mathcal{S}_{i-1} \parallel \mathcal{S}_{j-1}$ or $\mathcal{S}_i \parallel \mathcal{S}_{j-1}$, $\mathcal{S}_{i-1} \parallel \mathcal{S}_j$ because if these relations do not hold, a self-intersection of these edges occurs before $t = T$. Let $\eta_j := \text{sgn}(\theta_j - \theta_{j-1}) \in \{1, -1\}$. If $(\eta_i, \eta_j) = (1, -1)$ or $(-1, 1)$, then edge-edge contacts also occur and this cases reduces to the case (2) (see Fig. 7). Then, there remain two cases: $(\eta_i, \eta_j) = (1, 1)$ and $(-1, -1)$. We only consider the case $(\eta_i, \eta_j) = (1, 1)$ since we can discuss the other case in the same manner.

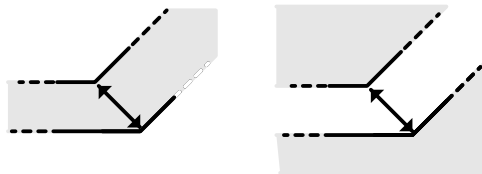


Fig. 7. Two possible positions of edges in the case where $(\eta_i, \eta_j) = (1, -1)$ or $(-1, 1)$ in vertex-vertex self contacting.

In Case (a) in Fig. 8, since all $d_j(T)$'s are positive, two intersections between \mathcal{S}_i and \mathcal{S}_{j-1} and between \mathcal{S}_{i-1} and \mathcal{S}_j must occur before $t = T$. Thus, Case (a) never appear. We next consider Case (b). Notice that $p_i(t) \in M_{i-1}(0) \cap M_i(0)$ and $p_j(t) \in M_{j-1}(0) \cap M_j(0)$ for $t > 0$. However, by the fact that $\chi_k \geq 0$ for $k = i, i-1, j, j-1$ and geometry, we have $(M_{j-1}(0) \cap M_j(0)) \cap (M_{i-1}(0) \cap M_i(0)) = \emptyset$. Thus, p_i and p_j never contact each other. Therefore, Case (b) does not occur and the case (3) also does not occur.

Hence, we have the assertion. \square

REMARK 5. In [7], they do not assume (W1) explicitly. If (W1) does not hold, the vertex-edge contacting may happen (see Fig. 2).

From this lemma, the solution polygon is admissible for $0 \leq t < T_1$. We next show $T_1 < \infty$.



Fig. 8. Two possible positions of edges in the case where $(\eta_i, \eta_j) = (1, 1)$ in vertex-vertex self contacting.

LEMMA 3. *Suppose the same assumptions as in the previous lemmas. Then, at least one edge of solution polygon of (1) disappears in a finite time.*

Proof. By the previous lemma, $\Omega(t)$ is admissible for $0 \leq t < T_1$. Then, for each $t \in [0, T_1)$, we can define a minimal convex admissible polygon $\tilde{\Omega}(t)$ which contains $\Omega(t)$ (see Fig. 9). By definition, $\Omega(t) \subseteq \tilde{\Omega}(t)$ for $0 \leq t < T_1$. By geometry, each edge of $\Omega(t)$ which touches $\partial\tilde{\Omega}(t)$ has a positive curvature and moves in its inward normal direction. Thus, each edge of $\tilde{\Omega}(t)$ also moves in its inward normal direction. Therefore, $\tilde{\Omega}(t_2) \subset \tilde{\Omega}(t_1)$ for $t_1 < t_2$. From this monotonicity, we have $d_j(t) \leq d^*$ for all j and $t > 0$, where $d^* = \sup_{x,y \in \tilde{\Omega}(0)} \text{dist}(x,y)$. By the monotonicity of g in (G2), any edge of $\tilde{\Omega}(t)$ has a positive inward normal velocity which is larger than $v_* := \min_j g(\mathbf{N}_j, l_\sigma(\mathbf{N}_j)/d^*) > 0$. Thus $\tilde{\Omega}(t)$ must collapse before or just at $t^* = d^*/v_*$. Since $\Omega(t) \subseteq \tilde{\Omega}(t)$, there exist k and $T_1 > 0$ such that $T_1 \leq t^*$ and $\liminf_{t \rightarrow T_1} d_k(t) = 0$. By Lemma 1, we have $\lim_{t \rightarrow T_1} d_k(t) = 0$. \square

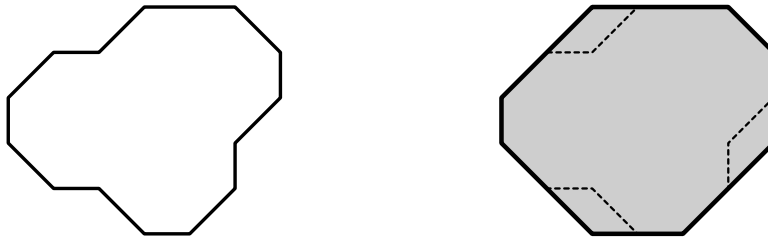


Fig. 9. $\Omega(t)$ (left) and $\tilde{\Omega}(t)$ (right). The dotted line in right figure shows $\Omega(t)$.

Here and hereafter, we always assume all conditions in Theorem 1. By Lemmas 3 and 2, at least one edge disappears at $t = T_1$ and $\Omega(t)$ is N_0 -sided admissible polygon for $0 \leq t < T_1$. If all edges disappear at the same time, T_1 is the extinction time and the solution polygon shrink to a single point (the case 1 of Theorem 1). The next lemma characterizes the disappearing edges when some edges remain at $t = T_1$ (the case 2 of Theorem 1).

PROPOSITION 1. *Suppose that there exists at least one edge whose length is positive at $t = T_1$. Then, only inflection edge may disappear at $t = T_1$. Namely, if $\lim_{t \rightarrow T_1} d_k(t) = 0$, then $\chi_k = 0$.*

The following result is due to M.-H. Giga and Y. Giga [4].

LEMMA 4. *Suppose that $|\theta_{j_0} - \theta_{j_1}| = \pi$ ($j_0 < j_1$) and $\chi_j \neq 0$ for $j \in Q := \{j_0 + 1, j_0 + 2, \dots, j_1 - 1\}$. If \mathcal{S}_k ($k = j_0, j_1$) satisfies $d_k(T_1) > 0$ unless $\chi_k = 0$, then the phenomena that $d_j(T_1) = 0$ for all $j \in Q$ does not happen.*

REMARK 6. When $\chi_j = 0$ ($j = j_0$ or j_1), $d_j(T_1) > 0$ is not necessary.

For reader's convenience, we give a proof.

Proof. Here and in the sequel we denote by C the various positive constants and $A \sim B$ means that $c_1 A \leq B \leq c_2 A$ for some positive constants c_1 and c_2 .

Suppose on contrary that $d_j(T_1) = 0$ for all $j \in Q$. Let $w(t)$ be the distance between $\mathcal{L}_{j_0}(t)$ and $\mathcal{L}_{j_1}(t)$. By geometry, $w(t) > 0$ for $0 \leq t < T_1$ and $\lim_{t \rightarrow T_1} w(t) = 0$. If $(\chi_{j_0}, \chi_{j_1}) = (0, 0)$, then $w(t) = w(0)$ since $\dot{w}(t) = -(V_{j_0} + V_{j_1}) = 0$. Thus, $(\chi_{j_0}, \chi_{j_1}) \neq (0, 0)$. We only consider the case where $\chi_Q, \chi_{j_0} = 1$ since the argument is symmetric.

Since $\chi_{j_0} = 1$, we have $V_{j_0-1} \geq 0$. Noting that V_{j_0} is bounded, $m := \sin(\theta_{j_0+1} - \theta_{j_0})$ and $\sin(\theta_{j_0} - \theta_{j_0-1})$ are positive constants and $\cot(\theta_{j_0+1} - \theta_{j_0}) + \cot(\theta_{j_0} - \theta_{j_0-1})$ is bounded, by virtue of (2), we have

$$\dot{d}_{j_0} \leq C - \frac{1}{m} V_{j_0+1}. \quad (7)$$

Since $\chi_{j_0+1} = 1$ and $d_{j_0+1}(T_1) = 0$, we have $\lim_{t \rightarrow T_1} V_{j_0+1}(t) = \infty$. Thus, $\dot{d}_{j_0}(t) \rightarrow -\infty$ as $t \rightarrow T_1$ and there exists $\delta > 0$ such that $\dot{d}_{j_0}(t) < 0$ for $t \in [T_1 - \delta, T_1]$.

Noting that $d_{j_0}(t) \sim 1$ and $d_{j_1}(t) \sim 1$ if $\chi_{j_1} = 1$, we have $\dot{w}(t) \sim -1$ in $[0, T_1]$. Thus,

$$I := \int_{T_1-\delta}^{T_1} \dot{d}_{j_0} \dot{w} dt \leq C \int_{T_1-\delta}^{T_1} (-\dot{d}_{j_0}) dt = C[d_{j_0}(T_1 - \delta) - d_{j_0}(T_1)] < \infty.$$

On the other hand, by geometry, we have $m d_{j_0+1}(t) \leq w(t)$. By monotonicity of g , we have $V_{j_0+1} \geq g(\mathbf{N}_{j_0+1}, ml_\sigma(\mathbf{N}_{j_0+1})/w)$. Thus, by virtue of (7), we have

$$I \geq \int_{T_1-\delta}^{T_1} \left(C - \frac{1}{m} V_{j_0+1} \right) \dot{w} dt \geq C - \frac{1}{m} \int_{T_1-\delta}^{T_1} g(\mathbf{N}_{j_0+1}, ml_\sigma(\mathbf{N}_{j_0+1})/w) \dot{w} dt.$$

By assumption (G4), we have $I \geq \infty$ which leads a contradiction. Thus, we have the assertion. \square

Proof of Proposition 1. By Lemma 2, there exist at least one edge disappearing at $t = T_1$ and $\Omega(t)$ is N_0 -sided admissible polygon in $[0, T_1]$. Because $\mathcal{P}(t)$ is a closed polygonal curve, there are at least two edges whose lengths are positive at $t = T_1$. Hence, for any i such that $\lim_{t \rightarrow T_1} d_i(t) = 0$, we can find \mathcal{S}_{j_0} and

\mathcal{S}_{j_i} ($j_0 < j_1$) such that $d_{j_0}(T_1), d_{j_1}(T_1) > 0$, $\lim_{t \rightarrow T_1} d_j(t) = 0$ for any $j \in Q := \{j_0 + 1, \dots, j_1 - 1\}$ and $i \in Q$. Let $p^* = (x^*, y^*)$ be the meeting point of $p_j(t)$'s ($j \in Q \cup \{j_1\}$) at $t = T_1$, that is, $\lim_{t \rightarrow T_1} p_j(t) = p^*$ for all $j \in Q \cup \{j_1\}$.

We will show that only inflection edges may disappear, that is, $Q = Q_0 := \{j \in Q \mid \chi_j = 0\}$. Suppose on contrary, we assume that there exists $i^* \in Q$ such that $\chi_{i^*} \neq 0$. Since the argument is symmetric, we only consider the case where $\chi_{i^*} = 1$ and without loss of generality, we may assume $\theta_{j_0} = 0$.

We divide the argument into the following three cases: the case where Q_0 is empty, the case where $\#Q_0 = 1$ and the case where $\#Q_0 \geq 2$. Here $\#Q_0$ denotes the number of elements in Q_0 .

Case 1: Q_0 is empty.

Note that $\chi_Q = 1$ since $i^* \in Q$. By geometry, there are four possibilities: $0 < \theta_{j_1} < \pi$, $\theta_{j_1} = \pi$, $\pi < \theta_{j_1} < 2\pi$ and $\theta_{j_1} \geq 2\pi$.

Subcase 1-1: $0 < \theta_{j_1} < \pi$.

Let \mathcal{L}^* be a line which is parallel to \mathcal{L}_{j_0} and passing through p^* and $q(t) = (x_q(t), y_q(t)) := \mathcal{L}_{j_0}(t) \cap \mathcal{L}_{j_1}(t)$. Choose any $i \in Q$. We introduce two new points: $z_i(t) = (x_{z_i}(t), y_{z_i}(t)) := \mathcal{L}^* \cap \mathcal{L}_i(t)$ and $z_q(t) = (x_{z_q}(t), y_{z_q}(t)) := \mathcal{L}^* \cap \{z \in \mathbb{R}^2 \mid (z - q) \cdot \mathbf{N}_i = 0\}$ and define $w(t) = y_{z_q}(t) - y_{z_i}(t)$. Then we have $x_{z_i}(t) = x_{z_q}(t) = x^*$, $y_q(t) > y_{z_i}(t) > y^*$ for $0 \leq t < T_1$ and $\lim_{t \rightarrow T_1} y_{z_q}(t) = \lim_{t \rightarrow T_1} y_{z_i}(t) = y^*$, from which it follows that $w(t) > 0$ for $t \in [0, T_1)$ and $\lim_{t \rightarrow T_1} w(t) = 0$. On the other hand,

$$\begin{aligned} \dot{w} &= \dot{y}_{z_q}(t) - \dot{y}_{z_i}(t) \\ &= -\frac{V_{j_1}}{\sin \theta_{j_1}} + \frac{V_i}{\sin \theta_i} \rightarrow \infty \quad \text{as } t \rightarrow T_1. \end{aligned}$$

This is a contradiction. Therefore, Subcase 1-1 does not happen.

Subcase 1-2: $\theta_{j_1} = \pi$.

By Lemma 4, this subcase does not happen.

Subcase 1-3: $\pi < \theta_{j_1} < 2\pi$.

By geometry, $x_{j_0}(t) = x_{j_0+1}(t)$ and $x_{j_1}(t) < x_{j_1+1}(t)$. From $p_{j_0+1}(T_1) = p_{j_1}(T_1) = p^*$ and $d_{j_1}(T_1) > 0$, we have $x_{j_0}(t) < x_{j_1+1}(t)$ near $t = T_1$. Since $d_{j_0}(T_1), d_{j_1}(T_1) > 0$, $\chi_Q = 1$ and $\sum_{j=j_0+1}^{j_1-1} d_j(t) \rightarrow 0$ as $t \rightarrow T_1$, a self-intersection of $\sum_{j=j_0}^{j_1} \mathcal{S}_j$ may occur strictly before $t = T_1$ and this leads the contradiction.

Subcase 1-4: $\theta_{j_1} \geq 2\pi$.

By geometry, $\sum_{j=j_0+1}^{j_1-1} d_j(t) > \min(d_{j_0}(t), d_{j_1}(t))$ is necessary to prevent a self-contacting before $t = T_1$. However, $\min(d_{j_0}(t), d_{j_1}(t)) \geq \inf_{0 \leq t \leq T_1} \min(d_{j_0}(t), d_{j_1}(t)) > 0$ and $\sum_{j=j_0+1}^{j_1-1} d_j(t) \rightarrow 0$ as $t \rightarrow T_1$. Thus, this leads a contradiction.

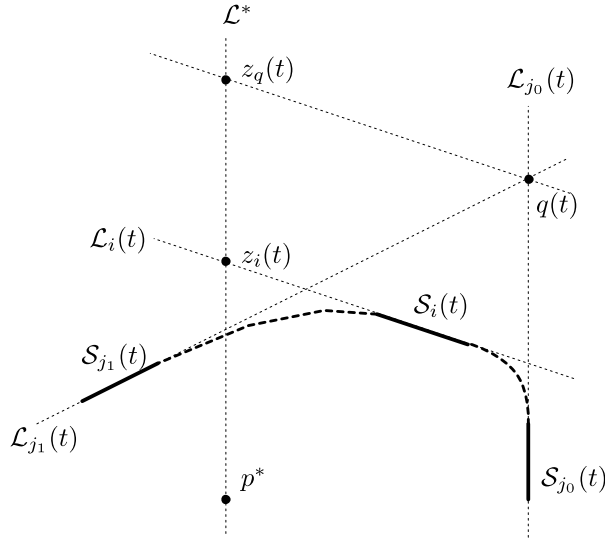


Fig. 10. Subcase 1-1.

Therefore, Case 1 does not happen.

Case 2: $Q_0 = \{k\}$.

There are two possibilities: the case where $k \neq j_0 + 1, j_1 - 1$ and the case where $k \in \{j_0 + 1, j_1 - 1\}$.

Subcase 2-1: $k \neq j_0 + 1, j_1 - 1$.

Let $Q_1 := \{j_0 + 1, \dots, k - 1\}$ and $Q_2 := \{k + 1, \dots, j_1 - 1\}$. We consider only the case $i^* \in Q_1$. Then, $\chi_{Q_1} = 1$ and $\chi_{Q_2} = -1$. Since $\chi_{k-1} = 1, \chi_{k+1} = -1$ and $\chi_k = 0$, we have $\mathcal{S}_k(t_2) \subset \mathcal{S}_k(t_1)$ for $t_1 < t_2$. Thus, $p^* \in \bigcap_{t \in [0, T_1]} \mathcal{S}_k(t) \subset \mathcal{S}_k(0)$. We also have $\chi_{j_0} = 1$ and $\chi_{j_1} = -1$ since Q_1 and Q_2 are not empty and $p_{j_0+1}(T_1) = p_{j_1}(T_1) = p^* \in \mathcal{S}_k(0)$.

If $\theta_k < \pi$, by the argument similar to that in Subcase 1-1 (we consider $\mathcal{L}_k(t)$ instead of $\mathcal{L}_{j_1}(t)$), we can show that this case does not happen. If $\theta_k = \pi$, we can also apply Lemma 4 and show that this subcase also does not happen.

Thus, $\theta_k > \pi$. We also have $\theta_k - \theta_{j_1} > \pi$ in the same way.

We first consider the case where $(2m - 1)\pi < \theta_k < 2m\pi$ for some integer $m \geq 1$. We introduce a new time variable $\tau = T_1 - t$ and define $\tilde{f}(\tau) = f(t)$ for some function f . Note that $\tilde{\mathcal{L}}_k(\tau) = \tilde{\mathcal{L}}_k(0)$ because of $\chi_k = 0$. Since $\tilde{p}_{j_0+1}(0) = p^*$ and $\tilde{d}_{j_0}(0) > 0$, $\tilde{p}_{j_0}(\tau)$ remains in the half space below $\tilde{\mathcal{L}}_k(0)$ in $\tau \in [0, \tau_0]$ for some $\tau_0 > 0$. Thus, if the condition that $\tilde{x}_{j_0}(\tau) > \tilde{x}_{k+1}(\tau)$ for $0 < \tau \leq \tau_0$ fails, the self-intersection of the curve $\sum_{j=j_0}^k \tilde{\mathcal{S}}_j(\tau)$ must happen for small $\tau > 0$ (see Fig. 11). We show below that this condition never be satisfied.

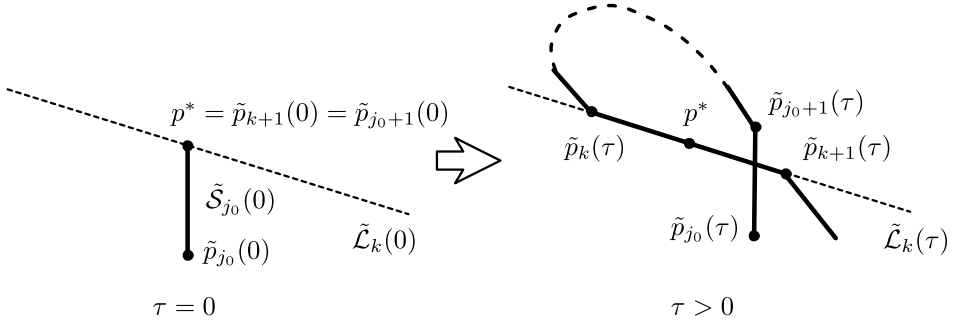


Fig. 11. Subcase 2-1.

Since $\mathcal{S}_{j_0}(t)$ does not disappear, we have $V_{j_0}(t) \sim +1$ and thus

$$\frac{d\tilde{x}_{j_0}(\tau)}{d\tau} \sim +1.$$

On the other hand, since $\lim_{t \rightarrow T_1} V_{k+1} = -\infty$ and $(2m - 1)\pi < \theta_k < 2m\pi$, we have

$$\left. \frac{d\tilde{x}_{k+1}(\tau)}{d\tau} \right|_{\tau=0} = +\infty.$$

Thus, put $\tilde{w}(\tau) = \tilde{x}_{k+1}(\tau) - \tilde{x}_{j_0}(\tau)$, then we have $\tilde{w}(0) = 0$ and

$$\left. \frac{d\tilde{w}(\tau)}{d\tau} \right|_{\tau=0} = +\infty.$$

Hence, $\tilde{w}(\tau) > 0$ in $(0, \tau_0)$ for some $\tau_0 > 0$. Thus, this case does not happen.

For the case $2m\pi < \theta_k < (2m + 1)\pi$, we consider $\tilde{x}_k(\tau)$ instead of $\tilde{x}_{k+1}(\tau)$. Then, we can prove in the same way.

For the cases $\theta_k = 2m\pi$ and $\theta_k = (2m + 1)\pi$, the arguments are similar to the above but slight different. We only mention the first case. By (W1) and $\theta_k - \theta_{j_1} > \pi$, there exists $j_2 \in Q_2$ such that $\mathbf{N}_{j_2} = -\mathbf{N}_{j_0}$. Since $\tilde{p}_{j_0+1}(0) = p^*$, $\tilde{d}_{j_0}(0) > 0$ and $\tilde{p}_j(0) = p^*$ for $j = k + 1, k + 2, \dots, j_2$, we have $\tilde{y}_{j_0}(\tau) < y^* - \tilde{d}_{j_0}(0)/2$ and $\tilde{y}_j(\tau) > y^* - \tilde{d}_{j_0}(0)/2$ for $j = k + 1, k + 2, \dots, j_2$ in $\tau \in [0, \tau'_0]$ for some $\tau'_0 > 0$. Thus, $\tilde{p}_{j_0}(\tau)$ is below the curve $\sum_{j=k+1}^{j_2-1} \tilde{\mathcal{S}}_j(\tau)$ in $\tau \in [0, \tau'_0]$. On the other hand, noting that $\frac{d\tilde{x}_{j_0}(\tau)}{d\tau} \sim +1$ and $\left. \frac{d\tilde{x}_{j_2}(\tau)}{d\tau} \right|_{\tau=0} = +\infty$ since $\chi_{j_2} = -1$ and $\tilde{d}_{j_2}(0) = 0$, we have $x^* < \tilde{x}_{j_0}(\tau) < \tilde{x}_{j_2}(\tau)$ near $\tau = 0$. Therefore, the curve $\sum_{j=k}^{j_2} \tilde{\mathcal{S}}_j(\tau)$ must intersect the curve $\sum_{j=j_0}^{k-1} \tilde{\mathcal{S}}_j(\tau)$. Thus, this case does not happen.

Hence, Subcase 2-1 never occur.

Subcase 2-2: $k \in \{j_0 + 1, j_1 - 1\}$.

We only consider the case $k = j_1 - 1$. Let $Q_1 := \{j_0 + 1, \dots, k - 1\}$. Since $\chi_{Q_1} = 1$ and $\chi_k = 0$, we have $\chi_{j_0} \geq 0$ and $\chi_{j_1} \leq 0$ and we also have $\mathcal{S}_k(t_2) \subseteq \mathcal{S}_k(t_1)$

for $t_1 < t_2$. Thus, $p^* \in \mathcal{S}_k(0)$. Since Q_1 are not empty and $p_{j_0+1}(T_1) = p^* \in \mathcal{S}_k(0)$, we have $\chi_{j_0} = 1$.

By the same argument as in the previous subcase, we have $\theta_k > \pi$. Then, by (W1), there exists $i_0 \in Q_1$ such that $\theta_{i_0} = \pi$. Since $\chi_k = 0$ and $\chi_{Q_1} = 1$, we have $\theta_{j_1} = \theta_{k-1} \geq \theta_{i_0}$ and thus $\theta_{j_1} \geq \pi$.

When $\theta_{j_1} > \pi$, by the same arguments as in Subcase 1-3 and 1-4, \mathcal{S}_{j_1} intersects \mathcal{S}_{j_0} or the subarc $\sum_{j=j_0+1}^{j_1-1} \mathcal{S}_j$ before $t = T_1$. Thus, only one possibility is the case where $\theta_{j_1} = \pi$, that is, \mathcal{S}_{j_1} is parallel to \mathcal{S}_{j_0} . Let $Q' = \{0, 1, \dots, N - 1\} \setminus \{j_0, j_0 + 1, \dots, j_1\}$. From $\chi_{k-1} = 1$ and $\chi_k = 0$, we have $\chi_{j_1} \leq 0$.

Subcase 2-2-1: $\chi_{j_1} = 0$.

Note that $p^* = p_{j_1}(0)$ and $\chi_{j_1+1} \geq 0$. If $\chi_{Q'} = 1$, then $\mathcal{S}_j(t)$ is located between $\mathcal{L}_{j_0}(t)$ and $\mathcal{L}_{j_1}(t)$ and is not parallel to \mathcal{S}_{j_0} and \mathcal{S}_{j_1} for any $j \in Q'$. Noting that $\text{dist}(\mathcal{L}_{j_0}(t), \mathcal{L}_{j_1}(t)) \rightarrow 0$ as $t \rightarrow T_1$, we have $d_j(T_1) = 0$ for all $j \in Q'$. By Lemma 4, we deduce that this motion is impossible since $d_{j_0}(T_1), d_{j_1}(T_1) > 0$. Thus, there is at least one inflection edge on the curve $C_1(t) := \sum_{j \in Q'} \mathcal{S}_j(t)$. Note that $\chi_{j_0} = 1$ and $\chi_{j_1+1} \geq 0$.

Step 1. Let $R_1(t)$ be the region enclosed by the curve $\sum_{j=j_0}^{j_1} \mathcal{S}_j(t)$, $\mathcal{L}_{j_0-1}(t)$ and $\mathcal{L}_{j_1+1}(t)$. We will show that there exists $\mu_1 > 0$ such that for any $j^* \in Q' \setminus \{j_1 + 1\}$, $\bigcup_{t \in (T_1 - \mu_1, T_1)} (R_1(t) \cap \mathcal{S}_{j^*}(t)) = \emptyset$ if $\chi_{j^*} = 0$.

Suppose on contrary that there exist a time sequence $\{t_k\}$ and $j^* \in Q' \setminus \{j_1 + 1\}$ such that $t_k \uparrow T_1$ ($k \rightarrow \infty$), $\chi_{j^*} = 0$ and $R_1(t_k) \cap \mathcal{S}_{j^*}(t_k) \neq \emptyset$ (see Fig. 12, (1)). Since $p_j(t)$'s ($j \in Q$) converge to $p^* = p_{j_1}(0)$ and $\text{dist}(\mathcal{L}_{j_0}(t), \mathcal{L}_{j_1}(0)) \rightarrow 0$, $R_1(t)$ collapses to a line segment $\Gamma_1 \subset \mathcal{L}_{j_1}(0)$ at $t = T_1$. Note that the area of $R_1(t)$ tends to zero as $t \rightarrow T_1$. Thus, $\text{dist}(\Gamma_1, \mathcal{S}_{j^*}(t_k)) \rightarrow 0$ as $k \rightarrow \infty$. On the other hand, noting that $\text{dist}(\mathcal{S}_{j^*}(t_1), \mathcal{S}_{j_1}(t_1)) > 0$ and $\Gamma_1 = \mathcal{S}_{j_0}(T_1) \cap \mathcal{S}_{j_1}(T_1) \subseteq \mathcal{S}_{j_1}(T_1) \subseteq \mathcal{S}_{j_1}(t_1)$ since $\chi_{j_1} = 0$, we have $\text{dist}(\Gamma_1, \mathcal{S}_{j^*}(t_1)) > 0$. From $\chi_{j^*} = 0$, we have $\mathcal{S}_{j^*}(t_k) \subseteq \mathcal{S}_{j^*}(t_1)$ for any k . Thus, $\text{dist}(\Gamma_1, \mathcal{S}_{j^*}(t_k)) \geq \text{dist}(\Gamma_1, \mathcal{S}_{j^*}(t_1)) > 0$ for any k . This leads a contradiction.

Step 2. The assertion in the previous step leads that $\chi_{j_0-1} = 0$ or $\chi_{j_1+1} = 0$ since the transition number of every edge never change till $t = T_1$. We will show that if $\chi_{j_0-1} = 0$, then $x^* \geq x_{j_0-1}(t)$ for any $t \in [0, T_1]$.

Suppose on contrary that $\chi_{j_0-1} = 0$ and there exists $t' \in [0, T_1]$ such that $x^* < x_{j_0-1}(t')$ (see Fig. 12, (2)). From $\theta_{j_0-2} = \theta_{j_0} = 0$ and $\chi_{j_0-2} \leq 0$, we have $\dot{x}_{j_0-1} \geq 0$. Since $d_{j_0-1}(t) > 0$ in $[0, T_1]$, we get $x^* < x_{j_0-1}(t') \leq x_{j_0-1}(t) < x_{j_0}(t)$ for any $t \in [t', T_1]$. This contradicts the fact that $\lim_{t \rightarrow T_1} x_{j_0}(t) = x^*$.

Step 3. If p^* is below $\mathcal{L}_{j_0-1}(t'')$ for some $t'' \in [0, T_1]$, then p^* is below $\mathcal{L}_{j_0-1}(t)$ for any $t \in [t'', T_1]$ since $\chi_{j_0-1} \geq 0$. However, this leads a contradiction because $p_{j_0+1}(t)$ is above $\mathcal{L}_{j_0-1}(t)$ for $t \in [0, T_1]$ and $\lim_{t \rightarrow T_1} p_{j_0+1}(t) = p^*$. Thus, p^* is above or on $\mathcal{L}_{j_0-1}(t)$ for any $t \in [0, T_1]$.

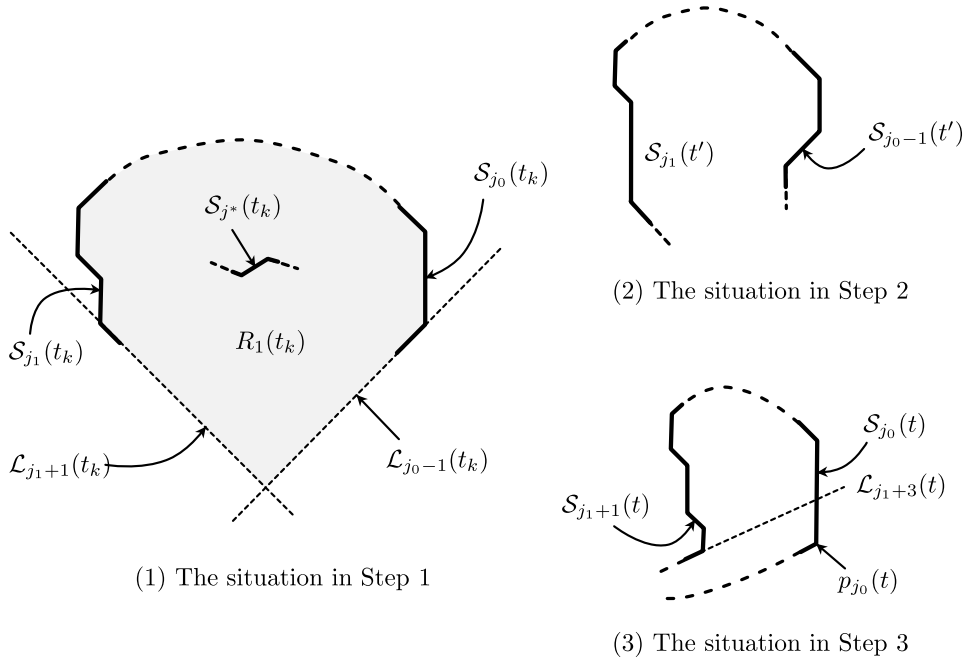


Fig. 12. Three situations in Subcase 2-2-1: (1) the situation in Step 1, (2) the situation in Step 2 and (3) the situation in Step 3.

Step 4. First, we will show $\chi_{j_1+1} = 0$. Suppose $\chi_{j_1+1} \neq 0$. Then, $\chi_{j_0-1} = 0$ by Step 1. By geometry, Step 2 and Step 3 lead that $p_{j_1+1}(t)$ is above $\mathcal{L}_{j_0-1}(t)$. This is a contradiction with Step 1 and $\chi_{j_1+1} \neq 0$. Hence, $\chi_{j_1+1} = 0$.

Note that $\chi_{j_1+2} \leq 0$. Let $\chi_{j_1+2} = 0$. Then $\mathcal{S}_{j_1+2} \parallel \mathcal{S}_{j_0}$ and $x^* < x_{j_1+2}(t) = x_{j_1+2}(0)$ for any $t \in [0, T_1)$. Since $\dot{x}_{j_0}(t) < 0$ and $x_{j_0}(t) \rightarrow x^*$ as $t \rightarrow T_1$, there exists $t_* \in [0, T_1)$ such that $x_{j_0}(t) < x_{j_1+2}(0) = x_{j_1+2}(t)$ for any $t \in [t_*, T_1)$. By geometry, it is needed to prevent a self-intersection of the boundary that $p_{j_0}(t)$ is above $\mathcal{L}_{j_1+1}(t) = \mathcal{L}_{j_1+1}(0)$ for any $t \in [t_*, T_1)$. This is a contradiction with Step 1. Thus, $\chi_{j_1+2} = -1$.

Next we will show that $p_{j_0}(t)$ is below $\mathcal{L}_{j_1+3}(t)$ for any t near T_1 . Suppose on contrary that there exists a time sequence $\{s_k\}$ such that $s_k \uparrow T_1$ ($k \rightarrow \infty$) and $p_{j_0}(s_k)$ is above or on $\mathcal{L}_{j_1+3}(s_k)$. Note that p^* is above or on $\mathcal{L}_{j_0-1}(t)$ for any $t \in [0, T_1)$. Then we have $x_{j_0-1}(s_k) > x^*$ for any s_k and thus $\chi_{j_0-1} = 1$ by Step 2. Hence, $C_1(s_k)$ has at least one inflection edge in $R_2(s_k)$ defined as the region enclosed by the curve $\sum_{j=j_0}^{j_1+2} \mathcal{S}_j(s_k)$ and $\mathcal{L}_{j_0-1}(s_k)$. Then, there exist a subsequence $\{s'_k\}$ and an inflection edge \mathcal{S}_{j^*} such that $s'_k \rightarrow T_1$ ($k \rightarrow \infty$) and $\mathcal{S}_{j^*}(s'_k)$ is in $R_2(s'_k)$. Since $x_{j_0}(s'_k) \rightarrow x^*$ as $k \rightarrow \infty$ and $p_j(t)$'s ($j \in Q$) converge to p^* , $R_2(s'_k)$ collapses to a line segment on $\mathcal{L}_{j_1}(0)$ as $k \rightarrow \infty$. Thus, by the similar argument to that in Step 1, we have a contradiction. Therefore, $\mathcal{S}_{j_0-1}(t)$ is below $\mathcal{L}_{j_1+3}(t)$ for any t near T_1 (see Fig. 12, (3)). Hence we have $x^* (= x_{j_1}(t)) < x_{j_1+2}(t) < x_{j_0}(t)$ and $x_{j_0}(T_1) = x_{j_1+2}(T_1) = x^*$. We have $d_{j_1}(T_1) + d_{j_1+2}(T_1) = y_{j_1}(T_1) - y_{j_1+3}(T_1) \leq$

$y_{j_0+1}(T_1) - y_{j_0}(T_1) = d_{j_0}(T_1)$. Thus, $d_{j_0}(t) > d_{j_1+2}(t)$ for any t near T_1 , from which it follows that $x_{j_0}(t) - x_{j_1+2}(t)$ increases for any t near T_1 . Thus, we get a contradiction. Consequently, Subcase 2-2-1 does not occur.

Subcase 2-2-2: $\chi_{j_1} = -1$.

We first show that $d_{j_0}(T_1) = d_{j_1}(T_1)$. Assume that $d_{j_0}(T_1) < d_{j_1}(T_1)$. Since $p_{j_0+1}(T_1) = p_{j_1}(T_1) = p^*$, we have that $d_{j_0}(t) < d_{j_1}(t)$ and $p_{j_0}(t)$ is above $\mathcal{L}_{j_1+1}(t)$ near $t = T_1$. Thus, by geometry, \mathcal{S}_{j_0-1} is an inflection edge or the curve $C_1(t)$ has at least one inflection edge $\mathcal{S}_{j_*}(t)$ in the interior region enclosed by the curve $\sum_{j=j_0}^{j_1} \mathcal{S}_j(t)$ and $\mathcal{L}_{j_0-1}(t)$. However, by the same argument as in the previous subcase 2-2-1, we see that the self-intersection of $\mathcal{P}(t)$ or the edge-disappearing occurs before $t = T_1$ and this leads a contradiction to the definition of T_1 . Thus, $d_{j_0}(T_1) \geq d_{j_1}(T_1)$. If $d_{j_0}(T_1) > d_{j_1}(T_1)$, then $V_{j_0}(t) < -V_{j_1}(t)$ near $t = T_1$, from which it follows that $\text{dist}(\mathcal{L}_{j_0}(t), \mathcal{L}_{j_1}(t))$ increases near $t = T_1$. This leads to a contradiction. Therefore, we have $d_{j_0}(T_1) = d_{j_1}(T_1)$, that is, $p_{j_0}(T_1) = p_{j_1+1}(T_1)$.

We next show that $d_{j_0}(t) > d_{j_1}(t)$ in $[T_1 - \delta, T_1)$ for some $\delta > 0$. Note that by convexity of the Wulff shape and (W1), $\theta_{j_0+1} - \theta_{j_0-1} \leq \pi$. In the case where $\theta_{j_0+1} - \theta_{j_0-1} = \pi$, it is obvious that $d_{j_0}(t) > d_{j_1}(t)$ for $t < T_1$ since $p_{j_0}(t)$ is below $\mathcal{L}_{j_1+1}(t)$ and $p_{j_0+1}(t)$ is above $\mathcal{L}_k(0)$. Thus, $\theta_{j_0+1} - \theta_{j_0-1} < \pi$. By using the new time variable τ , we have

$$\begin{aligned} \frac{d}{d\tau} \tilde{d}_{j_0}(\tau) &= -c_1 \tilde{V}_{j_0} + c_2 \tilde{V}_{j_0+1} + c_3 \tilde{V}_{j_0-1}, \\ \frac{d}{d\tau} \tilde{d}_{j_1}(\tau) &= c_1 \tilde{V}_{j_1} - c_2 \tilde{V}_k - c_3 \tilde{V}_{j_1+1}. \end{aligned}$$

Here $c_1 = \cot(\theta_{j_0+1} - \theta_{j_0}) + \cot(\theta_{j_0} - \theta_{j_0-1}) > 0$, $c_2 = 1/\sin(\theta_{j_0+1} - \theta_{j_0}) > 0$ and $c_3 = 1/\sin(\theta_{j_0} - \theta_{j_0-1}) > 0$. Note that $\mathcal{S}_{j_0-1} \parallel \mathcal{S}_{j_1+1}$. By geometry, it is needed that $\tilde{p}_{j_0}(\tau)$ is below $\tilde{\mathcal{L}}_{j_1+1}(\tau)$ for $\tau > 0$ to prevent the self-intersection for $\tau > 0$. Thus, we have

$$\begin{aligned} &\text{dist}(\tilde{\mathcal{L}}_{j_1+1}(0), \tilde{\mathcal{L}}_{j_0-1}(\tau)) - \text{dist}(\tilde{\mathcal{L}}_{j_1+1}(0), \tilde{\mathcal{L}}_{j_1+1}(\tau)) \\ &= \int_0^\tau (\tilde{V}_{j_0-1}(\tau') - (-\tilde{V}_{j_1+1}(\tau'))) d\tau' > 0 \end{aligned}$$

since $\chi_{j_0-1} \geq 0$ and $\chi_{j_1+1} \leq 0$. Integrating $\frac{d}{d\tau}(\tilde{d}_{j_0} - \tilde{d}_{j_1})$ from 0 to τ , we have

$$\begin{aligned} \tilde{d}_{j_0}(\tau) - \tilde{d}_{j_1}(\tau) &= \int_0^\tau (-c_1(\tilde{V}_{j_0} + \tilde{V}_{j_1}) + c_2(\tilde{V}_{j_0+1} + \tilde{V}_k) + c_3(\tilde{V}_{j_0-1} + \tilde{V}_{j_1+1})) d\tau' \\ &> \int_0^\tau (-c_1(\tilde{V}_{j_0} + \tilde{V}_{j_1}) + c_2 \tilde{V}_{j_0+1}) d\tau' \\ &> 0. \end{aligned}$$

for small $\tau > 0$. Here we use the facts that $\tilde{V}_k(\tau) = 0$, $\lim_{\tau \rightarrow 0} \tilde{V}_{j_0+1}(\tau) = +\infty$ and $\tilde{V}_{j_1}(\tau) + \tilde{V}_{j_0}(\tau)$ is bounded. Therefore, for small $\tau > 0$, we have $-\tilde{H}_{j_1}(\tau) > \tilde{H}_{j_0}(\tau)$ and thus $\frac{d}{d\tau} \tilde{x}_{j_1}(\tau) = -\tilde{V}_{j_1}(\tau) > \tilde{V}_{j_0}(\tau) = \frac{d}{d\tau} \tilde{x}_{j_0}(\tau)$. Note that $-\tilde{V}_{j_1}(0) = \tilde{V}_{j_0}(0)$

and $\tilde{x}_{j_1}(0) = \tilde{x}_{j_0}(0)$. Then, we have $\tilde{x}_{j_0}(\tau) < \tilde{x}_{j_1}(\tau)$ for small $\tau > 0$. Obviously, this leads the self-intersection. Hence, Subcase 2-2 never appear.

Case 3: $Q_0 = \{k_1, k_2, \dots, k_m\}$, $m \geq 2$.

By the argument similar to that in Subcase 2-1, we have $S_k(t_2) \subseteq S_k(t_1)$ for any $k \in Q_0$ and $t_1 < t_2$ since $\chi_k = 0$ and $\chi_{k-1}\chi_{k+1} \leq 0$. Thus, we have $p^* \in \bigcap_{k \in Q_0} S_k(0)$ and this is impossible when $m \geq 3$. Then, we have $m = 2$ and $k_2 = k_1 + 1$. Let $Q_1 := \{j_0 + 1, \dots, k_1 - 1\}$, $Q_2 := \{k_2 + 1, \dots, j_1 - 1\}$.

Subcase 3-1: $Q_1, Q_2 \neq \emptyset$.

If $i^* \in Q_1$, then $\chi_{Q_1} = 1$. Noting that $\chi_{k_1-1} = 1, \chi_{k_1} = \chi_{k_2} = 0$ and $\chi_{k_2+1} \neq 0$, we get $\chi_{k_2+1} = 1$. For the case $i \in Q_2$, we can show $\chi_{Q_1} = 1$ in the same way. Thus, $\chi_{Q_1} = \chi_{Q_2} = 1$.

By the same argument as in Case 2, we have $\theta_{k_1} - \theta_{j_0} > \pi$ and $\theta_{j_1} - \theta_{k_2} > \pi$. By the same argument as in the second paragraph of Subcase 2-2, we have $\theta_{k_2} - \theta_{j_0} \geq \pi$. Thus, we obtain $\theta_{j_1} - \theta_{j_0} > 2\pi$. Since $\sum_{j=j_0+1}^{j_1-1} d_j(t) \rightarrow 0$ as $t \rightarrow T_1$, we can show that the self-intersection must occur before $t = T_1$ by the same argument as in Subcase 1-4. Thus, this case does not happen.

Subcase 3-2: One of Q_1 and Q_2 is empty.

Let $Q_2 = \emptyset$. Then, $j_1 = k_1 + 2$ and $\theta_{j_1} = \theta_{k_1}$ since $\chi_{k_2} = 0$. By the same argument as in the previous subcase, we have $\theta_{j_1} > \pi$. By the same argument as in Subcase 1-3 and 1-4, we can show that this case does not happen.

Therefore, we have the assertion. \square

Proof of Theorem 1. Here we use the same notation in the proof of Proposition 1. By the same argument as in Case 3 of the previous proof, we have $\#Q \leq 2$, that is, $j_1 = j_0 + 2$ or $j_0 + 3$ (see Fig. 13).

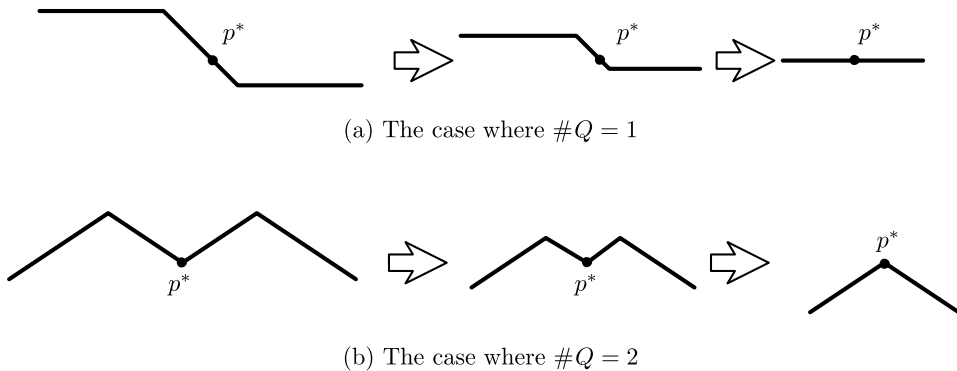


Fig. 13. Two patterns of edge-disappearing.

Case I: $\#Q = 1$.

Only \mathcal{S}_{j_0+1} disappears at $t = T_1$. Since $N_{j_0} = N_{j_1}$, \mathcal{S}_{j_0} and \mathcal{S}_{j_1} merge together and $\Omega(T_1)$ is still admissible.

Case II: $\#Q = 2$.

At $t = T_1$, \mathcal{S}_{j_0+1} and \mathcal{S}_{j_0+2} disappear. Since $N_{j_0} = N_{j_0+2}$ and $N_{j_1} = N_{j_0+1}$, we have

$$\frac{sN_{j_0} + (1-s)N_{j_1}}{|sN_{j_0} + (1-s)N_{j_1}|} = \frac{sN_{j_0+2} + (1-s)N_{j_0+1}}{|sN_{j_0+2} + (1-s)N_{j_0+1}|} \notin \mathcal{N}_{\mathcal{W}_\sigma}$$

for any $0 < s < 1$. Thus, $\Omega(T_1)$ is still admissible.

The proof is complete. \square

REMARK 7. For both edge-disappearing patterns, the extinction rate of the length of disappearing edge is exactly $(T_1 - t)$. Because $V_{j_0}, V_{j_1} \sim \pm 1$, thus $\dot{d}_j \sim -1$ for $j = j_0 + 1, j_1 - 1$.

Proof of Theorem 2. By Theorem 1, $\Omega(t)$ shrinks to a single point $q \in \mathbb{R}^2$ at $t = T_m$, that is, $\lim_{t \rightarrow T_m} p_j(t) = q$ for all j . Suppose on contrary that there exists at least one edge with negative curvature. By closedness of $\mathcal{P}(t)$ and the fact that there exist at least one inflection edge on the subarc between an edge with positive curvature and an edge with negative curvature, there exist at least two inflection edges on $\mathcal{P}(t)$. Then, we can find j_0 and j_1 ($j_0 < j_1$) which satisfy $\chi_{j_0} = \chi_{j_1} = 0$ and $\chi_Q = -1$, where $Q := \{j_0 + 1, \dots, j_1 - 1\}$. Note that Q is not empty.

By the same arguments as in Proposition 1, Case 2 and 3, we have $\mathcal{S}_k(t) \subseteq \mathcal{S}_k(0)$ for $k = j_0$ and j_1 . Thus, we have $q \in \mathcal{S}_{j_0}(0) \cap \mathcal{S}_{j_1}(0)$ which leads a contradiction to $Q \neq \emptyset$. Thus, all $H_j(t)$'s are non-negative for $t \geq T_{m-1}$.

Let N^0 be the number of inflection edges at $t = T_{m-1}$. Again by the same arguments as in Proposition 1, Case 3, we have $N^0 \leq 2$. However, the case where $N^0 = 1$ is impossible. Because, if $N^0 = 1$ holds, there exists k such that $\chi_k = 0$ and $\chi_{k\pm 1} = 1$. From $\chi_{k\pm 1} = 1$, we have $\text{sgn}(\theta_{k+1} - \theta_k) = 1$ and $\text{sgn}(\theta_k - \theta_{k-1}) = 1$, from which we obtain $\chi_k = (\text{sgn}(\theta_{k+1} - \theta_k) + \text{sgn}(\theta_k - \theta_{k-1})) / 2 = 1$. This contradicts $\chi_k = 0$. Therefore, we have the assertion. \square

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