JGSP **18** (2010) 93–106

JOURNAL OF
Geometry and Symmetry in Physics

BOOK REVIEW

A Gyrovector Space Approach to Hyperbolic Geometry, by Abraham A. Ungar, Morgan & Claypool Publ., 2009, ix + 182 pp., ISBN: 978-1-59829-822-2, eBook, ISBN: 978-1-59829-823-9

1. Introduction

In the years 1908 – 1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865 – 1942), professor and rector of Zagreb University, showed that this theory has a natural interpretation in the hyperbolic geometry of János Bolyai and Nikolai Ivanovich Lobachevsky [22]. However, much to his chagrin, Varičak had to admit in 1924 [23, p. 80] that the adaption of vector algebra for use in hyperbolic geometry was just not feasible. Fortunately, the author's studies of Einstein's velocity addition law of special relativity theory since 1988 [11] led him to discover the way of introducing into hyperbolic geometry both Cartesian coordinates and hyperbolic vector algebra. Hyperbolic vectors are called *gyrovectors* and their algebra is called gyroalgebra. The author's introduction of Cartesian coordinates and gyrovector gyroalgebra results in the *gyrovector space approach to hyperbolic geometry*, which is the title of the book under review, in a way fully analogous to the familiar vector space approach to Euclidean geometry. In order to elaborate a precise language for dealing with the resulting analytic hyperbolic geometry, which emphasizes analogies with classical notions, the author introduced the prefix "gyro", giving rise to *gyrolanguage*, the author's language of gyrogroups, gyrovector spaces and analytic hyperbolic geometry.

When I became familiar with the author's elegant work in analytic hyperbolic geometry several years ago, I invited him to publish some of his results in volumes that I have edited [12–14].

2. Einstein Addition

Let $\mathbb{R}^n$ be the Euclidean n-space equipped with Cartesian coordinates. These are all the n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers satisfying

$$x_1^2 + x_2^2 + \ \ldots \ + x_n^2 < \infty\,. \tag{1}$$

Similarly, let

$$\mathbb{R}_s^n = \{\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\,;\ \|\mathbf{v}\| < s\} \tag{2}$$

be the s-ball of $\mathbb{R}^n$ for an arbitrarily fixed constant $s > 0$. The ball $\mathbb{R}_s^n$ is equipped with Cartesian coordinates, which are all the n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers satisfying

$$x_1^2 + x_2^2 + \ \ldots \ + x_n^2 < s^2\,. \tag{3}$$

Points of the resulting Cartesian model of the ball $\mathbb{R}_s^n$ are n-tuples of real numbers. The point $\mathbf{0} = (0, 0, \ldots, 0)$ (n zeros) is the *origin* of the ball $\mathbb{R}_s^n$.

Einstein addition, $\oplus$, of relativistically admissible velocities is a binary operation in the ball $\mathbb{R}_s^n$ of all relativistically admissible velocities, where if $n = 3$, then $s = c$ is the speed of light in empty space. It was introduced by Einstein in his 1905 [1] paper that founded the special theory of relativity. In modern notation, Einstein's velocity addition law takes the form

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{4}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}} \tag{5}$$

in $\mathbb{R}_s^n$, and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{R}_s^n$ inherits from its space $\mathbb{R}^n$. Einstein subtraction, $\ominus$, is given by $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ so that $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, as expected.

Without loss of generality, one may select $s = 1$ for simplicity. However, the author prefers to leave s as a free positive parameter, allowing the limit as $s \to \infty$, when the modern and unfamiliar reduces to the classical and familiar. Thus, for instance, in that limit Einstein velocity addition (4) – (5) in $\mathbb{R}_s^n$ reduces to Newton velocity addition in $\mathbb{R}^n$, which is the common vector addition, and gamma factors reduce to 1.

Counterintuitively, Einstein addition is neither commutative nor associative. Einstein's failure to recognize and advance the rich structure of his velocity addition law contributed to the eclipse of his velocity addition of relativistically admissible three-velocities, creating a void that could be filled only with Minkowskian relativity, which is based on the Lorentz transformation group [16]. In 1988 the author discovered in [11] the rich nonassociative algebraic structure of Einstein addition, thus launching the fields of gyroalgebra, gyrogeometry and gyrotrigonometry that are studied in the book under review.

Owing to its nonassociativity, Einstein addition gives rise to an operator, gyr, called a *gyrator*

$$\mathrm{gyr} : \mathbb{R}_s^n \times \mathbb{R}_s^n \rightarrow \mathrm{Aut}(\mathbb{R}_s^n, \oplus) \tag{6}$$

given by

$$\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{7}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_s^n$. We recall that a groupoid $(\mathbb{R}_s^n, \oplus)$ is a nonempty set, $\mathbb{R}_s^n$, with a binary operation, $\oplus$, and its automorphism group, $\mathrm{Aut}(\mathbb{R}_s^n, \oplus)$, is the group of all bijections $f : \mathbb{R}_s^n \rightarrow \mathbb{R}_s^n$ of the groupoid $(\mathbb{R}_s^n, \oplus)$ that preserve its binary operation, that is, $f(\mathbf{u}\oplus\mathbf{v}) = f(\mathbf{u})\oplus f(\mathbf{v})$. The gyrator, gyr, generates *gyrations*, $\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]$, which are automorphisms of $(\mathbb{R}_s^n, \oplus)$ and, hence, are also called *gyroautomorphisms*.

3. Gyroalgebra

We now consider the gyroalgebra topic of the book. It is clear from (7) that the gyrations $\mathrm{gyr}\,[\mathbf{u},\mathbf{v}] : \mathbb{R}_s^n \rightarrow \mathbb{R}_s^n$ measure the extent to which Einstein addition deviates from associativity, where associativity corresponds to trivial gyrations (that is, gyrations given by the identity map of $\mathbb{R}_s^n$ onto itself). Surprisingly, gyrations turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}_s^n, \oplus)$ that repair the breakdown of both commutativity and associativity in Einstein addition. Indeed, gyrations give rise to the following *gyrocommutative* law and left and right *gyroassociative* laws of Einstein addition

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}\,[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) \\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{w} \\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}\,[\mathbf{v},\mathbf{u}]\mathbf{w}
\end{aligned} \tag{8}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_s$.

Accordingly, Einstein addition possesses a nonassociative structure regulated by gyrations. Additionally, the gyrations contribute to Einstein addition their own rich structure which is revealed, for instance, in the gyration identities

$$(\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}])^{-1} = \mathrm{gyr}\,[\mathbf{v}, \mathbf{u}], \qquad \mathrm{gyr}\,[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}\,[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$. Several other useful gyration identities are presented and employed in the book.

Guided by the group axioms and taking the key features of Einstein addition as axioms led the author to define the abstract gyrogroup as follows:

Definition 1 (Gyrogroups) . *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element,* 0*, called a left identity, satisfying*

G1) $\quad 0\oplus a = a$

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* G1) *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of a, satisfying*

G2) $\quad \ominus a\oplus a = 0$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}\,[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

G3) $\quad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}\,[a, b]c$

The map $\mathrm{gyr}\,[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}\,[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

G4) $\quad \mathrm{gyr}\,[a, b] \in \mathrm{Aut}(G, \oplus)$

and the automorphism $\mathrm{gyr}\,[a, b]$ *of G is called the gyroautomorphism, or the gyration, of G generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of G. Finally, the gyroautomorphism* $\mathrm{gyr}\,[a, b]$ *generated by any* $a, b \in G$ *possesses the left loop property*

G5) $\quad \mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a\oplus b, b]$.

The gyrogroup axioms G1) – G5) are classified into three classes:

1) The first pair of axioms, G1) and G2), is a reminiscent of the group axioms.
2) The last pair of axioms, G4) and G5), presents the gyrator axioms.
3) The middle axiom, G3), is a hybrid axiom linking the two pairs of axioms in 1) and 2).

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2 (Gyrocommutative Gyrogroups) . *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

G6) $a \oplus b = \text{gyr}\,[a, b](b \oplus a)$

for all $a, b \in G$.

The author presents in the book, and in [19], an elegant motivational approach to gyrogroups in terms of the group of all Möbius transformations of the complex open unit disc. It turns out that the set of all Möbius transformations of the disc without rotations of the disc does not form a subgroup but, rather, it forms a gyrocommutative gyrogroup.

A (commutative) group is a degenerate (gyrocommutative) gyrogroup whose gyroautomorphisms are all trivial. The algebraic structure of gyrogroups is, accordingly, richer than that of groups. Thus, without losing the flavor of the group structure the author has generalized it into the gyrogroup structure to suit the needs of Einstein addition in the ball.

Einstein addition admits a scalar multiplication, $\otimes$, given by

$$r\otimes\mathbf{v} = \frac{1 - (\gamma_{\mathbf{v}} - \sqrt{\gamma_{\mathbf{v}}^2 - 1})^{2r}}{1 + (\gamma_{\mathbf{v}} - \sqrt{\gamma_{\mathbf{v}}^2 - 1})^{2r}} \frac{\gamma_{\mathbf{v}}}{\sqrt{\gamma_{\mathbf{v}}^2 - 1}}\mathbf{v} \tag{10}$$

$r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$. Einstein scalar multiplication possesses some properties of vector space scalar multiplication as, for instance,

$$n\otimes\mathbf{v} = \mathbf{v}\oplus \;\ldots\; \oplus\mathbf{v} \qquad\qquad (n \text{ terms}) \tag{11}$$

for any positive integer n. However, Einstein scalar multiplication does not distribute over Einstein addition. Owing to the lack of a distributive law, Einstein addition $\oplus = \oplus_{\text{E}}$ in $\mathbb{R}_s^n$ gives rise to a distinct addition $\oplus_{\text{M}}$ in $\mathbb{R}_s^n$, called the Möbius addition, by the equation

$$\mathbf{u}\oplus_{\text{M}}\mathbf{v} = \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\text{E}}2\otimes\mathbf{v})\,. \tag{12}$$

Since Einstein addition does not distribute with scalar multiplication, Einstein addition, $\oplus_{\text{E}}$, and Möbius addition, $\oplus_{\text{M}}$, in (12) are distinct, but isomorphic, gyrocommutative gyrogroup additions. Both additions, $\oplus_{\text{E}}$ and $\oplus_{\text{M}}$, admit the same scalar multiplication, $\otimes$, that satisfies (11) for $\oplus = \oplus_{\text{E}}$ and $\oplus = \oplus_{\text{M}}$.

Let $A, B \in \mathbb{R}^n$ be two distinct points of the Euclidean n-space $\mathbb{R}^n$. Then the set of points L_{AB}^{euc} given by the equation

$$L_{AB}^{\mathrm{euc}} = A + (-A + B)t \tag{13}$$

$t \in \mathbb{R}$, forms a straight line, which is the unique geodesic that passes through the points A and B in the standard model of Euclidean geometry.

In full analogy, let $A, B \in \mathbb{R}_s^n$ be two distinct points in the s-ball of $\mathbb{R}^n$. Then, in full analogy with (13), let us consider the set of points given by each of the two equations

$$\begin{aligned} L_{AB}^{\mathrm{ein}} &= A \oplus_{\mathrm{E}} (\ominus_{\mathrm{E}} A \oplus_{\mathrm{E}} B) \otimes t \\ L_{AB}^{\mathrm{mob}} &= A \oplus_{\mathrm{M}} (\ominus_{\mathrm{E}} A \oplus_{\mathrm{M}} B) \otimes t \end{aligned} \tag{14}$$

$t \in \mathbb{R}$. Surprisingly, L_{AB}^{ein} forms a chord of the ball, and L_{AB}^{mob} forms a circle in the interior of the ball, which approaches the boundary of the ball orthogonally. What is surprising here is that the former is the unique geodesic that passes through the points A and B in the Beltrami-Klein ball model of hyperbolic geometry, and the latter is the unique geodesic that passes through the points A and B in the Poincaré ball model of hyperbolic geometry. The relationship between gyroalgebra and analytic hyperbolic geometry thus emerges in a way analogous to the relationship between algebra and analytic Euclidean geometry.

4. Gyrogeometry

We now consider the gyrogeometry topic of the book. Let $(\mathbb{R}_s^n, \oplus, \otimes)$ be a gyrovector space, The two special cases which are studied in the book are the Einstein gyrovector spaces $(\mathbb{R}_s^n, \oplus_{\mathrm{E}}, \otimes)$ and the Möbius gyrovector spaces $(\mathbb{R}_s^n, \oplus_{\mathrm{M}}, \otimes)$, $n \geq 2$. These, respectively, form the algebraic setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry and the Cartesian-Poincaré ball model of hyperbolic geometry. To demonstrate the link between Einstein gyrovector spaces $(\mathbb{R}_s^n, \oplus_{\mathrm{E}}, \otimes)$ and the Cartesian-Beltrami-Klein ball model of hyperbolic geometry and, similarly, the link between Möbius gyrovector spaces $(\mathbb{R}_s^n, \oplus_{\mathrm{M}}, \otimes)$ and the Cartesian-Poincaré ball model of hyperbolic geometry we need the notion of the gyrodistance and two well-known results from differential geometry.

The gyrodistance function in $(\mathbb{R}_s^n, \oplus, \otimes)$ is given by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} \ominus \mathbf{v}\| \tag{15}$$

satisfying the gyrotriangle inequality

$$d(\mathbf{u},\mathbf{w}) \le d(\mathbf{u},\mathbf{v}) \oplus d(\mathbf{v},\mathbf{w}) \tag{16}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_s^n$.

In a two dimensional Einstein gyrovector space $(\mathbb{R}_s^2, \oplus, \otimes)$ the squared gyrodistance between a point $\mathbf{v} = (x_1, x_2) \in \mathbb{R}_s^2$ and an infinitesimally nearby point $\mathbf{v} + \mathrm{d}\mathbf{v} \in \mathbb{R}_s^2$, where $\mathrm{d}\mathbf{v} = (\mathrm{d}x_1, \mathrm{d}x_2)$, is given by the equation

$$\mathrm{d}s^2 = \|\mathbf{v} \ominus (\mathbf{v} + \mathrm{d}\mathbf{v})\|^2 = E\mathrm{d}x_1^2 + 2F\mathrm{d}x_1\mathrm{d}x_2 + G\mathrm{d}x_2^2 \tag{17}$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$

$$E = s^2 \frac{s^2 - x_2^2}{(s^2 - r^2)^2}, \qquad F = s^2 \frac{x_1 x_2}{(s^2 - r^2)^2}, \qquad G = s^2 \frac{s^2 - x_1^2}{(s^2 - r^2)^2}. \tag{18}$$

The triple $(g_{11}, g_{12}, g_{22}) = (E, F, G)$ along with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [6]. It turns out to be the metric tensor of the Beltrami-Klein disc model of hyperbolic geometry [8, p. 220]. Hence, $\mathrm{d}s^2$ in (17) – (18) is the well-known Riemannian line element of the Beltrami-Klein disc model of hyperbolic geometry, linked to Einstein addition $\oplus$ and to Einstein gyrodistance function (15). An extension to higher dimensions is straightforward.

Similarly, in a two dimensional Möbius gyrovector space $(\mathbb{R}_s^2, \oplus, \otimes)$ the squared gyrodistance between a point $\mathbf{v} \in \mathbb{R}_s^2$ and an infinitesimally nearby point $\mathbf{v} + \mathrm{d}\mathbf{v} \in \mathbb{R}_s^2$, $\mathrm{d}\mathbf{v} = (\mathrm{d}x_1,\ \mathrm{d}x_2)$, is given by the equation

$$\mathrm{d}s^2 = \|\mathbf{v} \ominus (\mathbf{v} + \mathrm{d}\mathbf{v})\|^2 = E\mathrm{d}x_1^2 + 2F\mathrm{d}x_1\mathrm{d}x_2 + G\mathrm{d}x_2^2 \tag{19}$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$,

$$E = \frac{s^4}{(s^2 - r^2)^2}, \qquad F = 0, \qquad G = \frac{s^2}{(s^2 - r^2)^2}. \tag{20}$$

The triple $(g_{11}, g_{12}, g_{22}) = (E, F, G)$ along with $g_{21} = g_{12}$ turns out to be the well-known metric tensor of the Poincaré disc model of hyperbolic geometry [8, p. 226]. Hence, $\mathrm{d}s^2$ in (19) – (20) is the Riemannian line element of the Poincaré disc model of hyperbolic geometry, linked to Möbius addition $\oplus$ and to Möbius gyrodistance function (15). An extension to higher dimensions is straightforward.

We thus see that the gyroalgebra of gyrovector spaces forms the setting for Cartesian models of hyperbolic geometry just as the algebra of vector spaces forms the setting for the standard Cartesian model of Euclidean geometry. In this sense one can say that hyperbolic geometry is the gyro-counterpart of Euclidean geometry.

5. Gyrotrigonometry

We now consider the gyrotrigonometry topic of the book. Let ABC be a hyperbolic triangle, called a gyrotriangle, in a gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{R}^n_s$ and hyperbolic angles, called gyroangles, $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. These gyrotriangle gyroangles are defined in the book by the equations

$$\begin{aligned}\cos\alpha &= \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \\ \cos\beta &= \frac{\ominus B \oplus A}{\|\ominus B \oplus A\|} \cdot \frac{\ominus B \oplus C}{\|\ominus B \oplus C\|} \\ \cos\gamma &= \frac{\ominus C \oplus A}{\|\ominus C \oplus A\|} \cdot \frac{\ominus C \oplus B}{\|\ominus C \oplus B\|}\end{aligned} \tag{21}$$

and $\sin\alpha = \sqrt{1 - \cos^2\alpha} > 0$, etc. It turns out that $\alpha + \beta + \gamma < \pi$, as expected in hyperbolic geometry, giving rise to the gyrotriangle defect

$$\delta = \pi - (\alpha + \beta + \gamma)\,. \tag{22}$$

Let now ABC be a right-gyroangled gyrotriangle with $\gamma = \pi/2$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let the leg-gyrolengths of this gyrotriangle be a and b, and its hypotenuse-gyrolength be c, that is

$$a = \|C \ominus B\| \qquad b = \|C \ominus A\| \qquad c = \|B \ominus A\|\,. \tag{23}$$

Then, the side-gyrolengths a, b, c of the right-gyroangled gyrotriangle are related by the two *hyperbolic Pythagorean identities*

$$\left(\frac{a}{c}\right)^2 + \left(\frac{\gamma_b b}{\gamma_c c}\right)^2 = 1, \qquad \left(\frac{\gamma_a a}{\gamma_c c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1\,. \tag{24}$$

Moreover, the author shows in the book that (24) and (21) are related, resulting in the equations

$$\cos\alpha = \frac{b}{c}, \qquad \sin\alpha = \frac{\gamma_a a}{\gamma_c c} \tag{25}$$

and

$$\cos\beta = \frac{a}{c}, \qquad \sin\beta = \frac{\gamma_b b}{\gamma_c c} \tag{26}$$

which, in turn, imply the inequality $\alpha+\beta < \pi/2$, as expected in gyrotrigonometry.

In the limit $s \to \infty$, gamma factors tend to 1. Hence, in that limit the two hyperbolic Pythagorean identities (24) degenerate into the familiar single Euclidean Pythagorean identity

$$a^2 + b^2 = c^2 \,. \tag{27}$$

Furthermore, in that limit the basic gyrotrigonometric equations (25) and (26) reduce to their Euclidean counterparts,

$$\cos\alpha = \sin\beta = \frac{b}{c}, \qquad \sin\alpha = \cos\beta = \frac{a}{c} \tag{28}$$

implying $\alpha = \pi/2 - \beta$, as expected in trigonometry. Accordingly, more generally, in the limit $s \to \infty$, gyrotrigonometry reduces to trigonometry.

It is widely believed that hyperbolic geometry does not allow a definition of hyperbolic triangle area which is fully analogous to Euclidean triangle area. Greenberg states in [4, p. 321] that "It can be proved rigorously that in hyperbolic geometry the area of a triangle cannot be calculated as half the base times the height." Similarly, Moise states in [9, p. 349] that "it is impossible to define an area function [in hyperbolic geometry] which has even a minimal resemblance to the Euclidean area function." Accordingly, Ruoff presents an explanation in a paper entitled: *"Why Euclidean area measure fails in the noneuclidean plane"* [10].

However, the gyroalgebra, gyrogeometry and gyrotrigonometry developed in the book enable the gyrotriangle gyroarea to be defined in a way fully analogous to the definition of the triangle area. Indeed, let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with vertices A, B, C, gyroangles α, β, γ given by (21), gyroangular defect δ given by (22), side-gyrolengths a, b, c given by (23), and altitude-gyrolengths h_a, h_b, h_c. It is customary to define the hyperbolic triangle area as $K\delta$, where K is a fixed positive constant. However, guided by analogies, the author defines the gyrotriangle gyroarea slightly differently.

In this book the author defines the gyrotriangle gyroarea $|ABC|$ of gyrotriangle ABC in an Einstein gyrovector space by the equation

$$|ABC| = 2s^2 \tan\tfrac{\delta}{2} \tag{29}$$

and proves that

$$\begin{aligned}
|ABC| &= \frac{2}{1+\gamma_a+\gamma_b+\gamma_c}\gamma_a a \gamma_{h_a} h_a \\
|ABC| &= \frac{2}{1+\gamma_a+\gamma_b+\gamma_c}\gamma_b b \gamma_{h_b} h_b \\
|ABC| &= \frac{2}{1+\gamma_a+\gamma_b+\gamma_c}\gamma_c c \gamma_{h_c} h_c \,.
\end{aligned} \tag{30}$$

In the limit when $s \to \infty$ gamma factors tend to 1 and, accordingly, the right-hand-side of each of the three equations in (30) tends to the Euclidean triangle area, which is half the base times the corresponding height.

6. Gyrovectors

The author's Cartesian models of hyperbolic geometry, regulated algebraically by gyrovector spaces, admit gyrovectors which are fully analogous to the familiar Euclidean vectors. Let $A, B \in \mathbb{R}^n_s$ be two points in a gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. Then $\mathbf{v} = \ominus A \oplus B$ is a rooted gyrovector with tail A and head B. It has the value $\ominus A \oplus B$ and the magnitude $\|\ominus A \oplus B\|$. To liberate rooted gyrovectors from their roots the author defines gyrovectors to be equivalence classes of rooted gyrovectors, where two gyrovectors are equivalent if they have the same value. A point P is then identified with the gyrovector $\ominus O \oplus P$, where $O = \mathbf{0} = (0, \ldots, 0)$ is the arbitrarily selected origin of the ball. In order to determine gyrovector addition in a way analogous to vector addition one needs hyperbolic parallelograms, naturally called gyroparallelograms. On first glance it seems that the notion of the parallelogram cannot be extended from Euclidean into hyperbolic geometry since the latter denies parallelism. However, the task is not impossible. The author defines in the book a gyroparallelogram as a gyroquadrilateral the two gyrodiagonals of which intersect at their gyromidpoints. Once the concept of the gyroparallelogram has been established, the definition of gyrovector addition in terms of gyroparallelograms in full analogy with the definition of vector addition in terms of parallelograms is straightforward. Gyrovectors are, thus, equivalence classes that add according to the gyroparallelogram law just as vectors are equivalence classes that add according to the parallelogram law.

Along analogies that gyrovectors share with vectors, there is a remarkable disanalogy:

1) Einstein velocity addition in $\mathbb{R}^n_s$ leads in the book to the gyroparallelogram addition law, which is commutative, just as

2) Newton velocity addition law in $\mathbb{R}^n$, which is the common vector addition, leads to the common parallelogram addition law. However,

3) Newton velocity addition law and its resulting parallelogram addition law are coincident while, in contrast,

4) Einstein velocity addition law (which is noncommutative) and its resulting gyroparallelogram addition law (which is commutative) are not coincident.

This disanalogy raises a natural question: Does nature, within the frame of Einstein's special theory of relativity, obey *Einstein's velocity addition* or *Einstein's gyroparallelogram velocity addition* of relativistically admissible velocities?

To answer the question, the author presents in the book a study of the effect known as "relativistic stellar aberration", in which he employs the gyro-techniques of gyrotrigonometry and gyrovectors that he developed in the book, and in which he is guided by analogies with the familiar "classical stellar aberration".

Fortunately, the author obtains in the book the same relativistic stellar aberration formulas that are obtained in the literature by means of the Lorentz transformation group. Hence, if the relativistic stellar aberration formulas that are available in the literature are experimentally valid within the frame of special relativity, then the author's study demonstrates that, within the frame of special relativity (where gravitation is ignored), velocities are added according to Einstein's gyroparallelogram velocity addition law (which is commutative) rather than according to Einstein's 1905 velocity addition law (which is noncommutative).

It is therefore important to note that the special relativistic stellar aberration formulas have not yet been verified experimentally, but the experimental evidence might appear in the near future from the NASA's space gyroscopes experiment, known as GP-B [3]. In this experiment to measure the gyroscopic precession of gyroscopes in space orbit, the special relativistic stellar aberration formulas have been taken for granted. The expected success of GP-B, therefore, would amount to an experimental confirmation of the validity of the special relativistic stellar aberration formulas. Following this book, this experimental confirmation would amount to the confirmation that within the frame of special relativity, uniformly relativistically admissible velocities are gyrovectors that add according to Einstein's gyroparallelogram addition law.

The discovery of the commutative gyroparallelogram addition law of relativistically admissible velocities to which Einstein's noncommutative addition law gives rise is a major accomplishment that resolves an old problem posed by the famous mathematician Émil Borel (1871–1956) in 1913. According to the historian of relativity physics, Scott Walter [24, pp. 117–118], Borel considered the noncommutativity of Einstein's velocity addition as a defect, and he fixed it by proposing a velocity addition law that involves a significant modification of Einstein's addition. We learn from this book that in order to resolve Borel's 1913 problem of the noncommutativity of Einstein addition there is no need to modify Einstein's special theory of relativity. Rather, there is a need to extend Einstein's unfinished symphony.

7. Final Remarks

The book under review is quite successful in providing evidence that the mere introduction of gyrations turns groups into gyrogroups, vector spaces into gyrovector spaces, trigonometry into gyrotrigonometry and Euclidean geometry into hyperbolic geometry. Each gyro-structure is richer than its corresponding classical structure, and can be studied along analogies that it shares with its corresponding classical structure. Moreover, each gyro-structure degenerates into its corresponding classical structure in the special case when all the gyrations that are involved are trivial.

Owing to the analogies that a gyro-structure shares with its corresponding classical structure, a gyrovector space approach to hyperbolic geometry emerges in this book in a way fully analogous to the common vector space approach to Euclidean geometry [5]. The book is, indeed, accessible to anyone who is familiar with the common vector space approach to Euclidean geometry. Readers who enjoy reading this book are likely to enjoy reading the other books of the author [15, 17, 18, 20, 21] about gyroalgebra, gyrogeometry, gyrotrigonometry and gyrophysics.

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