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The Structure of PEC Networks

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Abstract. A packed exponential connections (PEC) network is a grid-based network with connectivity and routing results that are competitive with hypercubic networks. The prior results are all empirical, since the structure of the network has been understood only through an indirect existence proof. In this paper we provide the first direct characterization of a PEC network.

1. Introduction

A packed exponential connections (PEC) network is a network that tries to solve the scalability and connectivity problems of tightly coupled interconnection networks [Kirkman and Quammen 91]. It has been studied extensively and found to have connectivity problems very similar to the hypercube network. However, since it is defined as a set of mesh networks, it is essentially a two-dimensional design, and it has superior layout properties.

Nevertheless, the PEC network is not yet well understood analytically. There have been some loose routing and algorithmic results [Quammen et al. 96, Wong et al. 95]. However, most of the results for routing are empirical [Quammen et al. 96, Liao and Sun 99]; this is also true for other properties such as diameter and congestion. While the PEC definition is essentially based on a two-dimensional grid, it is possible to define a much simpler one-dimensional PEC network (by using only the first row of the 2-D construction). It is only for this simpler 1-D PEC network that exact results are known for routing and diameter [Lin and Prasanna 95, Raghavendra and Sridhar 96].

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In this paper, we study the properties of two-dimensional PEC networks. The PEC network has resisted analysis because the definition of the PEC network is indirect, by way of a construction. Unfortunately, the construction does not allow us to know the details of one part of the PEC network without essentially generating a large portion of the network. The main contribution of this paper is a new direct definition of the PEC network, which will lead to better and more precise knowledge of the network's properties.

The paper is organized as follows: Section 2 defines two matrices that characterize the PEC matrix but that are easy to understand and construct. Based on results in Section 2, we are able to provide an algorithm in Section 3 that can compute the PEC value of any entry directly, without computing PEC values for other entries. In Section 4 we will reexamine the definition of the PEC matrix and prove that the PEC matrix can be defined by other properties.

2. Definitions

Informally, the PEC network is an augmentation of a grid network (where each node is connected to its four neighbors, except on the edges). Each node has a label, say $i > 0$, and connections to four additional neighbors (N, S, E, W), each a distance 2^i away in the grid. The nodes labeled $i > 0$ form grids with edges of length 2^i, and each node is in exactly one such grid. Roughly $1/2^i$ of the nodes are in grids with the label i. This provides enough grid connections with long edges to allow efficient routing. The definition, given below, is for an infinite PEC network, but any implementation would use a finite truncation (not discussed here).

More formally, the PEC network consists of an infinite matrix of nodes $N[i, j]$, where $0 \leq i, j < \infty$. Each such node is connected by a bidirectional link to $N[i\pm1, j]$ and $N[i, j\pm1]$ (if they exist). Further, each such node has an associated value $P[i, j]$ and is connected by a bidirectional link to $N[i\pm2^{P[i,j]}, j]$ and $N[i, j\pm 2^{P[i,j]}]$ (if they exist). Clearly, the crux of the definition is the specification of P, which is called the *PEC matrix*.

An (i, j)-*tile* is a "properly aligned" submatrix of P of size $2^i \times 2^j$, and its upper left corner is the entry $P[k2^i, m2^j]$, for some integers k and m. An (i, j)-*tile* contains at least two entries, so $i > 0$ or $j > 0$. Note that a $(0, j)$-tile would be a portion of a row of the matrix and is called a *row-tile* of size 2^j. Given some two-dimensional matrix M, we will use $M[i_1..j_1][i_2..j_2]$ to denote the submatrix of M that consists of all entries $M[i, j]$ such that $i_1 \leq i < j_1, i_2 \leq j < j_2$. The ith "row" R of the matrix is $M[i..i+1][0..\infty]$. Given a row R, we will use $R[i..j]$ to represent the subarray of R that consists of all the entries $R[k]$, where $i \leq k < j$.

Definition 2.1. The PEC matrix P satisfies

1. $P[0,0]=\infty$,
2. in any (i,j)-tile there is exactly one entry $i+j$,
3. if $P[i,j]=n$, then $P[i+2^n,j]=n$ and $P[i,j+2^n]=n$.

It is easy to show that the following property follows from the definition of the PEC matrix [Shermer 02]. (Interestingly, it was part of the original definition.)

Lemma 2.2. [Shermer 02] *In any (i,j)-tile there is exactly one entry that is greater than $i+j$.*

Proof. Let T be any (i,j)-tile. Then T has 2^{i+j} entries altogether, and we will prove the lemma by showing that the total number of entries in T that are less than or equal to $i+j$ is $2^{i+j}-1$.

For any k such that $0<k\leq j$, each row of T can be divided into 2^{j-k} subtiles of size $(0,k)$, and in each such subtile, there is exactly one entry equal to k, so in T, there are altogether $2^{j-k}\times 2^i = 2^{i+j-k}$ entries equal to k.

For any k such that $j<k\leq i+j$, T can be divided into 2^{i+j-k} subtiles of size $(k-j,j)$, and in each such subtile, there is exactly one entry equal to k, so there are 2^{i+j-k} entries equal to k in T.

So in T, the total number of entries that are less than or equal to $i+j-1$ is $\sum_{k=1}^{i+j} 2^{i+j-k} = 2^{i+j}-1$. $\square$

It has been shown [Kirkman and Quammen 91] that the definition uniquely defines P. It is a constructive argument and can be used to fill in the P matrix one entry at a time, working away from the origin. The goal of this paper is to calculate $P[i,j]$ without calculating any other entries of P. Figure 1 shows the beginning of the P matrix; at first, it seems very regular, but on close inspection it becomes clear that any pattern is very obscure.

Definition 2.3. Consider any row R of P. The *encoding array* E_R of R is a binary array such that

$$E_R[i]=0 \text{ iff the maximum entry in } R[0..2^{i+1}] \text{ is in } R[0..2^i].$$

We say that E_R is an encoding array, since it contains enough information to reconstruct the row R, as the following theorem shows. First, we introduce some notation. We use $\langle x_n \ldots x_0\rangle$ to denote the binary representation of the

$$
\begin{array}{cccccccccccccccc}
\infty & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 \\
1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 5 \\
2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 6 & 1 & 2 & 1 & 3 & 1 \\
1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 \\
3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 7 & 1 & 2 & 1 \\
1 & 2 & 1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 \\
2 & 1 & 3 & 1 & 2 & 1 & 6 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 \\
1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 \\
4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 8 & 1 & 2 & 1 & 3 & 1 & 2 & 1 \\
1 & 2 & 1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 \\
2 & 1 & 6 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 \\
1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 \\
3 & 1 & 2 & 1 & 7 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\
1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 3 \\
2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 6 & 1 \\
1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 3 & 1 & 2
\end{array}
$$

Figure 1. The upper left corner of the P matrix.

integer $x = \sum_{i=0}^{n} x_i \cdot 2^i$, where $x_i \in \{0,1\}$. Given an integer $x = \langle x_n \dots x_0 \rangle$, we define the *reverse binary representation* $\mathrm{RB}(x)$ to be the infinite binary sequence $x_0 x_1 \dots x_n 000 \dots$. Further, we use $\oplus$ for the binary exclusive-or operation; if the operands are binary arrays (finite or infinite), then the operation is done bitwise.

Lemma 2.4. *Let R be any row of P, and E_R the encoding array of R. Let $e_i = E_R[i]$. For any integers $j \geq i \geq 0$, we divide row-tile $R[0..2^{j+1}]$ into 2^{j-i+1} smaller row-tiles of length 2^i and index them with $0, 1, \dots, 2^{j-i+1} - 1$. If m is the index of the row-tile that contains the unique entry that is greater than $j+1$, then $m = \langle e_j e_{j-1} \dots e_i \rangle$. Further, if m' is the index of the row-tile that contains the unique entry that is equal to $j+1$, then $m' = \langle (1 - e_j) e_{j-1} \dots e_i \rangle$.*

Proof. We prove the lemma by induction on $j - i$. For $j - i = 0$, by the definition of e_j, the lemma is true.

Now suppose the lemma is true for $j - i = k$. Consider the case of $j - i = k+1$. Let n be the index of the row-tile of size 2^i in $R[0..2^j]$ that contains the entry that is greater than j. Since $(j-1) - i = k$, by induction, $n = \langle e_{j-1} e_{j-2} \dots e_i \rangle$. For any entry $R[l]$ in the other row-tiles of size 2^i, we have $R[l] \leq j$, and by the definition of P, $R[l + 2^j] = R[l] \leq j$. Thus the only two possible values for m and m' are n and $n + 2^{j-i}$. By the definition of e_j, if $e_j = 0$, then $m \leq 2^{j-i}$, and $m = n = \langle e_j e_{j-1} \dots e_i \rangle$. If $e_j = 1$, then $m = n + 2^{j-i} = \langle e_j e_{j-1} \dots e_i \rangle$. In either case, the lemma is true. $\square$

$$
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

Figure 2. The upper left portion of the E array.

The following theorem shows how to calculate a row from its encoding array.

Theorem 2.5. *Let R be a row of P and let E_R be its encoding array. Then*

$$R[i] = 1 + \textit{the index of the first } 1 \textit{ in } E \oplus \mathrm{RB}(i).$$

Proof. Let $E_R[i] = e_i$. Suppose that $\mathrm{RB}(i) = i_0 i_1 \ldots i_k \ldots$, where $e_l = i_l$, for every $l < k$ and $e_k \neq i_k$. Applying Lemma 2.4 with $i = 0$, $j = k$, we have that the position of the unique $k+1$ in $R[0..2^{k+1}]$ is $\langle (1-e_k)e_{k-1} \ldots e_0 \rangle = \langle i_k i_{k-1} \ldots i_0 \rangle$. We will call it i'. There are two cases:

Case 1: $2^k \geq i$. Then $\langle i_k i_{k-1} \ldots i_0 \rangle$ equals i, and so $R[i] = k + 1$.

Case 2: $2^k < i$. Then $i - i' = m \cdot 2^{k+1}$ for some integer m, and by the definition of P, $R[i] = R[i'] = k + 1$. □

Definition 2.6. Define the *encoding matrix* E of the PEC matrix P as follows: the ith row of E is the encoding array of the ith row of P.

Figure 2 shows the E matrix; the reader should verify that each row is the encoding of the corresponding row of the P matrix. The following lemmas give various properties of the matrix E that will prove to characterize it.

Lemma 2.7. (Zero initial row property.) *The first row (indexed by 0) of E is composed of all zeros.*

Proof. The entry at position 0 is ∞, so the largest entry is always in the lower half. Therefore, $E[0, i] = 0$, for all i. $\square$

Lemma 2.8. (Distinct rows property.) *Let $m, i \geq 0$ and $j > 0$ be integers. Then all the rows in the submatrix $E[m \cdot 2^j..(m+1) \cdot 2^j][i..i+j]$ are different from one another.*

Proof. Denote the submatrix by E', and consider the corresponding tile of P, $P' = P[m \cdot 2^j..(m+1) \cdot 2^j][0..2^{i+j}]$. If we divide each row of P' into row-tiles of size 2^i, then by Lemma 2.4, each row of E' represents the index of the row-tile that contains the entry greater than $i + j$ in the corresponding row of P'.

Note that all the row-tiles with the same index from all the rows will form a (j, i)-tile. By Lemma 2.2, there will be only one entry in this tile that is greater than $i + j$, which implies that all the rows in E' must be distinct. $\square$

Lemma 2.9. (Repeating headers property.) *For any $m, n \geq 0$ and $i > 0$,*

$$E[m \cdot 2^i..(m+1) \cdot 2^i][0..i] = E[n \cdot 2^i..(n+1) \cdot 2^i][0..i].$$

Proof. First we prove the lemma for the case of $n = m + 1$.

Let R, R' be the rth and $(r + 2^j)$th rows of P and $m \cdot 2^i \leq r < (m+1) \cdot 2^i$. Suppose $R[k]$ is the unique entry that is greater than i in $R[0..2^i]$. Then for $l < 2^i$ and $l \neq k$, $R[l] \leq i$, we have by the definition of P that $R'[l] = R[l] \leq i$. This leaves $R'[k]$ as the only possible entry in $R'[0..2^i]$ that is greater than i.

Note that for a row L, $E_L[0..i]$ depends only on the comparison between the entries in $L[0..2^i]$. Since $R[0..2^i]$ and $R'[0..2^i]$ are all the same except for the value of their largest entry, we have $E_R[0..i] = E_{R'}[0..i]$.

So we have $E[m \cdot 2^i..(m+1) \cdot 2^i][0..i] = E[(m+1) \cdot 2^i..(m+2) \cdot 2^i][0..i]$.

For other cases, without loss of generality, we can assume that $m < n$. Then

$$\begin{aligned} E[n \cdot 2^i..(n+1) \cdot 2^i][0..i] &= E[(n-1) \cdot 2^i..n \cdot 2^i][0..i] \\ &\;\;\vdots \\ &= E[m \cdot 2^i..(m+1) \cdot 2^i][0..i]. \end{aligned}$$ $\square$

In fact, E is the only matrix that has all these three properties. Let F be a matrix that has the properties stated in Lemmas 2.7, 2.8, and 2.9. The following lemma shows that there is a unique way to fill in F, starting from the origin.

Lemma 2.10. *The first row of F consists of all 0's, and the second row of F consists of all 1's.*

Proof. By the zero initial row property, the first row of F consists of all 0's. For any integer $m \geq 0$, consider $F[0..2][m..m+1]$. By the distinct rows property and the fact that $F[0,m] = 0$, we have $F[1,m] = 1$. $\square$

Lemma 2.11. *Given $F[0..2^i][j..j+i+1]$ and $F[2^i..2^{i+1}][j..j+i]$, there is a unique way to fill in $F[2^i..2^{i+1}][j+i..j+i+1]$.*

Proof. Let B_1 be the set of binary strings in $F[0..2^i][j..j+i+1]$, and B_2 the set of the binary strings in $F[2^i..2^{i+1}][j..j+i+1]$. Then $B_1 \cup B_2$ contains all the 2^{i+1} binary strings of length $i+1$.

Note that $F[0..2^i][j..j+i]$ and $F[2^i..2^{i+1}][j..j+i+1]$ both consist of all the binary strings of length i.

For each string s in B_2, let t be its prefix of length i. Since $F[0..2^i][j..j+i]$ contains all binary strings of length i, it contains t and there is one string s' in B_1 that has prefix t. But s is not equal to s', and so they must be different in their last bit. Because the last bit of s' is given, there is only one choice for the last bit of s. So $F[2^i..2^{i+1}][j+i..j+i+1]$ is uniquely determined. $\square$

Now we are ready to prove the following theorem.

Theorem 2.12. *There is a unique matrix that satisfies Lemmas 2.7, 2.8, and 2.9.*

Proof. Let F be a matrix that has these three properties. For any $i, j \geq 0$, we need to prove that $F[i,j]$ is uniquely determined.

Take integers n such that $2^n > i$ and $n > j$. We will prove the following statement by induction on m: For $m \leq n$, $F[0..2^m][0..n]$ is uniquely determined.

First of all, by Lemma 2.10, the first two rows of F are uniquely determined. Specifically, $F[0..2][0..n]$ is uniquely determined. The statement is true for $m = 1$.

Now suppose we have filled in $F[0..2^k][0..n]$ for an integer $k < n$. By Lemma 2.9, $F[2^k..2^{k+1}][0..k] = F[0..2^k][0..k]$ is uniquely determined. By repeatedly applying Lemma 2.11, we can fill in $F[2^k..2^{k+1}][0..n]$ column by column. So $F[0..2^{k+1}][0..n]$ is uniquely determined.

Thus the statement is proved. Since $F[i,j]$ is an entry of $F[0..2^n][0..n]$, it is uniquely determined. $\square$

The irregular E matrix can be constructed from a compact matrix M that has a simple and regular definition.

$$
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Figure 3. The upper left portion of the M matrix.

Definition 2.13. Define the M matrix as follows:

1. $M[0, i] = 1$ for $i \geq 0$, $M[i, 0] = 0$ for $i > 0$, and
2. $M[i, j] = M[i, j-1] \oplus M[i-1, j-1]$ for $i, j > 0$.

The M matrix is illustrated in Figure 3. It is easy to show that for $i > j$, $M[i, j] = 0$. The following theorem describes the relation between E and M.

Theorem 2.14. *E can be computed from the matrix M in the following way:*

$$E_i = \bigoplus_{k=0}^{l} i_k \cdot M_k,$$

where E_i is the ith row of E, $i = \langle i_l \dots i_0 \rangle$, and M_k is the kth row of M.

Proof. Let E' be the matrix that is constructed using M, as above. By Theorem 2.12, we need to show only that E' has properties described in Lemmas 2.7, 2.8, and 2.9, and it follows that $E = E'$.

Zero Initial Row Property: By the construction of E', its first row is all 0's.

Repeating Headers Property: Let i be any integer. Consider the prefix of length i of the rows. It is sufficient to prove that for any m, $E'[m \cdot 2^i..(m+1) \cdot 2^i][0..i]$ is the same as $E'[0..2^i][0..i]$.

Consider the nth row of E' that lies in $E'[m \cdot 2^i..(m+1) \cdot 2^i][0..i]$. Let $\langle n_j \dots n_i \dots n_0 \rangle$ be the binary representation of n and

$$n' = n \bmod 2^i \ = \langle n_{i-1} \dots n_0 \rangle.$$

Then $E'_i = \oplus_{k=0}^{j} \ n_k \cdot M_k = (\oplus_{k=0}^{(i-1)} \ n_k \cdot M_k) \bigoplus (\oplus_{k=i}^{j} n_k \cdot M_k)$.

For each M_k in the second term, its prefix of length i consists of all zeros. So

$$E'_n[0..i] = \bigoplus_{k=0}^{i} i_k \cdot M_k[0..i] = E'_{n'}[0..i].$$

Thus the repeating headers property holds for E'.

Distinct Rows Property: To prove that E' has the distinct row property, view each row of M and E' as a vector over $\mathbb{Z}_2$. Then the rows of $E'[0..2^j][i..i+j]$ are all the vectors in the linear space spanned by $M[0..j][i..i+j]$. To show that all the rows of E' are distinct, we need to show only that the vectors in M are linearly independent, which can be proved by showing that the determinant $|M|$ is nonzero.

We prove this by induction on the order j of $|M|$. For $j = 1$, since the first row of M contains only 1's, we have $|M| = 1$.

Now suppose that for any i, $|M[0..j-1][i..i+j-1]| \neq 0$.

To calculate $|M[0..j][i..i+j]|$, apply the following linear transformation on it: for $i = 1, \dots, j-1$, add the ith column to the $(i-1)$st column. Because of condition (2) in Definition 2.13, it is easy to see that

$$|M[0..j][i..i+j]| = \begin{vmatrix} \mathbf{0} & 1 \\ M[0..j-1][i..i+j-1] & B \end{vmatrix} = |M[0..j-1][i..i+j-1]|,$$

where B is a $(j-1) \times 1$ binary matrix.

By induction, the determinant is not 0, and the distinct rows property is proved. $\square$

3. Algorithm

Based on the results in the last section, we can directly compute an entry of the PEC network using Algorithm 1.

What is the time complexity of this algorithm? The cost of steps 1 through 3 is $O(m+n)$. The body of the while loop will be executed $l = P[i,j] \leq m+n$ times, while the body of the for loop takes $O(m)$ steps. So the total time complexity of the algorithm is $O(l \cdot m+n)$, or $O(m^2+mn)$ in the worst case. Since the P matrix is symmetric, we can assume without loss of generality that $m \leq n$, so we can simplify the worst-case analysis to $O(mn)$, or alternatively, to $O(\log(i) \cdot \log(j))$.

Algorithm 1. (Computing an entry of a PEC network.)

Input: coordinates i and j.
Output: the value of the entry $P[i, j]$.

1. Compute the binary representation of $i = \langle i_m \dots i_0 \rangle$
2. Compute the binary representation of $j = \langle j_n \dots j_0 \rangle$
3. Create an array M' of m integers with initial value $100\dots0$ (this is column 0 of the M matrix)
4. $l \leftarrow 0$
5. $e_0 \leftarrow i_0$
6. **while** $e_l \neq j_l$ **do** (treat j_l as 0 if $l > n$)
   (a) $l = l + 1$ (let M' be the lth column of the M matrix)
   (b) **for** $k \leftarrow m$ **downto** 1 **do**
   $$M'[k] \leftarrow M'[k] \oplus M'[k-1]$$
   (c) $e_l = \bigoplus_{k=1}^{m} \; i_k \cdot M'[k]$
7. **return** $l + 1$

4. More on the Definition of the PEC Matrix

In this section we show that the original definition of P was too strong. After proving the following theorem we get a corollary that shows that the definition of P did not need to specify that each entry of value n was on a grid with spacing 2^n.

Theorem 4.1. *Let Q be a matrix that satisfies*

1. $Q[0, 0] = \infty$,
2. *in each (i, j)-tile of Q there is exactly one entry $i + j$.*

Then Q is uniquely determined by its first row and first column.

We will give the proof of Theorem 4.1 at the end of this section.

The main result of this section is the following corollary.

Corollary 4.2. *The PEC matrix P can also be defined as a matrix that satisfies*

1. $P[0,0] = \infty$,
2. *in each (i,j)-tile of P there is exactly one entry $i+j$,*
3. *the first row and first column are the one-dimensional PEC matrix.*

First of all, we investigate some properties of the matrices defined in Theorem 4.1.

Lemma 4.3. *Let Q be a matrix defined in Theorem 4.1. Then*

(a) *in an (m,n)-tile of Q, there are 2^{m+n-i} entries equal to i, for $1 \le i \le m+n$,*

(b) *all entries of Q are greater than* 0*,*

(c) *in an (m,n)-tile of Q, there is exactly one entry greater than $m+n$,*

(d) *the largest 2^i numbers in an (m,n)-tile are the numbers in that tile that are greater than $m+n-i$.*

Proof. We prove (a) by induction on $m+n$.

For the case $m+n=1$, by definition of Q, there is one entry equal to 1, so (a) is true for this case.

Suppose that (a) is true for $m+n=k \ge 1$, and consider the case of $m+n = k+1$. Since $k+1 \ge 2$, we can assume without loss of generality that $m > 0$. Split the tile into two $(m-1,n)$-tiles.

For $1 \le i \le m+n-1$, by induction, in each subtile there are $2^{m+n-1-i}$ entries equal to i, so there are altogether $2 \cdot 2^{m+n-1-i} = 2^{m+n-i}$ entries equal to i in the (m,n)-tile. For $i = m+n$, by definition of Q, there is one $m+n$ in this tile. Thus (a) is true for $m+n = k+1$.

(b) For any (i,j), take m,n such that $2^m > i$, $2^n > j$. The number of entries in $Q[0..2^m][0..2^n]$ that are greater than 0 is $1 + \sum_{i=1}^{m+n} 2^{m+n-i} = 2^{m+n}$, i.e., all entries in this (m,n)-tile are greater than 0. So $Q[i,j] > 0$.

(c) From (a) and (b), the number of entries in an (m,n)-tile that are greater than $m+n$ is $2^{m+n} - \sum_{i=1}^{m+n} 2^{m+n-i} = 1$.

(d) The number of entries in an (m,n)-tile that are greater than $m+n-i$ is

$$1 + \sum_{k=m+n-i+1}^{m+n} 2^{m+n-k} = 1 + \sum_{l=0}^{i-1} 2^l = 2^i. \qquad \square$$

The following lemma is just a corollary of Lemma 4.3.

Lemma 4.4. *For any $m, n \geq 0$, consider the tile $Q[2^m..2^{m+1}][2^n..2^{n+1}]$. In this tile, there is one $m+n+2$, and for $1 \leq i \leq m+n$, there are 2^{m+n-i} entries equal to i.*

So for any m, n, the set of integers in the submatrix $Q[2^m..2^{m+1}][2^n..2^{n+1}]$ are uniquely determined. Determining the submatrix is only a matter of determining the positions of these integers.

For a given entry e, we will use $\mathrm{FP}(e)$ to denote its feasible positions. For example, if e is the entry in $M[2^m..2^{m+1}][2^n..2^{n+1}]$ with value $m+n$, then without any other information, $\mathrm{FP}(e)$ is the whole block. Given the position of the entry $m+n+2$, since e cannot be in the same half-side with $m+n+2$, $\mathrm{FP}(e)$ will be a quarter of the tile.

Note that given an (m, n)-tile B, for any $s > 0$, there is a unique $(m+s, n)$-tile B_s that contains B. We call such a tile the *supertile* of B in the first direction. The two $(m-1, n)$-tiles contained in B are called the subtiles of B in the first direction. We also define similar terms for the second direction.

The following lemma shows that we can narrow down the feasible positions using the information from other entries.

Lemma 4.5. (Expand–shrink lemma.) *Let B be an (m, n)-tile, and* $\max$ *is the entry with the largest number in B. Suppose B' is a supertile of B in the first direction, and* $\max, \max'$ *are the largest two entries in B'. Given the position* $\max'$*, then for the two subtiles of B in the second direction, we can determine which one of them contains* $\mathrm{FP}(\max)$.

Proof. Split B' into two subtiles in the second direction. Then max and max$'$ must be in different subtiles. Since the position of max$'$ is given, we can determine which subtile of B' max must be in. Let it be A. Then $A \cap B$ is a subtile of B in the second direction and $\max \in A \cap B$. □

The positions of larger entries in a tile will restrict the feasible positions of smaller entries, as proved in the following lemma.

Lemma 4.6. *Let B be an (m, n)-tile of M, $m > n$, and suppose that the set of integers in B is known. For $1 \leq i \leq m+n$, if the positions of the entries with values greater than i are given, then for the 2^{m+n-i} entries with values i, we can determine 2^{m+n-i} tiles of size $h \times w$ such that each such tile has one entry with value i as the largest entry; here $h = \max\{1, 2^{i-1-n}\}$, $w = \max\{1, 2^{i-1-m}\}$.*

Proof. There are two cases, depending on i:

Case 1: $i > n$. Divide B into $2^{m+n-i+1}$ subtiles of size $2^{i-1-n} \times 2^n$. Each such subtile contains one entry that is greater than $i-1$. We know the positions of the $2^{m+n-i-1}$ entries with value greater than i, so we know the subtiles they are in, and we can determine the subtiles that contain i.

Take one subtile that contains an entry max with value i. We can expand it in the first direction, and the maximum entry max$'$ in its supertile will be an entry with value greater than i, so we know the position of max$'$. Applying Lemma 4.5, we can shrink the "width" of FP(max) to 2^{n-1}. Applying Lemma 4.5 repeatedly, we can further reduce the "width."

This procedure will terminate for one of two reasons:

1. FP(max) cannot expand to encounter another max$'$ whose position is known, which will happen after applying the lemma for $m-(i-1-n) = m+n-i+1$ times.

2. The "width" of FP(max) becomes 1 and cannot be reduced any further, which will happen after n times.

If $i > m$, $m+n-i+1 \leq n$, the procedure will terminate after $m+n-i-1$ steps, and the final width of FP(max) is $2^{n-(m+n-i+1)} = 2^{i-1-m}$.

If $n < i \leq m$, then $m+n-i+1 > n$, the procedure will terminate after n steps, and the final width of FP(max) is 1.

For both cases, the lemma is true.

Case 2: $i \leq n$. Divide B into subtiles of size $1 \times 2^{i-1}$. Using a similar argument to that in (1), in this case we can apply Lemma 4.5 for $i-1$ times, and the feasible position for one entry with value i will be reduced to a point, which means that its position is uniquely determined. □

The following lemma shows how to fill in M uniquely.

Lemma 4.7. *Let*

$$B = M[0..2^{m+1}][0..2^{n+1}] = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}\right],$$

where $m, n \geq 1$ and the B_{ij} are matrices of size $2^m \times 2^n$, for $i, j \in \{1, 2\}$. Then B is uniquely determined by the three blocks B_{11}, B_{12}, and B_{21}.

Proof. Without loss of generality, we can assume that $m \geq n$.

Suppose B_{11}, B_{12}, and B_{21} are determined. As we noted before, determining B_{22} is to determine the positions for its entries.

First we will show that there is a unique feasible position for $m+n$. Then for $i = m+n, \ldots, 1$, we will determine the positions of the entries with value i based on the positions of entries with value greater than i, as well as B_{11}, B_{12}, and B_{21}.

For the entry $m+n$, it is the maximum entry in B_{22}, and B_{22} can be expanded in the first direction. Since all entries in B_{12} are determined, Lemma 4.5 can apply n times, and $\mathrm{FP}(m+n)$ is narrowed down to a column of B_{22}. Similarly, in the other direction, $\mathrm{FP}(m+n)$ can be narrowed down to a row. Thus its position is uniquely determined.

Suppose that all entries with value greater than i have determined positions. Consider the entries with value i. By Lemma 4.6, each entry's feasible position is a subtile of size $h \times w$. If h, w are not 1, then we can apply Lemma 4.5 in the appropriate direction and find the unique position for that entry. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Given the first row and first column of Q, we show that for any i, j, we can fill in $Q[i,j]$ uniquely.

Take m, n such that $2^n > j$ and $2^m > i$. We prove the following statement by induction on k:

$$Q[0..2^k][0..2^n] \text{ is uniquely determined.}$$

For $k = 0$, since the first row is given, the statement is true.

Suppose the statement is true for k, and consider the case of $k+1$. By induction, $Q[0..2^k][0..2^n]$ is known and $Q[0..2^{k+1}][0..1]$ is given by the boundary condition; by Lemma 4.7, $Q[0..2^{k+1}][0..2]$ is uniquely determined. Repeatedly applying Lemma 4.7, we can fill in $Q[0..2^{k+1}][0..2^n]$ uniquely. Thus the statement is true for $k+1$. □

5. Conclusion

We have shown how to calculate the entries of the PEC matrix directly. The method first constructs some entries of the compact M matrix and then some entries of a row of the E matrix. The algorithm is very efficient.

The paper [Kirkman and Quammen 91] anticipated our approach. The authors provided a table that is essentially the same as our M matrix and used it to compute "shift values" that relate successive rows to the top row. This makes sense only for finite truncations of P. (Our work shows that most rows of the infinite P matrix are not a finite shifted copy of the top row; our approach never considered shift values.) Their discussion of how to compute the shift values was only for one finite case with no proof or technique for generalization.

There remain many open problems. For example, even though it is clear by construction that $P[i, j] = P[j, i]$, the algorithm is very asymmetric in terms of i and j. Also, the M matrix is the well-known "Sierpiński gasket," and there are probably further relationships to be discovered.

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