Experimental Mathematics, 21(1):57–64, 2012

ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2011.594668

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Smooth Structures on Eschenburg Spaces: Numerical Computations

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CONTENTS

This paper numerically computes the topological and smooth invariants of Eschenburg–Kruggel spaces with small fourth cohomology group, following Kruggel's determination of the Kreck–Stolz invariants of Eschenburg spaces satisfying condition C [Eschenburg 82, Kruggel 05, Kreck and Stolz 91]. It is shown that each topological Eschenburg–Kruggel space with small fourth cohomology group has each of its 28 oriented smooth structures represented by an Eschenburg–Kruggel space. Our investigations also suggest that there is an action of $\mathbb{Z}_{12}$ on the set of homotopy classes of Eschenburg–Kruggel spaces, the nature of which remains to be understood.

The calculations are done in `C++`with the GNU GMP arbitrary-precision library and Jon Wilkening's `C++` wrapper.

1. INTRODUCTION

In [Aloff and Wallach 75], the authors introduced a family of 7-manifolds that are homogeneous spaces of SU_3 as follows: let p, q be coprime integers and let $U_{p,q} \subset \mathrm{SU}_3$ be the subgroup of diagonal matrices of the form $\mathrm{diag}(z^p, z^q, z^{-p-q})$ for $z \in S^1$. The Aloff–Wallach 7-manifold M_{pq} is equal to $\mathrm{SU}_3/U_{p,q}$. Aloff and Wallach showed that a bi-invariant metric on SU_3 induces a positively curved submersion metric on the quotient $M_{p,q}$. In [Kreck and Stolz 91, Kreck and Stolz 88], the authors studied the topological and smooth classification of Aloff–Wallach spaces. Among other things, they showed that there are diffeomorphic Aloff–Wallach spaces that are not SU_3 equivariantly diffeomorphic: the "smallest" example occurs with (p, q) equal to $(-4\,638\,661, 582\,656)$ and $(-2\,594\,149, 5\,052\,965)$ [Kreck and Stolz 88, p. 468]. Each of these spaces has a finite cyclic fourth integral cohomology group; they showed, through a computer search, that if the order of $H^4(M_{p,q};\mathbb{Z})$ is less than $r = 2\,955\,27\,597$, then the topological structure determines the smooth structure. Additional computer searches, attributed to Zagier and Odlyzko, revealed homeomorphic, but not diffeomorphic, Aloff–Wallach spaces with rank $H^4(M_{p,q};\mathbb{Z})$ between the above number and roughly

2000 AMS Subject Classification: Primary 57R55; Secondary 53C25
Keywords: Smooth structures, Eschenburg spaces, Gnu Gmp

2×10^{20} [Kreck and Stolz 88, p. 467]. In all cases, there were no reported examples of a topological Aloff–Wallach space whose 28 distinct smooth structures are themselves diffeomorphic to Aloff–Wallach spaces.

In [Eschenburg 82], the author introduced a family of 7-manifolds that generalize Aloff–Wallach spaces [Kruggel 05]. Let $U \cong \mathrm{U}_1$ be a subgroup of $\mathrm{U}_3 \times \mathrm{U}_3$ such that the natural action of U on U_3 defined by

$$\forall u = (u_1, u_2) \in U, g \in \mathrm{U}_3 : u \cdot g = u_1 g u_2^{-1} \qquad (1\text{–}1)$$

stabilizes SU_3 and is free. The group U is conjugate to a diagonal subgroup characterized by two integer vectors k and $l \in \mathbb{Z}^3$ such that $k_0 + k_1 + k_2 = l_0 + l_1 + l_2$:

$$U_{kl} = \left\{ \mathrm{diag}(z^{k_0}, z^{k_1}, z^{k_2}) \oplus \mathrm{diag}(z^{l_0}, z^{l_1}, z^{l_2}) \ : \ z \in S^1 \right\}. \qquad (1\text{–}2)$$

The freeness of the action (1–1) is equivalent to the property that

$$\forall \text{ permutations } \sigma : k - \sigma(l) \text{ is a primitive vector in } \mathbb{Z}^3. \qquad (1\text{–}3)$$

Eschenburg defined k, l to be *admissible* if

$$\begin{gathered} \gcd(k_0 - l_0, k_1 - l_1),\ \gcd(k_0 - l_0, k_1 - l_2), \\ \gcd(k_0 - l_1, k_1 - l_0),\ \gcd(k_0 - l_1, k_1 - l_2), \\ \gcd(k_0 - l_2, k_1 - l_0),\ \gcd(k_0 - l_2, k_1 - l_1) \end{gathered} \qquad (1\text{–}4)$$

are all equal to 1.

Definition 1.1. Let $k, l \in \mathbb{Z}^3$ satisfy $k_0 + k_1 + k_2 = l_0 + l_1 + l_2$ and the admissibility conditions (1–4) and define U_{kl} as in (1–2). The 7-manifold $E_{kl} := \mathrm{SU}_3/U_{kl}$ is called an *Eschenburg* space.

Eschenburg computed the integral cohomology ring of $E_{k,l} = \mathrm{SU}_3/U_{k,l}$ and proved that these spaces are strongly inhomogeneous in most cases. He also showed that under certain conditions on k, l, a bi-invariant metric on SU_3 induces a positively curved submersion metric on $E_{k,l}$.

In [Kruggel 05], the author computed the Kreck–Stolz invariants of a broad number of Eschenburg spaces—henceforth an Eschenburg–Kruggel space—and obtained a classification of these Eschenburg–Kruggel spaces up to homotopy, homeomorphism, and diffeomorphism. In [Chinburg et al. 07], the authors implemented a computer search for homeomorphic, but not diffeomorphic, positively curved (respectively 3-Sasakian) Eschenburg–Kruggel spaces. They found that for $\#H^4(E_{k,l}; \mathbb{Z}) < 8000$, there is a unique pair of homeomorphic, but not diffeomorphic, positively curved Eschenburg–Kruggel spaces. In [Butler 09], the present author proved that the existence of a real-analytically completely integrable convex Hamiltonian is a nontrivial smooth invariant of the configuration space, and proved the complete integrability of geodesic flows on all Eschenburg–Kruggel spaces. That work motivated the following question.

Question 1.2. Let E be a topological Eschenburg space. Is each smooth structure on E diffeomorphic to an Eschenburg space $E_{k,l}$?

One knows from the work of Kreck and Stolz that each topological Eschenburg space admits 28 distinct oriented smooth structures, but one does not know whether each structure is represented by an Eschenburg space. From the above-mentioned results, it is not clear whether each distinct oriented smooth structure on a topological Eschenburg space is represented by an Eschenburg space or whether such representatives are rather sparse, as for Aloff–Wallach spaces. This note attempts to cast some light on this question.

Theorem 1.3. *Let $I = [-850, 850]$ and $J = [1, 101]$. Among the Eschenburg–Kruggel spaces with $(k, l) \in I^3 \times I^3$, for each odd $|r| = \#H^4(E; \mathbb{Z})$ in the interval J, columns 2 and 9 of Table 1 show a lower bound on the number of oriented homeomorphism classes. For $|r| \leq 9$, each oriented homeomorphism class of Eschenburg–Kruggel spaces has each of its 28 distinct oriented smooth structures represented by an Eschenburg–Kruggel space $E_{k,l}$ with $(k, l) \in I^3 \times I^3$.*

Remark 1.4. Columns 3–7 and 10–14 of Table 1 list the numbers of topological Eschenburg–Kruggel spaces, for a fixed $|r|$, that have the stated number of oriented smooth structures represented by Eschenburg–Kruggel spaces.

The smooth structures on a topological Eschenburg–Kruggel space are an orbit of the group of homotopy 7-spheres ($\cong \mathbb{Z}_{28}$). The Kreck–Stolz invariant s_1 is additive under this action: if Σ is a homotopy 7-sphere and E is an Eschenburg–Kruggel space, then $s_1(E\#\Sigma) = s_1(E) + s_1(\Sigma)$ and $28 \cdot s_1(\Sigma) \equiv 0 \bmod 1$. This implies that each topological Eschenburg–Kruggel space has 28 distinct oriented smooth structures [Kruggel 05]. The difficulty is that the surgery description of the smooth structure $E\#\Sigma$ does not appear to contain information about the structure of $E\#\Sigma$ as an Eschenburg–Kruggel space.

Tables 5–7 of [Butler 09] list representative Eschenburg–Kruggel spaces for each smooth structure on each topological Eschenburg–Kruggel space with $|r| \leq 5$ that was found in constructing Table 1. It seems likely that all topological and smooth Eschenburg–Kruggel spaces with $|r| \leq 5$ are enumerated in those tables.

This note is structured as follows: Section 2 reviews Kruggel's condition C; Section 3 reviews Kruggel's computation of the Kreck–Stolz invariants; Section 4 explains how the Kreck–Stolz invariants were computed in software; and Sections 5.1–5.3 discuss several tables.

2. ESCHENBURG–KRUGGEL SPACES

To compute the Kreck–Stolz invariants of Eschenburg spaces, Kruggel observed that the projection of an $x \in \mathrm{SU}_3$ onto its first two columns in the Stiefel manifold $\mathrm{V}_2(\mathbb{C}^3)$ is a diffeomorphism. From the embedding of $\mathrm{V}_2(\mathbb{C}^3) \subset \mathbb{C}^{2\cdot 3}$, Kruggel constructed an 8-manifold W' with boundary $\mathrm{V}_2(\mathbb{C}^3)$. The action of U_{kl} descends naturally to $\mathbb{C}^{2\cdot 3}$ and W', but the action on W' has three singular orbits. One can cut away these three singular orbits to construct a cobordism between E_{kl} and a union of three lens spaces, provided that the matrix

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix} \tag{2–1}$$

has a column or row containing nonzero pairwise coprime entries.

Definition 2.1. (Kruggel 2006.) The Eschenburg space $E_{k,l}$ satisfies *condition C* if the matrix A has a column or row containing nonzero pairwise coprime entries. An Eschenburg space that satisfies condition C is called an *Eschenburg–Kruggel space*.

Remark 2.2. Note that the coprimality conditions (1–4) do not imply that all entries of A are nonzero. The Eschenburg space E_{kl} with $k = (-1, -1, 2)$ and $l = (-2, 0, 2)$ has

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & -1 & -3 \\ 4 & 2 & 0 \end{bmatrix}. \tag{2–2}$$

This defines an Eschenburg–Kruggel space according to Definition 2.1. Indeed, the coprimality conditions (1–4) are satisfied, since they are

$$\begin{gathered} \gcd(A_{00}, A_{11}),\ \gcd(A_{00}, A_{12}), \\ \gcd(A_{01}, A_{10}),\ \gcd(A_{01}, A_{12}), \\ \gcd(A_{02}, A_{10}),\ \gcd(A_{02}, A_{11}) \end{gathered}$$

that is,

$$\begin{gathered} \gcd(1, -1),\ \gcd(1, -3), \\ \gcd(-1, 1),\ \gcd(-1, -3), \\ \gcd(-3, 1),\ \gcd(-3, -1), \end{gathered}$$

which are all unity; and condition C is satisfied by the leftmost column of A. See Remark 3.1 for more.

3. INVARIANTS OF ESCHENBURG–KRUGGEL SPACES

Let E_{kl} be an Eschenburg space. Let u be the Chern class of the bundle $S^1 = U_{kl} \hookrightarrow \mathrm{SU}_3 \to E_{kl}$. Eschenburg proved that the nontrivial parts of the integral cohomology ring of $E_{k,l}$ have the following structure:

$$H^2(E_{kl}; \mathbb{Z})\mathbb{Z} \cdot u, \quad H^4(E_{kl}; \mathbb{Z}) = \mathbb{Z}_r \cdot u^2. \tag{3–1}$$

The integer r is equal to $\sigma_2(k) - \sigma_2(l)$, where σ_j is the jth elementary symmetric polynomial, $\sigma_j(x) = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$. The linking form of $E_{k,l}$ is plainly determined by the linking number of u^2 with itself. Kruggel showed that this equals

$$\mathrm{Lk}\left(u^2, u^2\right) = -\frac{s^{-1}}{r} \bmod 1, \tag{3–2}$$

where $s = \sigma_3(k) - \sigma_3(l)$ and s^{-1} is the multiplicative inverse of $s \bmod r$. Kruggel also showed that the first Pontryagin class of E_{kl} equals

$$p_1(E_{kl}) = p_1 \cdot u^2 \bmod r, \tag{3–3}$$

where

$$p_1 = 2\sigma_1(k)^2 - 6\sigma_2(k).$$

Although this expression appears to be asymmetric in k and l, the sum condition plus the definition of r ensures that it is well defined.

In addition to the above invariants, Kruggel was able to compute the Kreck–Stolz invariants for Eschenburg–Kruggel spaces. To explain, let $p \neq 0$ be coprime to the nonzero integers $p_0, \ldots, p_3$, and let

$$L = L(p; p_0, p_1, p_2, p_3) = S^7/C, \tag{3–4}$$

where

$$C = \left\{ \mathrm{diag}\left(e^{2\pi i k p_0/p}, e^{2\pi i k p_1/p}, e^{2\pi i k p_2/p}, e^{2\pi i k p_3/p}\right) : k = 0, \ldots, p-1 \right\},$$

		Counts							Counts				
—r—	#Top.	28	27	14−26	2−13	1	—r—	#Top.	28	27	14−26	2−13	1
1	12	12					3	8	8				
5	48	48					7	120	120				
9	24	24					11	360	354	4	2		
13	576	542	22	12			15	32	32				
17	1152	988	68	96			19	1512	1216	86	204	6	
21	80	64	10	6			23	2640	1726	276	598	40	
25	240	240					27	72	72				
29	4704	1656	814	2212	22		31	5760	1506	794	3080	380	
33	240	114	30	92	4		35	480	230	90	160		
37	8634	904	918	5728	1072	12	39	384	118	58	176	32	
41	11988	376	636	8778	2176	22	43	12600	272	500	9412	2414	2
45	96	60	20	16			47	17108	82	248	10950	5812	16
49	1848	1028	310	510			51	768	44	26	522	176	
53	22456	46	122	11414	10836	38	55	1440	320	200	752	168	
57	1008	28	36	666	278		59	29902	10	32	10662	19034	164
61	32874	22	76	9468	22764	544	63	240	60	10	164	6	
65	2304	178	220	1332	574		67	39854	12	28	7890	31108	816
69	1756			864	878	14	71	47544			6596	39738	1210
73	48034	2	10	6090	40864	1068	75	160	50	42	68		
77	3600	332	112	1914	1234	8	79	59046		4	4508	51962	2572
81	216	188	22	6			83	67340			3544	59816	3980
85	4602	28	82	2670	1800	22	87	3128			580	2522	26
89	78944			2068	70090	6786	91	5740	256	154	2502	2734	94
93	3788			468	3182	138	95	6016	18	22	2836	3054	86
97	91772			1484	79690	10598	99	720	12	40	474	194	
101	100490			742	87290	12458							

TABLE 1. $|r| = \operatorname{rank} H^4(E;\mathbb{Z})$ versus the number of homeomorphism classes (#Top.), and the number of homeomorphism classes with the n smooth structures represented by Eschenburg–Kruggel spaces, for $n = 28, 27$, $14 \le n \le 26$, $2 \le n \le 13$, and $n = 1$.

be a lens space. Define the following functions:

$$s_1(L) = \frac{1}{2^7 \cdot 7 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=0}^{3} \cot\left(\frac{k\pi p_j}{p}\right) + \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=0}^{3} \csc\left(\frac{k\pi p_j}{p}\right) \mod 1 \tag{3–5}$$

$$s_2(L) = \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \left(e^{2\pi i k/p} - 1\right) \prod_{j=0}^{3} \csc\left(\frac{k\pi p_j}{p}\right) \mod 1.$$

These are the Kreck–Stolz invariants of the lens space L in (3–4), and they take values in $\mathbb{Q}/\mathbb{Z}$.

Assume that the leftmost column of the matrix A has pairwise coprime nonzero entries (the remaining cases are described below). The above-described cobordism exhibits E_{kl} as cobordant to the disjoint union of the three lens spaces

$$\begin{aligned} L_0 &= L(A_{00}; A_{10}, A_{20}, A_{11}, A_{21}), \\ L_1 &= L(A_{10}; A_{00}, A_{20}, A_{01}, A_{21}), \\ L_2 &= L(A_{20}; A_{00}, A_{10}, A_{01}, A_{11}). \end{aligned} \tag{3–6}$$

Let us see that L_0 is indeed a lens space. By condition C, the integers A_{j0} are pairwise coprime and nonzero. The primitivity condition (1–3) implies that A_{00} is coprime to A_{11} and A_{21}. For example, suppose that A_{00} and A_{11}

have a divisor $d > 1$, so one can write $A_{00} + A_{11} = dc$. The condition that $\sum k_i = \sum l_i$ in Definition 1.1 is equivalent to $A_{00} + A_{11} + A_{22} = 0$, so $A_{22} = -dc$. If $c = 0$, then the vector $k - l = (A_{00}, A_{11}, A_{22})$ is not primitive; if $c \neq 0$, then the same vector is also not primitive. (This argument also shows that if $A_{22} = 0$, then $A_{00} = -A_{11} = \pm 1$.) The remaining verifications for L_1 and L_2 are similar.

Kruggel showed that the Kreck–Stolz invariants are equal to

$$s_1(E_{kl}) = \frac{\operatorname{sign}(w)}{2^5 \cdot 7} - \frac{q^2}{2^7 \cdot 7 \cdot w} - \sum_{i=1}^{3} s_1(L_i) \mod 1 \tag{3–7}$$

$$s_2(E_{kl}) = \frac{q-2}{2^4 \cdot 3 \cdot w} - \sum_{i=1}^{3} s_2(L_i) \mod 1, \tag{3–8}$$

where

$$q = A_{00}^2 + A_{10}^2 + A_{20}^2 + A_{01}^2 + A_{11}^2 + A_{21}^2 - (l_0 - l_1)^2, \tag{3–9}$$

$$w = r \cdot A_{00} A_{10} A_{20}. \tag{3–10}$$

These invariants are transcendental functions of the variables k, l. This fact, plus the fact that the sums can have a rather large number of terms, means that showing that two Eschenburg–Kruggel spaces are homeomorphic or diffeomorphic is rather difficult. However, since s_i is a rational integer, one can use a few numerical tricks to prove equality of these invariants.

Remark 3.1. The well-definedness of Kruggel's formulas (3–7)–(3–8) amounts to the statement that if $E_{k,l}$ satisfies condition C, then w (3–10) does not vanish. Indeed, from the remark above, the lens-space invariants $s_j(L_i)$ (3–5)–(3–6) are well defined if $w \neq 0$. Since condition C is assumed to hold for the leftmost column of A, it follows that $A_{00}A_{10}A_{20} \neq 0$. In addition, Kruggel proved that r must be odd [Kruggel 05, p. 572] (in fact, since $H^3(E_{k,l}; \mathbb{Z})$ vanishes, Poincaré duality implies $r \neq 0$). Therefore, $w \neq 0$. Compare Remark 2.2.

The results of this note rely on the following theorem

Theorem 3.2. (Kruggel 2005.) *Two Eschenburg–Kruggel spaces, $E_{k,l}$ and $E_{k',l'}$ are orientation-preserving* homeomorphic *if $|r|$, s, p_1, and s_2 coincide. If, in addition, s_1 coincides with these, then they are orientation-preserving* diffeomorphic.

3.1. Automorphisms and Invariants

To compute the Kreck–Stolz invariants of Eschenburg–Kruggel spaces in general, one uses the extension of the natural action of the Weyl group of $SU_3 \times SU_3$ by the automorphism that interchanges factors. Concretely, let S_3 be the symmetric group acting naturally on $\mathbb{Z}^3$ by permutations, let τ be the involutive automorphism of $\mathbb{Z}^3 \oplus \mathbb{Z}^3$ that acts by $(k, l) \mapsto (l, k)$, and let $\eta : (k, l) \mapsto (-k, -l)$.

The group generated by $S_3 \times S_3$, τ (respectively $S_3 \times S_3$, τ, and η) is denoted by $\mathfrak{G}^+$ (respectively $\mathfrak{G}$). The group $\mathfrak{G}$ is of order 144, and $\mathfrak{G}^+$ is an index-2 subgroup.

Proposition 3.3. [Eschenburg 82] *For each $\sigma \in \mathfrak{G}$, the Eschenburg spaces $E_{k,l}$ and $E_{\sigma(k,l)}$ are diffeomorphic. If $\sigma \in \mathfrak{G}^+$, they are orientation-preserving diffeomorphic.*

Remark 3.4. With the above proposition, the formulas for the Kreck–Stolz invariants can be extended to all Eschenburg–Kruggel spaces as follows. The Eschenburg space E_{kl} is orientation-preserving diffeomorphic to $E_{\alpha(k),\beta(l)}$ for any permutations $\alpha, \beta \in S_3$. In addition, $E_{k,l}$ is orientation-preserving diffeomorphic to $E_{l,k}$. The permutation α permutes the rows (respectively β permutes the columns) of A, while the diffeomorphism $E_{kl} \to E_{lk}$ induces $A \mapsto -A'$.

It follows that if the column j (respectively row j) of A has nonzero pairwise coprime entries, then the leftmost column of $A_{k,\beta(l)}$ (respectively $A_{l,\beta(k)}$) has nonzero pairwise coprime entries and E_{kl} is orientation-preserving diffeomorphic to $E_{k,\beta(l)}$ (respectively $E_{l,\beta(k)}$) where $\beta = (0\,j)$. By this observation, one can compute the Kreck–Stolz invariants of any Eschenburg–Kruggel space by means of the formulas (3–7), (3–8).

The proposition also implies that each Eschenburg space $E_{k,l}$ has a representative, up to orientation, where $k_0 \le k_1 \le k_2$, $l_0 \le l_1 \le l_2$, and $k_0 \le l_0$.

4. METHODOLOGY

The search for homeomorphic smooth Eschenburg–Kruggel spaces neatly divides into three separate searches:

1. search over a domain of parameters $(k, l) \in \mathbb{Z}^3 \times \mathbb{Z}^3$ for Eschenburg–Kruggel spaces;
2. computation of the invariants r, s, p_1 and s_1, s_2 in terms of the parameters (k, l);

3. search the data generated for matching invariants.

Due to the size of the sample space considered, it was decided to do the first two steps in compiled code. The structure of the problem led to C++ as the language of choice.

The computations to generate all of the tables in this note and [Butler 09] took approximately six weeks of continuous CPU time on a single core of a 2-core 3.0-GHz Intel Core Duo E6850 CPU with 4 MB cache and 3.3-GB DDRAM 4.0-GB swap. The operating system was RHEL with the 2.6.8 Linux kernel.

4.1. The Search over Parameter Space

Let us define the parameter space and explain how the search is conducted.

4.1.1. The Parameter Space

Let $\mathbf{1} \in \mathbb{Z}^3$ be the vector all of whose elements are unity. If $E_{k,l}$ is an Eschenburg space, then $E_{k+n\mathbf{1},l+n\mathbf{1}}$ is the same Eschenburg space for any $n \in \mathbb{Z}$. There is, therefore, a unique representative of $(k,l) + \mathbb{Z}(\mathbf{1},\mathbf{1})$ such that $\sum k_i = \sum l_i \in [0,2]$. All searches were conducted with this constraint.[1]

4.1.2. The Search

The speed of the arithmetic in the native `signed long int` class of integers in C++ argued in favor of testing the admissibility condition (1–4) and condition C (Definition 2.1) in `signed long int`.

The coprimality tests are conducted by a two-part process. First, an $N \times N$ lookup table is created. The (i,j) entry of the lookup table is 1 if i and j are coprime and $ij \neq 0$; otherwise, it is 0. If $|i|$ or $|j|$ exceeds N, the Euclidean algorithm is first employed to reduce both i and j until the lookup table can be used. The parameter N is chosen at compile time; in our tests, $N = 2000$ was chosen, so that all coprimality tests required only a lookup.

4.2. Computation of the Invariants

The computation of the invariants is broken into two parts.

4.2.1. Integer Invariants

If (k,l), define an Eschenburg–Kruggel space. Then the rank of $H^4(E;\mathbb{Z})$, $|r|$, and the first Pontryagin class p_1 were computed using `signed long int` arithmetic. Since the set of `signed long ints` equals $[-2^{31}, 2^{31}-1] \cap \mathbb{Z}$, and both r and p_1 are quadratic forms in (k,l), `signed long int` arithmetic does not run into under/overflow errors for $|k_i| < 10922$. For the purposes of this note, all computations of r and p_1 were done in `signed long int` arithmetic.

Since s is cubic in (k,l), under/overflow does not affect computation for $|k_i|, |l_i| < 1023$. This relatively small bound led us to use Gmp arbitrary-precision floats[2] to compute s (see below).

4.2.2. Rational Invariants

From the definition of the Kreck–Stolz invariants (3–5), one can see that individual terms in each summand can be $O(1/p^3)$.

The Gnu Gmp package, along with its Gmpfrxx front end[3] for C++, permits one to do arbitrary-precision arithmetic from within C++. Since Gnu Gmp can compute the trigonometric functions to arbitrary precision, we elected to use this package to compute the Kreck–Stolz invariants of an Eschenburg–Kruggel space.

The relative slowness of software-implemented arithmetic also indicated a need to permit computation with machine-native floating-point arithmetic. The template facility of C++ made it possible to use the same code for both machine-native and software-implemented floating-point arithmetic and allow the user to choose the precision at run time rather than compile time.

4.3. Matching Invariants

The final step was to match the topological and smooth invariants that are computed for different Eschenburg–Kruggel spaces. This was accomplished, in essence, by multiple sorts. In the first step, a C++ program computed and sorted approximately 2 GB of the polynomial Eschenburg–Kruggel space invariants (r, s, p_1). These data were stored in text files, and these were sorted and split according to the value of $|r|$. The Kreck–Stolz invariants of these spaces were computed with 130 bits of precision and stored in a second database. The resulting data were imported into a second C++ program, where homeomorphism and diffeomorphism classes were

[1] In [Butler 09, Tables 5–7], one finds that the sums reported lie in $[-2,2]$. Those spaces with sum reported in $[-2,-1]$ are obtained by reversing the orientation of a space whose sum lies in $[1,2]$.

[2] Available at http://gmplib.org/.

[3] At http://math.berkeley.edu/~wilken/code/gmpfrxx/.

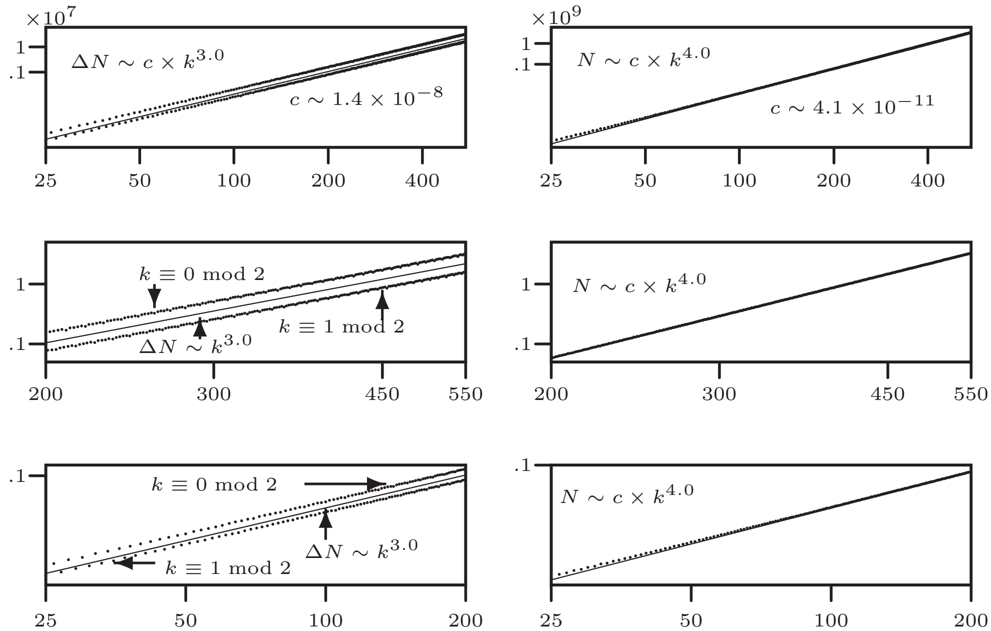

FIGURE 1. [Log-log scale]. The number of Eschenburg–Kruggel spaces, $N = N(k)$, and the marginal number, $\Delta N = \Delta N(k)$, in the cube $[-k, k]^6$. Left column, descending: the marginal number for k in the intervals $[25, 550]$, $[200, 550]$, and $[25, 200]$; Right column, descending: the total number for the same intervals. A least-squares regression line is also displayed on each graph.

computed. The data for Table 1 and [Butler 09, Tables 2–7] were generated in this way.

4.3.1. Testing

To ensure the accuracy of the computations, several tests were designed, including the following:

1. Replication of each of the published computations in [Astey et al. 97, Section 4], [Kruggel 05, Table 1], [Chinburg et al. 07, Tables 1–6],[4] and [Kruggel 05, Table 1].
2. Replication, up to a numerical $\epsilon \approx 2^{-130}$, of closed-form answers for the invariants of some Eschenburg–Kruggel space.
3. Replication, up to a numerical $\epsilon \approx 2^{-130}$, of the `C++` computed results in Maple, Maxima, and BC.

[4] In replicating these results, differing conventions for the projection map $x \mapsto \bar{x} \in (-\frac{1}{2}, \frac{1}{2}]$ became apparent. The Chinburg–Escher–Ziller code uses the convention that x is reduced modulo 1, and then $[0, \frac{1}{2}]$ is mapped to itself by the identity and $(\frac{1}{2}, 1]$ is mapped to $(-\frac{1}{2}, 0]$ by a constant shift. In our `C++` code, x is reduced modulo 1, then shifted by $-\frac{1}{2}$.

5. APPENDICES

5.1. Appendix A

The graph in Figure 1 shows the number N of Eschenburg–Kruggel spaces in the cube $[-k, k]^6$, as a function of k, with the constraint that $\sum k_i = \sum l_i \in [0, 2]$. A rough heuristic indicates that $N = O(k^4)$ for large k and $\Delta N = O(k^3)$, which is nicely captured here. It is also apparent that $\Delta N(k)$ grows like $c_\pm k^3$, where $c_\pm$ depends only on the parity of k.

5.2. Appendix B

We observed several unexplained phenomena. For fixed invariants r, s, and p_1, the Kreck–Stolz invariant s_2 appears to lie in the orbit of $\mathbb{Z}_n$ acting by $x \mapsto x + \frac{1}{n}$ mod 1, where $n = 4$ or 12. We also observed that the values taken on by s_1 appear to depend only on $|r|$, s, and p_1.

The first columns of [Butler 09, Tables 2 and 3] show these group actions on the Kreck–Stolz invariants when $|r| = 1, 3$. Table 4 in that same reference abstracts the picture from Tables 2 and 3, and shows the group actions on s_2 and s_1. It appears that $\mathbb{Z}_{12}$ acts effectively except when $r \equiv 0$ mod 3, $r \not\equiv 0$ mod 3^2, in which case $\mathbb{Z}_4$ acts effectively.

5.3. Appendix C

[Butler 09, Tables 5–7] list homeomorphism classes of Eschenburg–Kruggel spaces with the rank of the fourth integral cohomology group equal to $|r| = 1, 3, 5$ respectively. Each smooth structure in each such homeomorphism class is represented by an Eschenburg–Kruggel space; these tables list the "smallest" representatives.

ACKNOWLEDGMENTS

The author thanks T. Chinburg, C. Escher, and W. Ziller for helpful comments and providing their Maple code; J. Wilkening for releasing his GMPFRXX code; M. Kreck for discussions of [Kruggel 05]; and C. Peterson.

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Received November 22, 2010; accepted May 22, 2011.