Surface Realization with the Intersection Segment Functional

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CONTENTS

Deciding realizability of a given polyhedral map on a (compact, connected) surface belongs to the hard problems in discrete geometry from the theoretical, algorithmic, and practical points of view.

In this paper, we present a heuristic algorithm for the realization of simplicial maps, based on the intersection segment functional. This heuristic was used to find geometric realizations in $\mathbb{R}^3$ for *all* vertex-minimal triangulations of the orientable surfaces of genera $g = 3$ and $g = 4$. Moreover, for the first time, examples of simplicial polyhedra in $\mathbb{R}^3$ of genus 5 with 12 vertices have been obtained.

1. INTRODUCTION

A *polyhedral map* on a surface is a (finite) set of polygons (with at least three sides) that are glued together (topologically) along edges to form the surface in such a way that there are no self-identifications on the boundaries of the polygons, and two polygons are disjoint or intersect in exactly one edge or one vertex. We thus can think of a polyhedral map as a combinatorial model for a surface.

For a given polyhedral map it is natural to try to visualize it as a *polyhedron* in three-space or in some higher-dimensional space $\mathbb{R}^d$ such that every polygon is the convex hull of its vertices and two polygons are disjoint in $\mathbb{R}^d$, intersect in a common edge and are not coplanar, or intersect in a common vertex. Such a realization is usually called a *geometric* or *polyhedral realization*, with straight edges, plane polygons, and no nontrivial intersections (with neighboring polygons being not coplanar).

Example 1.1. A polyhedral map on the 2-sphere S^2 consisting of the polygons 123, 12478, 13568, 2354, 4567, and 678 together with a corresponding realization in $\mathbb{R}^3$ is displayed in Figure 1.

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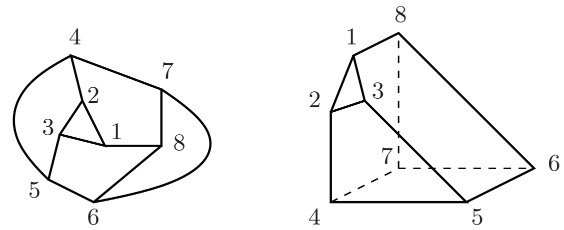

FIGURE 1. A polyhedral map on S^2 and a corresponding geometric realization in $\mathbb{R}^3$.

Realizability of maps on the 2-sphere S^2 was proved by Steinitz [Steinitz 22, Steinitz and Rademacher 34]; see also [Grünbaum 03, Chapter 13], [Ziegler 95, Lec. 4]): Every polyhedral map on the 2-sphere S^2 is geometrically realizable in $\mathbb{R}^3$ as the boundary complex of a convex 3-polytope.

However, not all polyhedral maps are realizable. For example, *simple polyhedral maps* (i.e., maps with all vertices of valence three) on surfaces different from the 2-sphere S^2 are not realizable in any $\mathbb{R}^d$ (see [Grünbaum 03, Exercises 11.1.7, 13.2.3]).

Example 1.2. All *equivelar maps* of type 6–3 on the torus (i.e., maps consisting of only 6-gons with every vertex belonging to exactly three 6-gons) are simple and therefore cannot be realized in any $\mathbb{R}^d$. The smallest example (see Figure 2) of the family is the combinatorial dual of Möbius's 7-vertex triangulation of the torus [Möbius 86]. A "realization" of this 6–3 torus with flat but nonconvex 6-gons, the *Szilassi torus*, was given in [Szilassi 86].

Betke and Gritzmann found a further combinatorial obstruction to geometric realizability [Betke and Gritzmann 82]: Let W be any subset of the set of odd valent vertices of a polyhedral map M^2 and let F_W be the set of facets containing some vertex of W. If $2|F_W| \leq |W|$, then M^2 is not realizable in any $\mathbb{R}^d$. Again, the Betke–Gritzmann obstruction rules out realizability of 6–3 equivelar maps on the torus, but the obstruction was also used in [McMullen et al. 82] to show nonrealizability for other, nonsimple, families of equivelar maps.

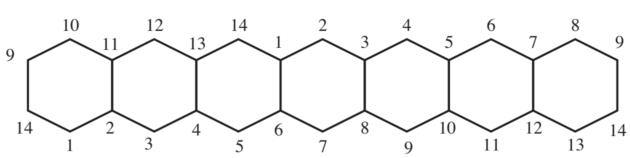

FIGURE 2. The nonrealizable 6–3 equivelar map with 14 vertices on the torus.

In a *simplicial map* (i.e., a triangulation of a surface as a simplicial complex) every triangle contains at most three odd valent vertices, from which it can be deduced that $|F_W| \geq |W|$ for every subset W of odd valent vertices. Thus, the Betke–Gritzmann obstruction cannot be applied to simplicial maps to show nonrealizability.

In 1967 Grünbaum proposed cf. [Grünbaum 03, Chapter 13] that every triangulated torus should be geometrically realizable in $\mathbb{R}^3$. This famous conjecture remained open for forty years and was proved in 2007 in [Archdeacon et al. 07]. For the class of triangulations of the projective plane with one face removed, geometric realizability in $\mathbb{R}^3$ was proved in [Bonnington and Nakamoto 08]. However, not every triangulated Möbius strip needs to be realizable in $\mathbb{R}^3$. A counterexample appears in [Brehm 83].

Until rather recently, no computational tools were available to actually find realizations for simplicial surfaces. In the past thirty years, the most promising approach to obtaining a polyhedral realization in $\mathbb{R}^3$ for a given triangulation was to try to build a physical model, for example by exploiting symmetries or by employing the *rubber-band technique* of [Bokowski 08].

Very basic heuristic procedures for finding realizations in $\mathbb{R}^3$ for larger classes of examples were used in [Fendrich 03] (to show that all triangulated tori with up to 11 vertices are realizable via embeddings in the 2-skeletons of random 4-polytopes) and in [Lutz 08] (to obtain realizations for triangulations of the orientable surface of genus 2 via choosing coordinates randomly). These methods, although useful in processing larger numbers of examples, are less powerful than human imagination in leading to a physical model.

For example, 864 of the 865 vertex-minimal triangulations of the orientable surface of genus 2 were realized with the random realization method in [Lutz 08], with a total computation time of 30 CPU months. However, it was impossible to randomly realize the remaining example (with the highest combinatorial symmetry of the 865 examples). A computer search was run unsuccessfully for more than a month before the example was realized within a day by Bokowski with the rubber-band method [Bokowski 08].

In this paper, we present a heuristic algorithm for finding polyhedral realizations for (closed, orientable) simplicial surfaces in $\mathbb{R}^3$ that for the first time surpasses the physical approach with respect to its processing time and its qualitative range of examples. In particular, we show that all vertex-minimal triangulations of the orientable surfaces of genera $g = 3$ and $g = 4$ are realizable. We also provide examples of vertex-minimal simplicial polyhedra of genus 5 with 12 vertices.

In the following section we give a brief survey of realizability results for surfaces, vertex-minimal triangulations, and algorithmic aspects of deciding realizability. Section 3 is devoted to our realization heuristic based on the *intersection segment functional*. Computational results are presented in Section 4. An extension of our approach to convex realizations of triangulated spheres is discussed in Section 5.

2. REALIZABILITY OF POLYHEDRAL SURFACES AND POLYHEDRAL COMPLEXES

In general, every d-dimensional simplicial complex (with n vertices) is polyhedrally embeddable in $\mathbb{R}^{2d+1}$, since it can be realized as a subcomplex of the boundary complex of the cyclic polytope $C(n, 2d+2)$; cf. [Grünbaum 03, Exercise 4.8.25, p. 67]. However, it is shown in [van Kampen 43] and [Flores 33] that d-dimensional simplicial complexes cannot always be embedded topologically in $\mathbb{R}^{2d}$, e.g., the d-skeleton $\mathrm{Sk}_d(\Delta_{2d+2})$ of the $(2d+2)$-simplex Δ_{2d+2} is not embeddable in $\mathbb{R}^{2d}$. For further examples and references see [Matoušek 03, 5.1], [Novik 00], and [Schild 93].

For smooth d-manifolds, it is proved in [Whitney 44] that they can be smoothly embedded in $\mathbb{R}^{2d}$, and [Penrose et al. 61] shows that for $0 < 2(k+1) \leq d$, every k-connected PL (i.e., piecewise linear) d-manifold has a PL embedding in $\mathbb{R}^{2d-k}$. In particular, surfaces have PL embeddings in $\mathbb{R}^4$. Orientable surfaces (with or without boundary) and nonorientable surfaces with boundary are even PL embeddable in $\mathbb{R}^3$ (which follows from the classification of surfaces in [Dehn and Heegaard 07]). Closed nonorientable d-manifolds cannot be embedded topologically in $\mathbb{R}^{d+1}$; cf. [Bredon 97, p. 353].

Thus, for triangulated orientable surfaces (with or without boundary) and for triangulated nonorientable surfaces with boundary we have

- PL embeddability in $\mathbb{R}^3$ and
- polyhedral realizability in $\mathbb{R}^5$.

Triangulations of closed nonorientable surfaces are

- not (topologically) embeddable in $\mathbb{R}^3$,
- but are PL embeddable in $\mathbb{R}^4$,
- and are polyhedrally realizable in $\mathbb{R}^5$.

Perles showed (cf. [Grünbaum 03, 11.1.8]) that a polyhedral map is realizable in some $\mathbb{R}^d$ if and only if it is realizable in $\mathbb{R}^5$.

A natural approach to establishing geometric realizability in $\mathbb{R}^3$ for polyhedral maps on orientable surfaces of genus $g \geq 1$ is to identify a given polyhedral map as a subcomplex of the boundary complex of a convex 4-polytope P. The Schlegel diagram of P then yields coordinates for the realization in $\mathbb{R}^3$; see, for example, [McMullen et al. 82] for realizations of equivelar maps obtained this way, and [Altshuler 71a, Altshuler 71b] for combinatorial properties on maps that guarantee realizability via Schlegel diagrams.

[Altshuler and Brehm 84] gives a polyhedral map T_8 on the torus with only eight vertices that is realizable in $\mathbb{R}^3$ (cf. also [Simutis 77]), but not via the Schlegel diagram of a convex 4-polytope. In fact, the map T_8 is not isomorphic to a subcomplex of the boundary complex of any convex polytope [Altshuler and Brehm 84].

Realizability (via subcomplexes of convex 5-polytopes) of triangulations of the torus and the projective plane in $\mathbb{R}^4$ was proved in [Brehm and Schild 95], thereby sharpening the result of [Barnette 83] on the geometric realizability of triangulations of the projective plane in $\mathbb{R}^4$.

Polyhedral surfaces that are obtained by projections (of 2-dimensional subcomplexes) of higher-dimensional polytopes together with obstructions to projectability are discussed in [Rörig 09, Rörig and Sanyal 09, Rörig and Ziegler 09]. Knotted realizations of triangulated tori are studied in [Lutz and Witte 07]. For further results and references on polyhedral maps see [Brehm and Wills 93, Brehm and Schulte 97, Ziegler 08b].

2.1 Simplicial Maps

Let M be a (closed) triangulated surface with $n = f_0$ vertices, f_1 edges, and f_2 triangles, i.e., M has *face-vector* $f = (n, f_1, f_2)$. If M has Euler characteristic $\chi(M)$, then by Euler's equation,

$$n - f_1 + f_2 = \chi(M).$$

Double counting of the incidences between edges and triangles of the triangulation yields $2f_1 = 3f_2$. So together,

$$f = (n, 3n - 3\chi(M), 2n - 2\chi(M)).$$

A triangulated surface with n vertices obviously has at most $f_1 \leq \binom{n}{2}$ edges. By plugging in $f_1 = 3n - 3\chi(M)$, we obtain Heawood's bound [Heawood 90] from 1890 that a triangulation of a 2-manifold M of Euler characteristic

$\chi(M)$ has at least

$$n \geq \left\lceil \frac{1}{2}\left(7+\sqrt{49-24\chi(M)}\right) \right\rceil$$

vertices. Heawood's bound is sharp for all surfaces, except for the orientable surface of genus 2, the Klein bottle, and the nonorientable surface of genus 3, where an extra vertex has to be added to the lower bound.

Corresponding *vertex-minimal triangulations* (i.e., triangulations with the minimal possible number of vertices) of the real projective plane $\mathbb{R}\mathbf{P}^2$ with six vertices and of the 2-torus with seven vertices (Möbius's torus [Möbius 86]) were already known in the nineteenth century, but it took until 1955 to complete the construction of (series of) examples of vertex-minimal triangulations for all nonorientable surfaces [Ringel 55] and until 1980 for all orientable surfaces [Jungerman and Ringel 80].

If a given triangulation of an orientable surface is realizable in $\mathbb{R}^3$, then so are subdivisions of it that are obtained by successively subdividing edges and triangles. Hence, vertex-minimal triangulations apparently are good candidates for nonrealizable simplicial maps. Here triangulations with

$$n = \frac{1}{2}\left(7+\sqrt{49-24\chi(M)}\right) \tag{2–1}$$

are of particular interest (cf. [Császár 50]), since for these we have $f_1 = \binom{n}{2}$, that is, the respective triangulations are *neighborly* with complete 1-skeleton (which should make realizability difficult).

A polyhedral realization of the combinatorially unique vertex-minimal 7-vertex triangulation of the torus with $f = (7, 21, 14)$ was given in [Császár 50] (although realizability possibly was known already to Möbius; cf. [Möbius 86, p. 553], [Reinhardt 85], and see [Lutz 02] for additional comments).

The next case of equality in (2–1) yields 59 examples of vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 [Altshuler et al. 96]; see below.

2.2 Realizability versus Nonrealizability of Simplicial Maps

For every individual triangulation of an orientable surface, realizability (in $\mathbb{R}^3$) can be decided algorithmically by the following two-step procedure; cf. [Bokowski 01], [Bokowski and Sturmfels 89, Chapter VIII]:

1. Enumerate all oriented matroids compatible with the given triangulation. If there are none, then the triangulation is not realizable.
2. Otherwise, decide realizability of the oriented matroids from step 1 via solving associated polynomial systems of inequalities.

Theoretically, the second step can be done algorithmically (for example, with Collins's *cylindrical algebraic decomposition* algorithm [Collins 75]). In practice, however, there are no methods known that work sufficiently fast to yield results even for small examples. See [Bokowski 01] and [Bokowski and Sturmfels 89, Chapter VIII] for more comments on this and on algebraic tools such as final polynomials. For general polyhedral maps on orientable surfaces, Brehm proved that the realizability problem is NP-hard (as a consequence of his universality theorem for realization spaces of maps; cf. [Ziegler 08a]). The complexity of the realization problem restricted to simplicial maps is unknown. In fact, it was open for a long time whether there are any nonrealizable examples at all.

In a major breakthrough, Bokowski and Guedes de Oliveira showed [Bokowski and Guedes de Oliveira 00] (using ten CPU years) that one of the 59 vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 has no compatible orientable matroid and therefore is not realizable.

Schewe substantially improved the enumeration of compatible orientable matroids [Schewe 07, Schewe 10] and was able to verify that in fact, all 59 vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 are nonrealizable. Moreover, he found three examples of nonrealizable vertex-minimal 12-vertex triangulations of the orientable surface of genus 5.

At least for one of these examples it is possible to remove a triangle from the triangulation while maintaining nonrealizability. Connected sums with other triangulations then still are nonrealizable. Hence, for every orientable surface of genus $g \geq 5$, there are triangulations that cannot be realized geometrically in $\mathbb{R}^3$.

Apart from the approach via oriented matroids, nonrealizability results (for simplicial maps in $\mathbb{R}^3$) seem to be difficult to achieve: [Novik 00] associates an integer program with a given triangulation, which, if it has no solution, yields nonrealizability. Improved systems have been proposed in [Timmreck 08]. So far, however, all tested systems for orientable surfaces either had solutions or turned out to be computationally intractable. In a different approach, [Brehm 83] uses a linking number argument to show that there is a nonrealizable triangulation of the Möbius strip with nine vertices.

2.3 Heuristics for the Realization of Simplicial Maps

Until recently, it was considered to be rather difficult and time-consuming to find realizations for given triangulations. Examples of polyhedral realizations of vertex-minimal triangulations of the orientable surfaces of genera 3 and 4 with 10 and 11 vertices, respectively, were constructed *by hand* in [Brehm 81, Brehm 87] and [Bokowski and Brehm 87, Bokowski and Brehm 89]. Some of these examples were found by exploiting combinatorial symmetries of the triangulations, others with the *rubber-band technique* of [Bokowski 08].

A simple computer heuristic (by choosing coordinates randomly) was used in [Lutz 08] to show that 864 of the 865 examples of vertex-minimal triangulations of the orientable surface of genus 2 are realizable. The remaining case then was settled by Bokowski with the rubber-band method [Bokowski 08]. All 865 examples were later found to have realizations with *small coordinates* [Hougardy et al. 07a], i.e., all these examples are realizable with integer coordinates in general position in the $(4 \times 4 \times 4)$ cube. Moreover, realizations in the $(5 \times 5 \times 5)$ cube were obtained for 17 of the 20 vertex-minimal triangulations with 10 vertices of the orientable surface of genus 3 by isomorphism-free enumeration of possible coordinate configurations in general position [Hougardy et al. 07b].

In the following, we will discuss an improved heuristic to obtain polyhedral realizations in $\mathbb{R}^3$ for triangulations of orientable surfaces. In particular, we will show that all vertex-minimal triangulations of the orientable surfaces of genera $g = 3$ and $g = 4$ are realizable and that there are examples of simplicial polyhedra of genus 5 with 12 vertices.

3. REALIZATION WITH THE INTERSECTION SEGMENT FUNCTIONAL

As mentioned in the previous section, there have been so far three major heuristics for the realization of simplicial surfaces (of genus $g \geq 1$) in $\mathbb{R}^3$:

- by explicit geometric construction [Bokowski and Brehm 87, Bokowski and Brehm 89, Brehm 81, Brehm 87] (e.g., via the rubber-band technique of [Bokowski 08]);
- by choosing coordinates randomly [Lutz 08];
- by enumeration of realizations with small coordinates [Hougardy et al. 07a, Hougardy et al. 07b, Hougardy et al. 08].

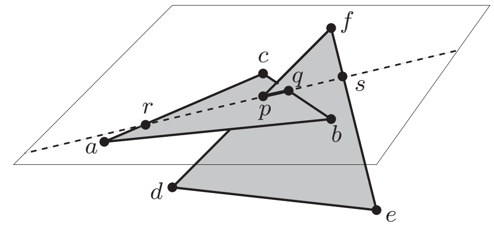

FIGURE 3. Two intersecting triangles.

As a more sophisticated approach we suggest to proceed as follows. For a given triangulation (of an orientable surface of small genus with few vertices):

1. start with random coordinates for the vertices of the triangulation,
2. and then "move vertices around" to eventually obtain a realization.

For the second step we take as an objective to minimize the *intersection segment functional*:

> Let M^2 be a triangulated orientable surface with vertex set V and let $V_{\mathbb{Z}^3}$ be a set of $|V|$ integer vertices in general position in $\mathbb{R}^3$. Then every pair of nonneighboring triangles of M^2 coordinatized with the coordinates of $V_{\mathbb{Z}^3}$ either has empty intersection in $\mathbb{R}^3$, intersects in one vertex, or intersects in a segment; see Figure 3 for the intersection segment p–q of two triangles. The sum of the lengths of the intersection segments over all pairs of nonneighboring triangles is the *intersection segment functional*.

We require that the points be *in general position*, i.e., no three points lie on a line and no four points lie in a plane, in order to avoid degenerate intersections of triangles. Further, we use integer coordinates and therefore move the points in the second step above on the integer grid $\mathbb{Z}^3$ only.

Our aim then will be to find integer coordinates in general position for which the intersection segment functional vanishes for the given triangulation.

From an initial set of random coordinates we proceed to minimize the intersection segment functional by a local search of *hill-climbing* type:

> In every step, we randomly pick a vertex $v \in V_{\mathbb{Z}^3}$ and a coordinate direction $\pm x$, $\pm y$, or $\pm z$, and then move the vertex v one integer step in the respective direction. If the resulting set of coordinates is in general position and the new value

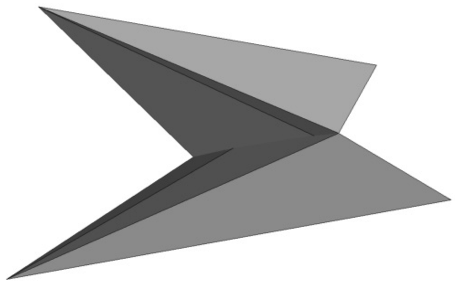

FIGURE 4. A "nonrealization" of the octahedron with locally minimal functional.

of the intersection segment functional is strictly smaller than before, the move is accepted and the next step is executed. Otherwise, the move is discarded and we start anew from the previous set of coordinates.

If all possible choices of moves have been tested for some set of coordinates without improvement, then we are stuck in a *local minimum*. In this case, for one step only, we choose one of the *admissible moves*, i.e., a move that yields a set of coordinates in general position, but that not necessarily decreases the intersection segment functional. From there, we then try to continue to decrease the intersection segment functional in a new direction.

Example 3.1. Local minima with a positive value of the intersection segment functional can occur even for small triangulations. For example, the boundary of the octahedron with triangles

$$123, \quad 124, \quad 135, \quad 145, \quad 236, \quad 246, \quad 356, \quad 456,$$

and furnished with coordinates (in general position)

$$1 : (4, 4, 6), \quad 2 : (5, 6, 6), \quad 3 : (9, 7, 4),$$
$$4 : (5, 9, 1), \quad 5 : (4, 6, 3), \quad 6 : (1, 5, 7),$$

attains a local minimum for the intersection segment functional with value 3.17; see Figure 4 for a visualization.

3.1 Details of the Algorithm

Initially, the vertices of the triangulation are placed randomly at general positions in a cube of size $50 \times 50 \times 50$. This cube is chosen at the center of a larger $(250 \times 250 \times 250)$ cube that we take as the *bounding box* for all possible positions of the vertices during the local search.

After the choice of starting positions, the smaller cube is not used.

- Thus, we allow the diameter of the vertex set to increase moderately (which possibly helps to decrease the intersection segment functional by unfolding the initial shape).
- At the same time there is a fixed lower bound for the change, at every step, of the intersection segment functional (determined by the size of the bounding box and the fact that we admit integer coordinates only). In this way, we avoid the sequence of improvements for the functional converging to zero.

An *admissible step*, then, is a movement of one vertex by one integer in one of the coordinate directions such that the resulting set of coordinates is in general position and is within the bounding box.

- If the intersection segment functional becomes zero, a realization for the given triangulation is found.
- If a realization is not found within a fixed period of time T, the whole process is restarted for the triangulation, beginning with the random selection of the starting coordinates (in the smaller cube). In so doing we try to overcome situations in which the process cycles between different local minima.

A standard problem with local search algorithms is the appropriate choice of the parameters that govern the procedure. For some of the 20 examples of vertex-minimal 10-vertex triangulations of the orientable surface of genus 3 we tried the following variants:

- We chose different sizes for the initial cube, ranging from $5 \times 5 \times 5$ to $500 \times 500 \times 500$.
- We allowed the bounding box to be between one and eight times the size of the initial cube.
- If the segment functional decreased by moving one vertex in one direction, we moved the vertex as far as possible in that direction (until the intersection segment functional began to increase again).
- In case of a local minimum, we determined all pairs of vertices for which the exchange of their positions decreased the intersection segment functional. We then executed one such exchange at random. If there was no such pair, we randomly exchanged two arbitrary vertices.

- Instead of minimizing the intersection segment functional we tried to minimize the *normalized intersection segment functional*, which is obtained from the intersection segment functional by dividing by the total length of the edges of the coordinatized triangulation.

- We first generated ten thousand sets of initial coordinates, from which we selected the set with the smallest functional before starting the local search.

From all these variants the previously described one turned out to have the best performance. This variant then was used to find realizations for other triangulations.

3.2 Test Sets of Minimal Triangulations

If some triangulation of an orientable surface is realizable, then so are all subdivisions of it that result from the starting triangulation by an iterative sequence of *elementary subdivisions* of triangles and edges (see Figure 5). For every geometrically realized triangulation in such a sequence, we can always place the new vertex slightly "above" or "below" the respective triangle or the respective edge of the previous realization. Alternatively, we could choose all new vertices on the original surface and then slightly perturb the coordinates of the new vertices into general position.

A triangulation of a surface is *minimal* if it does not result from a triangulation with fewer vertices by a sequence of elementary subdivisions. If all minimal triangulations of a surface are realizable, then all triangulations of the surface are realizable. Unfortunately, for surfaces of genus $g \geq 1$, the set of minimal triangulations is infinite: it comprises the infinite set of triangulations with all vertices of degree at least 5, since for any such triangulation we can replace the *star* of a vertex (i.e., all triangles that contain the vertex) with a patch that has more vertices, but all of degree at least 5; see Figure 6. Equivelar triangulations of surfaces of genus $g \geq 1$ are

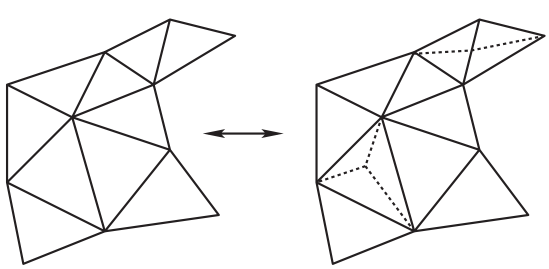

FIGURE 5. Subdivisions of a triangle and of an edge.

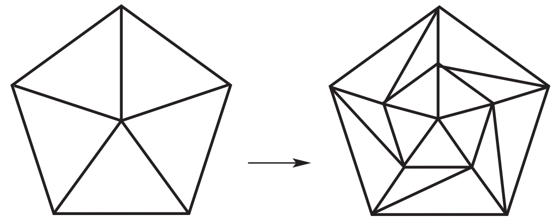

FIGURE 6. Replacement of the star of a vertex of degree 5.

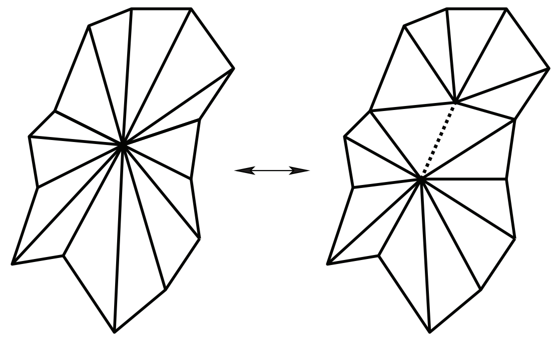

FIGURE 7. Expansion (respectively contraction) of an edge.

minimal. For $g = 1$ there are infinitely many equivelar triangulations, whereas for each $g > 1$ there are only finitely many examples; cf. [Sulanke and Lutz 09].

A finite subset (of the set of minimal triangulations) of particular interest for testing realizability is the set of vertex-minimal triangulations. If these are realizable, then this should give a strong indication that in fact, all triangulations of the surface are realizable.

A larger, but still finite, set of minimal triangulations that contains all vertex-minimal triangulations is defined as follows: If we allow *edge expansions* (with *edge contractions* as inverses (see Figure 7) instead of elementary subdivisions, then for every surface there is only a finite set (see [Barnette and Edelson 88]) of *irreducible* triangulations for which no edge can be contracted without changing the topological type of the triangulation. Unfortunately, it is unclear a priori whether a realizable triangulation of an orientable surface of genus $g \geq 1$ remains realizable after the expansion of an edge.

At least for every explicit polyhedral realization it can easily be tested whether a particular edge expansion can be carried out (via a system of linear constraints on the link of the respective vertex; we thank the anonymous referee for pointing this out to us).

It follows from [Steinitz and Rademacher 34, Section 46] that every triangulated 2-sphere can be reduced to the boundary of the tetrahedron by a sequence of edge contractions; that is, the boundary of the tetra-

g	$n_{\min}$	Types
0	4	1
1	7	1
2	10	865
3	10	20
4	11	821
5	12	751,593
6	12	59

TABLE 1. Numbers of vertex-minimal triangulations of the orientable surfaces of genus $g \leq 6$.

hedron is the only irreducible triangulation of the 2-sphere. The number of irreducible triangulations of the torus was determined by Grünbaum and independently in [Lavrenchenko 87]: there are 21 such examples with up to 10 vertices, and they are all realizable. It is shown in [Sulanke 06a, Sulanke 06b, Sulanke 05] by enumeration that there are exactly 396,784 examples of irreducible triangulations (with up to 17 vertices) of the orientable surface of genus 2.

Although it might be desirable to test realizability for a larger set of irreducible triangulations, we restricted ourselves to vertex-minimal ones. There is only one unique vertex-minimal triangulation of the torus, i.e., Möbius's 7-vertex torus [Möbius 86], for which [Császár 50] gives an explicit polyhedral model. Vertex-minimal triangulations of the orientable surfaces of genera 2 and 3 were enumerated in [Lutz 08], those of genera 4 and 5 in [Sulanke and Lutz 09], and the vertex-minimal examples of genus 6 in [Altshuler et al. 96]; see Table 1 for the corresponding minimal numbers of vertices $n_{\min}$ and the respective numbers of combinatorial types of triangulations.

4. COMPUTATIONAL RESULTS

4.1 Genus 2

In [Lutz 08], geometric realizations for 864 of the 865 vertex-minimal 10-vertex triangulations of the orientable surface of genus 2 were found with the random realization approach in a total computation time of 30 CPU months on a 2.8-GHz processor; the remaining example then was realized with the rubber-band method [Bokowski 08]. For realizations of the 865 examples with small coordinates, see [Hougardy et al. 07a] and the comments above. With our new heuristic algorithm, based on the intersection segment functional, realizations for the 865 triangulations were obtained in a total time of 218 CPU minutes on a 3.5-GHz processor.

4.2 Genus 3

Realizations for 5 of the 20 vertex-minimal 10-vertex triangulations of the orientable surface of genus 3 were constructed by hand in [Bokowski and Brehm 87, Brehm 81, Brehm 87]. The random realization approach of [Lutz 08], however, produced no results for these 5 (and for the other 15) examples, where we stopped the search after one CPU week each. Therefore, the basic random realization approach is not suitable for triangulations of surfaces of higher genus (or with more vertices). For 17 of the 20 triangulations, realizations with small coordinates in the $(5 \times 5 \times 5)$ cube were obtained in [Hougardy et al. 07b]; this search was run (in total) for 2 CPU years on a 3.5-GHz processor. Thus, the first task for our new program was to realize the remaining three examples.

Theorem 4.1. *All* 20 *vertex-minimal* 10*-vertex triangulations of the orientable surface of genus* 3 *are geometrically realizable in* $\mathbb{R}^3$.

Sets of coordinates for the realizations are available online; see [Hougardy et al. 07b, Lutz 10]. In total, it took 28 CPU hours on a 3.5-GHz processor to realize the 20 examples with the help of the intersection segment functional. For two of the last three of the 20 examples, we later found realizations in the $(6 \times 6 \times 6)$ cube; cf. [Lutz 10].

4.3 Genus 4

A first example of a polyhedron of genus 4 with 11 vertices was described in [Bokowski and Brehm 89]. With our intersection segment functional algorithm we found realizations for 626 of the 821 vertex-minimal 11-vertex triangulations of the orientable surface of genus 4. In an effort to speed up the search, realizations for the remaining 195 triangulations were obtained by *recycling of coordinates*, that is, whenever a new realization was found, we tried to reuse the respective set of coordinates for other triangulations. We also slightly distorted the coordinates and then tried to use these coordinates for other triangulations; see [Lutz 08] for additional comments.

Theorem 4.2. *All* 821 *vertex-minimal* 11*-vertex triangulations of the orientable surface of genus* 4 *are geometrically realizable in* $\mathbb{R}^3$.

We needed a total of $9.51 \cdot 10^{11}$ steps of the local search process to realize the 626 triangulations. As time interval T we chose 15 minutes, so if after 15 minutes

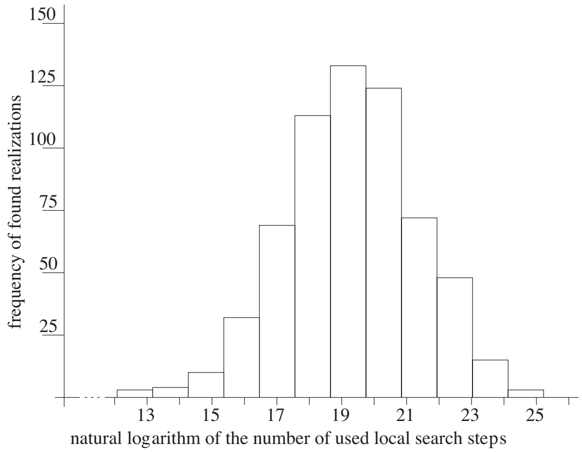

FIGURE 8. Histogram of the natural logarithms of the used local search steps for 626 realizations of genus 4.

(about $5.4 \cdot 10^6$ steps) a realization was not reached, the search was restarted with new initial coordinates.

Figure 8 displays a histogram of the natural logarithms of the number of used steps. The picture indicates that the logarithms of the used steps are normally distributed, i.e., the used steps underlie a log-normal distribution. To confirm this, we ran as a goodness-of-fit-test [Stuart and Ord 87, Chapter 30], the Anderson–Darling test (cf. [Everitt 98, p. 10]). The test estimates the mean to be 19.3 and the standard deviation to be 2. It yields a p-value of 0.5, which is far above the rejection value of 0.05. Therefore we can view the logarithms of the used steps to be normally distributed with the estimated parameters.

Our implementation of the local search process is performing about $3.6 \cdot 10^5$ steps per minute on a 3.5-GHz processor. Therefore, we needed a total of 5 CPU years to realize all triangulations. On average, it took 2.9 CPU days to find a realization for a single triangulation.

4.4 Genus 5

As mentioned in Section 2, [Schewe 07, Schewe 10] showed that there are at least three examples of vertex-minimal 12-vertex triangulations of the orientable surface of genus 5 that cannot be realized geometrically in $\mathbb{R}^3$.

In order to complement Schewe's result, we tried to find realizations for at least some of the vertex-minimal 751,593 triangulations. To this end we started our process on randomly selected triangulations out of all the 751,593 triangulations. If after 15 minutes a realization was not found, a new triangulation was selected at random. In this way, we tried about 94,000 triangulations, using a total of $7.52 \cdot 10^{11}$ local search steps, a CPU time of approximately four years, and succeeded in realizing 15 triangulations.

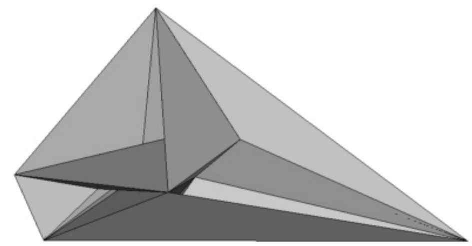

FIGURE 9. A polyhedron of genus 5.

Theorem 4.3. *At least* 15 *of the* 751,593 *vertex-minimal* 12*-vertex triangulations of the orientable surface of genus* 5 *are geometrically realizable in* $\mathbb{R}^3$.

Since the 94,000 triangulations we tested were chosen randomly from the list of the 751,593 genus-5 triangulations, probably at least 120 (and perhaps many more) of the examples are realizable.

Example 4.4. Figure 9 displays one of the polyhedra of genus 5 with 12 vertices, which has triangles

1 2 3, 1 2 4, 1 3 5, 1 4 6, 1 5 7, 1 6 8, 1 7 9, 1 8 10,
1 9 10, 2 3 6, 2 4 5, 2 5 8, 2 6 10, 2 8 11, 2 9 11, 2 9 12,
2 10 12, 3 5 11, 3 6 8, 3 7 8, 3 7 10, 3 9 10, 3 9 11, 4 5 9,
4 6 11, 4 7 8, 4 7 12, 4 8 9, 4 10 11, 4 10 12, 5 6 9,
5 6 10, 5 7 10, 5 8 12, 5 11 12, 6 7 9, 6 7 12 6 11 12,
8 9 12, 8 10 11

and coordinates

$$1 : (137, 124, 141) \quad 2 : (107, 118, 143) \quad 3 : (132, 130, 125)$$
$$4 : (122, 127, 129) \quad 5 : (124, 129, 132) \quad 6 : (126, 130, 124)$$
$$7 : (126, 129, 129) \quad 8 : (122, 125, 138) \quad 9 : (124, 128, 136)$$
$$10 : (119, 133, 134) \quad 11 : (120, 130, 135) \quad 12 : (121, 128, 133)$$

within the $(250 \times 250 \times 250)$ cube.

For the coordinates for the other 14 examples, see [Lutz 10].

Combining the result of [Schewe 07, Schewe 10] (that there are nonrealizable triangulations of the orientable surface of genus 5) with our finding (that all vertex-minimal triangulations of surfaces of genus $g \leq 4$ are realizable) gives rise to the following conjecture.

Conjecture 4.5. *Every triangulation of an orientable surface of genus $g \leq 4$ is geometrically realizable.*

The conjecture holds for genus 0 [Steinitz 22, Steinitz and Rademacher 34] and for genus 1 [Archdeacon et al. 07]).

4.5 Examples with More Vertices

We also tried our program on some triangulations of tori with more vertices. It turned out that it still is possible to find realizations, although it takes much longer for every step of the local search process: There are $O(|V|^2)$ pairs of triangles that have to be considered for the computation, respectively for the update, of the intersection segment functional. Moreover, there are $6|V|$ possible moves from a current set of coordinates that lead to a new set of coordinates. In the worst case, we are forced to test almost all these moves just to carry out a single improvement step. Finally, the initial value of the intersection segment functional will be larger for triangulations with more vertices, thus forcing us to perform more steps.

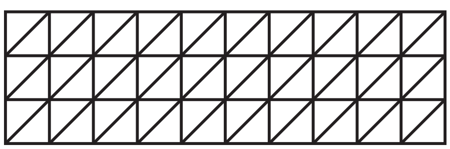

FIGURE 10. The standard (3×10) torus.

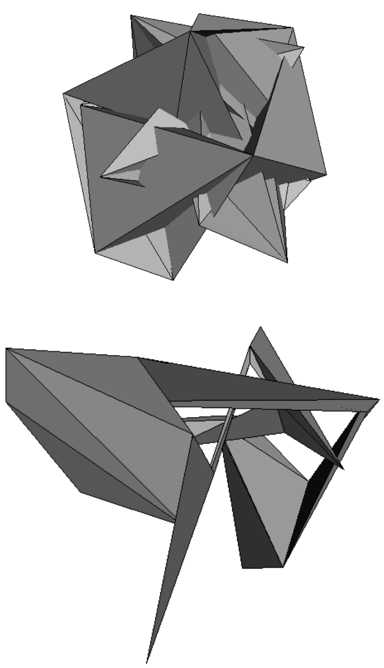

FIGURE 11. The standard (3×10) torus with random coordinates and a proper realization.

Example 4.6. For the standard (3×10) torus (Figure 10), we started with random coordinates (Figure 11, top) and an initial value 7924.26 of the intersection segment functional. It then took 3042 local search steps to obtain a proper realization (Figure 11, bottom).

5. CONVEX REALIZATIONS OF TRIANGULATED 2-SPHERES

According to [Steinitz 22, Steinitz and Rademacher 34], every polyhedral map on the 2-sphere S^2 is geometrically realizable in $\mathbb{R}^3$ as the boundary complex of a convex 3-polytope. Tutte's equilibrium method [Tutte 63] (see also [Richter-Gebert 96, Section 12.2]) allows us to obtain corresponding realizations algorithmically via first constructing a planar equilibrium representation of the edge graph of a given map. The resulting planar graph can then be interpreted as the Schlegel diagram of a 3-polytope.

Triangulated 2-spheres are realizable as boundary complexes of simplicial 3-polytopes. However, simplicial 3- and higher-dimensional spheres need not be polytopal. The Brückner–Grünbaum 3-sphere [Grünbaum and Sreedharan 67] and the Barnette 3-sphere [Barnette 73], both with eight vertices, are the smallest examples of nonpolytopal simplicial spheres.

In the following, we give a simple modification of our realization heuristic in order to obtain *convex realizations* of triangulated 2-spheres. (Using an *intersection area functional* one might generalize this approach to search for convex realizations of simplicial 3-spheres in $\mathbb{R}^4$.)

A *nonface* of a triangulated 2-sphere with n vertices is a two-element subset (an edge) or a three-element subset (a triangle) of the ground set of n vertices that does not constitute a face of the triangulation. In any convex realization (in general position) of the triangulated 2-sphere as the boundary complex of a simplicial 3-polytope, every nonface intersects the interior of the respective polytope. In particular, every face of the boundary 2-sphere either has no intersection with a given nonface or intersects the nonface in an edge or a vertex of the boundary sphere.

Hence, by adding to the intersection segment functional the lengths of intersection segments for all pairs of (nonneighboring) triangles consisting of a triangle of the triangulation and a triangle that does not belong to the triangulation, the resulting *extended intersection segment functional* can be used to obtain convex realizations for triangulated 2-spheres. To be more precise, we state the following proposition.

Proposition 5.1. *If a triangulated 2-sphere has no vertex of degree 3, then the extended intersection segment functional is zero if and only if a convex realization (with vertices in general position) has been reached.*

Proof: If a convex realization has been reached, then obviously the extended intersection segment functional is zero.

For the other direction, assume that the functional is zero and that the vertices are in general position. In case some vertex v is contained in the convex hull of the other $n-1$ vertices, then v is contained in the convex hull of some subset $\{v_1, v_2, v_3, v_4\}$ of four of the $n-1$ vertices: Pick any vertex v_1 on the boundary of the convex hull of the n vertices; then by the general position assumption, there is a unique triangle $\{v_2, v_3, v_4\}$ (opposite to v_1 with respect to v) on the boundary of the convex hull such that the tetrahedron $\{v_1, v_2, v_3, v_4\}$ contains v.

Without loss of generality, we may assume that no other of the $n-1$ vertices is contained in the tetrahedron $\{v_1, v_2, v_3, v_4\}$: If there is an additional such vertex, say v', then the convex hull of v' with the triangle on the boundary of the tetrahedron $\{v_1, v_2, v_3, v_4\}$ opposite to v' with respect to v is a tetrahedron (of smaller volume) that contains v. Moreover, since the smallest example of a triangulated 2-sphere without a vertex of degree 3 is the boundary complex of the octahedron with six vertices, there is at least one vertex of the triangulation that lies outside the tetrahedron $\{v_1, v_2, v_3, v_4\}$.

If the vertex v has degree larger than 4, then at least one of the triangles of the star of v intersects nontrivially some boundary triangle (a face or a nonface of the triangulation) of the tetrahedron $\{v_1, v_2, v_3, v_4\}$. The line segment in which the two triangles intersect contributes a positive value to the extended intersection segment functional, a contradiction.

If v has degree 4, there are two cases. If the star of the vertex v contains a vertex different from v_1, v_2, v_3, v_4, then at least one of the triangles in the star of v intersects some triangle of the boundary of the tetrahedron $\{v_1, v_2, v_3, v_4\}$, a contradiction. Otherwise, let v_5 be a vertex outside the convex hull of the vertices v_1, v_2, v_3, v_4. Then v lies in the convex hull of v_5 and some triangle $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ of the tetrahedron $\{v_1, v_2, v_3, v_4\}$. But then the vertex star of v contains the vertex v_4, which lies outside the tetrahedron spanned by the vertices $v_{i_1}, v_{i_2}, v_{i_3}, v_5$. This again leads to a contradiction. □

In case a triangulation has vertices of degree 3, nonconvex realizations of the triangulation can have vanishing extended intersection segment functional. The smallest such example is the bipyramid over a triangle with one apex pushed inside the convex hull of the other four vertices.

Nevertheless, we can recursively remove vertices of degree 3 from a given triangulation. The resulting triangulation then is either the boundary of a tetrahedron or a triangulation with vertices all of degree at least four. After obtaining a realization for the simplified triangulation, the removed vertices can be added back by placing them suitably "above" the triangles that they subdivide.

We successfully tested our approach for some small triangulations of S^2: there are 233 triangulations of S^2 with 10 vertices, of which 12 examples have no vertices of degree 3. It took, on average, about five minutes to obtain convex realizations for these 12 triangulations.

Remark 5.2. Although our main focus in this paper was on the realization of closed, orientable triangulated surfaces, the intersection segment functional can, of course, also be used to search for realizations in three-space for other 2-dimensional simplicial complexes. For example, the functional was modified in [Leopold 09] for a search for immersions of (orientable or nonorientable) triangulated surfaces, as well as for a search for symmetric realizations and immersions.

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